Quantum Complexity Theory, UPB
Summer 2019, Assignment 2

To be completed by: Friday, May 3, start of lecture

Notes: A $^{(**)}$ indicates a particularly important exercise.

1 Exercises

1. Prove that for any normalized vectors $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$,
$$\| \, |\psi\rangle - |\phi\rangle \, \|_2 = \sqrt{2 - 2 \cdot \text{Re}(\langle\psi|\phi\rangle)}.$$
Why does it not matter if we replace $\langle\psi|\phi\rangle$ with $\langle\phi|\psi\rangle$ in this equation?

2. Use the spectral decompositions of X and Z to prove that $HXH^\dagger = Z$. (Do not simply write out the matrices and multiply!) Why does this immediately also yield that $HZH^\dagger = X$?

3. Write down a quantum circuit which maps the Bell basis $\mathcal{B} = \{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$ to the standard basis $\mathcal{B}' = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ for $\mathbb{C}^2$.

4. Prove that a Hermitian matrix $A \in \mathcal{L}(\mathbb{C}^d)$ is positive semi-definite if and only if for all $|\psi\rangle \in \mathbb{C}^d$, $\langle\psi|A|\psi\rangle \geq 0$. (Hint: Use spectral decompositions.)

5. Define bipartite state $|\psi\rangle = \alpha|01\rangle - \beta|10\rangle$. Let $\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\psi\rangle\langle\psi|$. Compute $\text{Tr}_B(\rho)$.

6. Let $|\psi\rangle = |-\rangle \in \mathbb{C}^2$. Define the Z basis $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and X basis $B' = \{|+\rangle\langle +|, |-\rangle\langle -|\}$. If we keep alternating measurements in the Z and X bases, what measurement statistics will we get for each of the respective outcomes (i.e. what will be the probabilities of each possible outcome each time a measurement is made)?

7. In this exercise, you will show that an operator U is unitary if and only if there exists a Hermitian operator H such that $U = e^{iH}$ for complex number i. This ties back to one of the most important equations in quantum mechanics, the *Schrödinger* equation, which roughly says that quantum systems evolve in time according to some "Hamiltonian" H, whose action on the system is given by e^{iH}; this is how the notion of unitary evolution actually comes about.

   (a) Let $H \in \text{Herm}(\mathbb{C}^n)$ and $c \in \mathbb{C}$. Using the Taylor series definition of e^{cH}, what does the spectral decomposition of e^{cH} look like?

   (b) Prove that for any $H \in \text{Herm}\left(\mathbb{C}^d\right)$, e^{iH} is unitary.

   (c) Next, characterize the set of possible eigenvalues for a unitary matrix.

   (d) Now prove that for any unitary $U \in \text{U}(\mathbb{C}^n)$, there exists an $H \in \text{Herm}(\mathbb{C}^n)$ such that $U = e^{iH}$.