Introduction to Quantum Computation, UPB
Summer 2021, Assignment 5

Due: Friday, May 21, at start of tutorial

1 Exercises

1. Define $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$, and consider projective measurement $M = \{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$ with labels corresponding to outcomes $S = \{1, -1\}$, respectively. Suppose state $|0\rangle \in \mathbb{C}^2$ is measured via C. What is the expected value for the measurement?

2. In this question, we consider how well the CHSH game strategy from class fares if we use a *less* entangled state as a shared resource between Alice and Bob. Specifically, imagine we use the same observables as before, but now we replace $|\Phi^+\rangle$ as a shared state with $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$. Intuitively, as α gets closer to 1, this state becomes less entangled, and for $\alpha = 1$, it becomes a product state (i.e. non-entangled).

   (a) What is the probability of winning the CHSH game with shared state $|\psi\rangle$? (Hint: Recall from Lecture 5 that for *any* $|\psi\rangle$, the quantity $\text{Tr}(A \otimes B|\psi\rangle\langle\psi|)$ equals Pr(output same bits) $-$ Pr(output different bits), i.e. the interpretation of this quantity does not depend on our choice of $|\psi\rangle$.)

   (b) Based on your answer above, what is the probability of Alice and Bob winning with this strategy if $\alpha = 1$, i.e. $|\psi\rangle$ is unentangled?

3. This question studies a 3-player non-local game called the *GHZ game*. There are now three players, Alice, Bob and Charlie, each of which receives a question q_a, q_b, or q_c, respectively, such that $q_a q_b q_c \in \{000, 011, 101, 110\}$. The players each return a bit $r_a, r_b, r_c \in \{0, 1\}$, respectively, and win if

   $$q_a \vee q_b \vee q_c = r_a \oplus r_b \oplus r_c,$$

   where $\vee$ denotes the binary OR operation.

   An analysis similar to the CHSH game shows that the optimal winning classical strategy yields success probability 3/4. Your task in this question is to analyze an optimal quantum strategy.

   The 3-qubit state the players share is

   $$|\psi\rangle = \frac{1}{2}\left(|000\rangle - |011\rangle - |101\rangle - |110\rangle\right) \in \mathbb{C}^8.$$

   Each player will use the same measuring strategy: Given input bit 0, they will apply local unitary $U_0 = I$, and if they get input 1, they apply local unitary $U_1 = H$. They then measure their qubit in the standard basis, and return the answer (0 or 1). As for CHSH, we assume the labels for the measurement outcomes are $+1, -1$ for measurement outcomes $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, respectively.

   (a) Suppose Alice gets input 0. What is her observable? What if she gets input 1?

   (b) Suppose the questions are $q_A q_B q_C = 000$. What is the probability the players win?

   (c) Suppose the questions are $q_A q_B q_C = 011$. What is the probability the players win?