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Multiobjective Optimization of the Flow Around a Cylinder Using Model Order Reduction

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In this article an efficient numerical method to solve multiobjective optimization problems for fluid flow governed by the Navier Stokes equations is presented. In order to decrease the computational effort, a reduced order model is introduced using Proper Orthogonal Decomposition and a corresponding Galerkin Projection. A global, derivative free multiobjective optimization algorithm is applied to compute the Pareto set (i.e. the set of optimal compromises) for the concurrent objectives *minimization of flow field fluctuations* and *control cost*. The method is illustrated for a 2D flow around a cylinder at $Re = 100$.

1 Introduction

In many applications, one is interested in simultaneously optimizing several - possibly contradicting - criteria. Therefore, the task of computing the set of optimal compromises between the conflicting objectives arises which leads to a multiobjective optimization problem (MOP). When the underlying model is described by a (nonlinear) PDE, as is the case for fluid flow, the computation of this so-called *Pareto set* can quickly become numerically infeasible. A reduction of the computational effort can be achieved by approximating the PDE by a reduced order model of low dimension.

In this article, we show how reduced order modelling can be used to compute the Pareto set for the conflicting objectives *minimization of flow field fluctuations* and *control cost* for the flow around a cylinder at $Re = 100$ (Fig. 2a). The flow field $y(x,t)$ is described by the 2D incompressible Navier Stokes equations within the domain Ω. Dirichlet boundary conditions (BC) with a constant horizontal velocity are imposed on the inflow as well as on the upper and lower boundary, whereas a standard no-shear Neuman BC is imposed on the outflow boundary. On the cylinder Γ_c we prescribe a time dependent Dirichlet BC such that it performs a rotation around its center with the tangential velocity $v_c(t)$. This, together with a no-slip condition, serves as the control mechanism for the flow: $u(t) = v_c(t)$.

2 Multiobjective Optimization

We are interested in minimizing the fluctuations $\widetilde{y}(x,t) = y(x,t) - \overline{y}(x)$ around the mean flow field $\overline{y}(x) = 1/T \int_0^T y(x,t)dt$ and minimizing the control cost at the same time by means of controlling the cylinder rotation which leads to a multiobjective optimal control problem. By defining the control according to $u(t;\omega,f) = d/2\omega\cos(2\pi f\ t)$, where d is the diameter of the cylinder and ω and f are the amplitude and the frequency of the sinusoidal cylinder rotation respectively, we obtain the following objective function $J : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\min_{\omega,f} J(\omega,f) = \min_{\omega,f} \begin{pmatrix} g_1(\omega,f) \\ g_2(\omega,f) \end{pmatrix} = \min_{\omega,f} \begin{pmatrix} \int_0^T \|\widetilde{y}(x,t)\|_{L^2}^2\, dt \\ \int_0^T u^2(t;\omega,f)\, dt \end{pmatrix}. \tag{MOP}$$

Thus, we have transformed the problem into a classical multiobjective optimization problem. Note that each choice of ω and f provides a unique control $u(t)$ and a unique solution of the flow field $y(x,t)$.

Many approaches exist to compute the Pareto set of multiobjective optimization problems, cf. [1]. Here, we use a global, derivative free algorithm contained in the software package GAIO [2]. The algorithm computes a nested sequence of increasingly fine box coverings of the Pareto set, also in the situation where the set is disconnected. First, we define an n-dimensional box B_{max} with a sufficiently large radius such that it contains the Pareto set, where n is the dimension of the parameter space (here: $n = 2$). We define a box collection $\mathcal{B}_0$ with $\cup_{B\in\mathcal{B}_0} B = B_{max}$. Then, the following three steps are repeated until a sufficiently close covering of the Pareto set is achieved: (1) A subdivision step is performed such that $\cup_{B\in\mathcal{B}_k} B = \cup_{B\in\mathcal{B}_{k-1}} B$ and max(diam($\mathcal{B}_k$)) < max(diam($\mathcal{B}_{k-1}$)). By construction, diam($\mathcal{B}_k$) $\to 0$ as $k \to \infty$. (2) The objective function J is evaluated at multiple sampling points in each box $B \in \mathcal{B}_k$. (3) A nondominance test over all sampling points is performed and all boxes containing only dominated points are eliminated from the box collection.

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3 Reduced Order Model

Since the algorithm for solving (MOP) requires a considerable number of function evaluations, the PDE describing the flow field is replaced by a controllable reduced order model in order to decrease the computational effort. Therefore, follwing [3], it is assumed that the flow field can be decomposed into a mean flow field $\overline{y}(x)$, a fluctuating flow field $\widetilde{y}(x,t)$ and a so-called control function $y_c(x)$:

$$y(x,t;\gamma(t)) = \overline{y}(x) + \gamma(t) y_c(x) + \widetilde{y}(x,t) \qquad \text{for } (x,t) \in (\Omega \times [0,T]).$$

The decomposition is performed in such a manner that the fluctuations $\widetilde{y}(x,t)$ satisfy homogeneous BC while $y(x,t;\gamma(t))$ still satisfies the BC on all boundaries. The control parameter $\gamma(t)$ introduces a homotopy between the uncontrolled flow field ($\gamma(t) = 0 \Leftrightarrow u(t) = 0$) and a flow field controlled by a constant control u_c ($\gamma(t) = 1 \Leftrightarrow u(t) = u_c$). Using this ansatz, $y(x,t;\gamma(t))$ inherits the boundary control of the PDE system. We then replace the fluctuations $\widetilde{y}(x,t)$ by a Galerkin ansatz:

$$y(x,t;\gamma(t)) \approx \widehat{y}(x,t;\gamma(t)) = \overline{y}(x) + \gamma(t) y_c(x) + \textstyle\sum_{j=1}^{K} \alpha_j(t)\psi_j(x) \qquad \text{for } (x,t) \in (\Omega \times [0,T]), K \in \mathbb{N}^+,$$

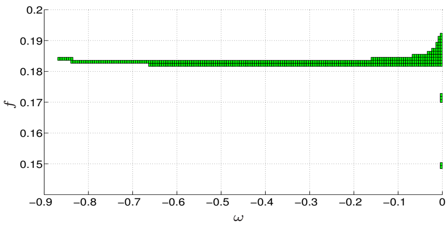
where $\alpha(t)$ and $\psi(x)$ are time dependent coefficients and global basis functions, respectively. The basis functions are computed by the well known Proper Orthogonal Decomposition (POD) [4]. For this purpose, snapshots are taken from $\widetilde{y}(x,t)$ calculated at a reference point with a reference control $u(t) = 0$. The data is generated using a detailed finite element (FEM) simulation. Inserting $\widehat{y}(x,t;\gamma(t))$ into the Navier Stokes equations yields a controllable reduced order model of dimension eight for the time dependent basis coefficients $\alpha(t;\gamma(t;\omega,f))$. The numerical solution of the reduced model is approximately 4500 times faster than the solution via an FEM discretization on a triangular mesh with 17800 degrees of freedom.

The multiobjective optimization problem can now be reformulated in terms of the reduced order model:

$$\min_{\omega,f} \begin{pmatrix} \widehat{g_1}(\omega,f) \\ \widehat{g_2}(\omega,f) \end{pmatrix} = \min_{\omega,f} \begin{pmatrix} \int_0^T \| \sum_{i=1}^{K} \alpha_i(t)\,\psi_i(x) \|_{L^2}^2 \, dt \\ \int_0^T \gamma^2(t;\omega,f)\, dt \end{pmatrix} \text{ s.t. } \gamma(t;\omega,f) = \omega\cos(2\pi f\, t) \qquad \text{(MOP-R)}$$

4 Results

Figure 1 shows the Pareto set and Pareto front of (MOP-R). The set is mostly restricted to a narrow region around the frequency $\overline{f} \approx 0.183$ which is not surprising since this corresponds to the frequency of the von Kármán vortex street of the uncontrolled flow (cf. Fig. 2a).


a)

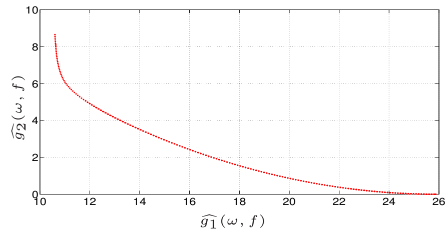
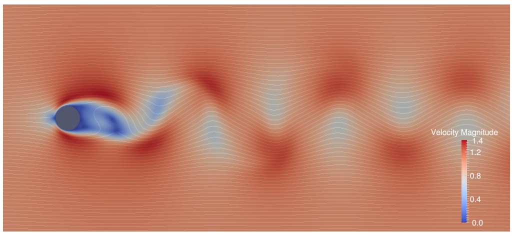

b)

Fig. 1: a) Pareto set of (MOP-R); b) Pareto front (i.e. the image of the Pareto set: flow field fluctuations and control cost) of (MOP-R)

It is worth mentioning that close to the minimal value of $\widehat{g_1}(\omega,f)$, a strong increase in the control cost is necessary in order to achieve further, yet small, improvements in flow stabilization. This illustrates the usefulness of the knowledge of the entire Pareto set. In fact, this particularly enables optimal switching of the system control during operation, for instance, in response to external influences.



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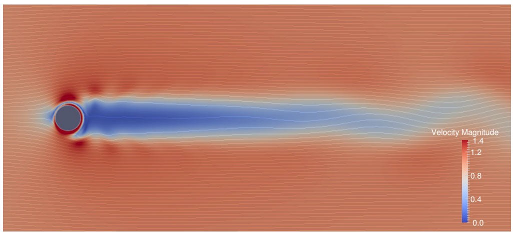

b)

Fig. 2: a) Flow field without boundary control; b) flow field corresponding to the parameters for maximal stabilization

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