Quantum Complexity Theory, UPB
Winter 2020, Assignment 6

To be completed by: Tuesday, January 26, start of tutorial

This assignment assumes the notation and terminology from Lectures 7 and 8.

1 Exercises

1. **Gentle measurement.** In this exercise we state and prove a variant of the Gentle Measurement Lemma seen in class. Let $|\psi\rangle$ be a pure quantum state, and consider any $0 \preceq M \preceq I$. (For example, M could be a projector, but need not be.) Suppose $\text{Tr}(M|\psi\rangle\langle\psi|) \geq 1 - \epsilon$ for $\epsilon > 0$. Prove that

$$\left\| |\psi\rangle\langle\psi| - \frac{\sqrt{M}|\psi\rangle\langle\psi|\sqrt{M}}{\text{Tr}(M|\psi\rangle\langle\psi|)} \right\|_{\text{tr}} \leq 2\sqrt{\epsilon}.$$

2. **SDPs.** In our standard form for SDPs, we used Hermiticity-preserving maps Ψ and Ψ^*, where recall the latter is the *adjoint* of the former.
   (a) Let $\Phi(X) = UXU^\dagger$ for unitary U. What is Φ^*?
   (b) Let $\Phi(X_{AB}) = \text{Tr}_B(X_{AB})$. What is Φ^*?

3. **QIP=PSPACE.** Recall in the correctness analysis, and particularly in Lemma 40, that we defined dual feasible solution

$$Y = \frac{1+2\epsilon}{T} \sum_{t=0}^{T-1} \frac{1}{\beta_t} \Pi_t.$$

   (a) Show that $Y \succeq 0$.
   (b) Prove that

$$\text{Tr}(W_T) = \text{Tr}\left(e^{-\epsilon\delta \sum_{j=0}^{T-1} \Phi^*\left(\frac{1}{\beta_j}\Pi_j\right)} \right) \geq e^{-\epsilon\delta\lambda_{\min}\left(\Phi^*\left(\sum_{j=0}^{T-1} \frac{1}{\beta_j}\Pi_j\right)\right)}.$$

   (c) For any $t \in 1, \ldots, T$, use the Golden-Thompson inequality to show that

$$\text{Tr}(W_t) \leq \text{Tr}(W_{t-1} e^{-\epsilon\delta\Phi^*(\Pi_{t-1}/\beta_{t-1})}).$$

   (d) Use the fact that for any Hermitian M satisfying $0 \preceq M \preceq I$, and every real $r > 0$,

$$e^{rM} \preceq I + re^r M \quad \text{and} \quad e^{-rM} \preceq I - re^{-r} M,$$

   to obtain that

$$e^{-\epsilon[\delta\Phi^*(\Pi_{t-1}/\beta_{t-1})]} \preceq I - \epsilon\delta e^{-\epsilon}\Phi^*(\Pi_{t-1}/\beta_{t-1}).$$

   (Hint: Use submultiplicativity of the spectral norm to show that $\| \Phi^*(\Pi_{t-1}) \|_\infty \leq \| Q^{-1} \|_\infty$.)