Introduction to Quantum Comp utation, UPB
Summer 2021, Assignment 4

To be completed by: Friday, May 14, start of tutorial

1 Exercises

1. (a) Let $A, B \in \mathcal{L}(\mathbb{C}^d)$ be positive semi-definite matrices. Prove that $A + B$ is positive semi-definite.

   (b) Prove that if ρ and σ density matrices, then so is $p_1\rho + p_2\sigma$ for any $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$.

2. Suppose that with probability 1/3, I give you state $|0\rangle \in \mathbb{C}^2$, and with probability 2/3, I give you state $|-\rangle$. Write down (i.e. as a 2×2 matrix) the density matrix describing the state in your possession.

3. Define bipartite state $|\psi\rangle = \alpha|01\rangle - \beta|10\rangle$. Let $\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\psi\rangle\langle\psi|$. Compute $\mathrm{Tr}_B(\rho)$. (Hint: Use the fact that the partial trace is linear, and that you already know $\mathrm{Tr}_B(|\Phi^+\rangle\langle\Phi^+|)$ from class.)

4. Let $|\psi\rangle = \alpha_0|a_0\rangle|b_0\rangle + \alpha_1|a_1\rangle|b_1\rangle$ be the Schmidt decomposition of a two-qubit state $|\psi\rangle$. Prove that for any single qubit unitaries U and V, $|\psi\rangle$ is entangled if and only if $|\psi'\rangle = (U \otimes V)|\psi\rangle$ is entangled. (Hint: Prove that that the Schmidt rank of $|\psi\rangle$ equals that of $|\psi'\rangle$. Also, you might find Lemma 1 of the Lecture 3 notes useful.)