

# Introduction to Quantum Computation, UPB

## Winter 2022, Assignment 2

To be completed by: Thursday, October 27

### 1 Exercises

1. Use the spectral decompositions of  $X$  and  $Z$  to prove that  $HXH^\dagger = Z$ . (Do not simply write out the matrices and multiply!) Why does this immediately also yield that  $HZH^\dagger = X$ ?
2. (a) Write out the 4-dimensional vector for  $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$ .  
(b) Let  $\mathcal{B}_1 = \{|\psi_1\rangle, |\psi_2\rangle\}$  and  $\mathcal{B}_2 = \{|\phi_1\rangle, |\phi_2\rangle\}$  be two orthonormal bases for  $\mathbb{C}^2$ . Prove that

$$\mathcal{B}_3 = \{|\psi_1\rangle \otimes |\phi_1\rangle, |\psi_1\rangle \otimes |\phi_2\rangle, |\psi_2\rangle \otimes |\phi_1\rangle, |\psi_2\rangle \otimes |\phi_2\rangle\}$$

is an orthonormal basis for  $\mathbb{C}^4$ . In other words, show that for each  $|v\rangle \in \mathcal{B}_3$ ,  $\| |v\rangle \|_2 = 1$ , and for all pairs of distinct  $|v\rangle, |w\rangle \in \mathcal{B}_3$ ,  $\langle v|w\rangle = 0$ .

3. (a) Prove that  $(Z \otimes Y)^\dagger = Z \otimes Y$ . Do not write out any matrices explicitly; rather, you must use the properties of the tensor product, dagger, and  $Y$ .  
(b) In class, we saw a quantum circuit which, given starting state  $|0\rangle \otimes |0\rangle$ , prepared the Bell state  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ . In fact, that circuit is a change of basis matrix, mapping the standard basis  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$  to the Bell basis  $|\Phi^+\rangle, |\Psi^+\rangle, |\Phi^-\rangle, |\Psi^-\rangle$ .
  - i. Write down a quantum circuit which maps  $|\Phi^+\rangle$  to  $|0\rangle \otimes |0\rangle$ . (Hint: Write out the circuit from class as a sequence of matrix operations. Then, recalling that the inverse of any unitary operation  $U$  is given by  $U^\dagger$ , take the inverse of this entire sequence by taking the dagger.)
  - ii. Your circuit from 3(b)(i) is actually a change of basis which maps the Bell basis *back* to the standard basis. To verify this, run your circuit from 3(b)(i) on input  $|\Psi^-\rangle$  and check that the output is  $|1\rangle \otimes |1\rangle$ .