Introduction to Quantum Computation, UPB
Summer 2021, Assignment 2

To be completed by: Friday, April 30

1 Exercises

1. Use the spectral decompositions of X and Z to prove that $HXH^\dagger = Z$. (Do not simply write out the matrices and multiply!) Why does this immediately also yield that $HZH^\dagger = X$?

2. (a) Write out the 4-dimensional vector for $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$.

   (b) Let $\mathcal{B}_1 = \{|\psi_1\rangle, |\psi_2\rangle\}$ and $\mathcal{B}_2 = \{|\phi_1\rangle, |\phi_2\rangle\}$ be two orthonormal bases for $\mathbb{C}^2$. Prove that

   $$\mathcal{B}_3 = \{|\psi_1\rangle \otimes |\phi_1\rangle, |\psi_1\rangle \otimes |\phi_2\rangle, |\psi_2\rangle \otimes |\phi_1\rangle, |\psi_2\rangle \otimes |\phi_2\rangle\}$$

   is an orthonormal basis for $\mathbb{C}^4$. In other words, show that for each $|v\rangle \in \mathcal{B}_3$, $\| |v\rangle \|_2 = 1$, and for all pairs of distinct $|v\rangle, |w\rangle \in \mathcal{B}_3$, $\langle v|w\rangle = 0$.

3. (a) Prove that $(Z \otimes Y)^\dagger = Z \otimes Y$. Do not write out any matrices explicitly; rather, you must use the properties of the tensor product, dagger, and Y.

   (b) In class, we saw a quantum circuit which, given starting state $|0\rangle \otimes |0\rangle$, prepared the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. In fact, that circuit is a change of basis matrix, mapping the standard basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ to the Bell basis $|\Phi^+\rangle, |\Psi^+\rangle, |\Phi^-\rangle, |\Psi^-\rangle$.

      i. Write down a quantum circuit which maps $|\Phi^+\rangle$ to $|0\rangle \otimes |0\rangle$. (Hint: Write out the circuit from class as a sequence of matrix operations. Then, recalling that the inverse of any unitary operation U is given by $U^\dagger$, take the inverse of this entire sequence by taking the dagger.)

      ii. Your circuit from 3(b)(i) is actually a change of basis which maps the Bell basis *back* to the standard basis. To verify this, run your circuit from 3(b)(i) on input $|\Psi^-\rangle$ and check that the output is $|1\rangle \otimes |1\rangle$.