Quantum Complexity Theory, UPB
Winter 2020, Assignment 5

To be completed by: Tuesday, January 12, start of tutorial

This assignment assumes the notation and terminology from Lecture 6.

1 Exercises

1. **Local Hamiltonians.** Recall that in the NP-complete problem MAX CUT, the input consists of a simple, undirected graph $G = (V, E)$, and a threshold parameter t. The output is to decide whether the maximum cut in G has size at least t. Here, a *cut* in G is a partition $\{V_1, V_2\}$ of V, and the *size* of the cut is the number of edges in E with precisely one endpoint in each of V_1 and V_2. Give a 2-local Hamiltonian H on n qubits whose ground state energy $\lambda(H)$ equals precisely $|E| - \text{OPT}_1$, for OPT_1 the size of the optimal cut in G. (Hint: For each edge $(i, j) \in E$, start by thinking about 2-local quantum constraint $H_{ij} = Z_i \otimes Z_j$; what is the matrix representation of H_{ij}?)

2. **Containment of** k-LH **in** QMA**.**

   (a) In class, given k-local Hamiltonian $H = \sum_{i=1}^{m} H_i$, we saw how to put k-LH in QMA for $k \in O(1)$ via a simple measurement strategy: Given the ground state $|\psi\rangle$ as a proof, we picked a uniformly random local constraint H_i and measured it (as an observable) against our state $|\psi\rangle$. The expected value of this measurement was $\lambda_{\min}(H)/m$. We then converted this to a high-probability statement via the Höffding bound.

   There is, however, a more complicated "off-the-shelf" way of achieving the same containment, which we explore in this question: The phase estimation algorithm. You may assume for this question that $\| H_i \|_\infty \leq 1$ for each i.

   i. Suppose we could simulate $U = e^{icH}$ perfectly in polynomial time for $c \in O(1/\operatorname{poly}(n))$. Show how to use the quantum phase estimation algorithm (QPE) from Lecture 4 to verify the ground state energy of H in polynomial time. Two points you will need to consider: (1) To which accuracy must the phase be implemented, and how does this affect how many times we call U as a black box? (2) In the NO case (i.e. when $\lambda_{\min}(H) \geq \beta$), why does your approach correctly reject any proof $|\psi\rangle$?

   ii. We are left with the task of simulating $U = e^{icH}$. For simplicity, assume we have $H = H_1 + H_2$ (for H_i each acting on k qubits). A tool we may use for the simulation is the *Trotter formula*, which says that for evolution time $t \in \mathbb{R}^+$,

   $$e^{i(H_1+H_2)t} = \left(e^{\frac{iH_1 t}{r}} e^{\frac{iH_2 t}{r}}\right)^r + O(t^2/r).$$

   In other words, we can break up the time evolution as a *product* of evolving H_1 and H_2 independently for short time slices, t/r. The error ϵ so attained (with respect to spectral norm) then scales as $\epsilon \in O(t^2/r)$, assuming $\left\| e^{ic(H_1+H_2)t} \right\|_\infty \leq 1$ (why does this hold for all $t \in \mathbb{R}$?).

   A. Recall that our goal is to simulate $U = e^{ic(H_1+H_2)}$. Since k-LH has an inverse polynomial completeness-soundness gap, it suffices to simulate U to within error $\epsilon := 1/p(n)$ for some sufficiently large polynomial p; call this simulation U_ϵ. What choices of t and n suffice to implement U_ϵ according to the Trotter formula?

B. Show how to use the Trotter formula to implement U' for your choice of t and n. (Hint: How many qubits does $e^{iH_j t}$ act on?)

3. **QMA-hardness of** k-LH**.** This question will get you to work through some of the missing details in the proof of QMA-hardness presented in class. Again, we assume the notation introduced therein.

(a) Prove that $\langle\psi_{\text{hist}}|H_{\text{out}}|\psi_{\text{hist}}\rangle = \frac{1}{m+1}\Pr[V \text{ rejects } |\psi\rangle] \leq \frac{\epsilon}{m+1}$ in the YES case.

(b) Prove that

$$UH_{\text{prop}}U^\dagger = \sum_{t=0}^{m-1} -I\otimes|t+1\rangle\langle t|_D - I\otimes|t\rangle\langle t+1|_D + I\otimes|t\rangle\langle t|_D + I\otimes|t+1\rangle\langle t+1|_D.$$

Convince yourself that $UH_{\text{prop}}U^\dagger = I_{A,B,C}\otimes\Lambda_D$ indeed has representation

$$\Lambda := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ 0 & 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & 0 & -1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

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(c) In the NO case, prove that $\max_{\substack{|y\rangle\in\text{Null}(H'_{\text{prop}}) \\ \||y\rangle\|_2=1}}\langle y|\Pi_{N_2}|y\rangle = \frac{m-1}{m+1}$.

(d) In the NO case, prove that $\langle y|\Pi_{N_1}+\Pi_{N_3}|y\rangle \leq \frac{1+\sqrt{\epsilon}}{m+1}$.