Fundamental Algorithms
Chapter 7: Linear Programming

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WS 2019

Outline

1. Definitions
2. Applications
3. Duality theory
4. Solving LPs

References

- CLRS Chapter 29
- Convex Optimization (Boyd and Vandenberghe):
  `https://web.stanford.edu/~boyd/cvxbook/`
- Luca Trevisan lecture notes:
  `http://theory.stanford.edu/~trevisan/cs261/lecture15.pdf`

Motivation

- Studied shortest paths, matchings, network flow, etc.
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- . . . have long history of algorithms for them (Simplex Method, Ellipsoid Method, Interior Point Methods).
- . . . can be generalized further to SDPs, cone programs, etc. Here, we focus on LPs.

Example

- Konditorei X produces 3 types of cake: Kirschtorte, Mohnkuchen, Sachertorte.
- X sells a whole cake of each type for 30,20,40 EUR, respectively.
- Assume the production of each cake requires:

Cake	Flour	Cocoa Powder	Butter
Kirschtorte	2	3	1
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 & K, M, S & \geq & 0 & \text{(negative cakes = bad)}
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Standard form linear program (LP)

Input:

- (Cost function) $c_1, \ldots, c_n \in \mathbb{R}$.
- (Constraints) $a_{ij} \in \mathbb{R}$ for $i \in [m], j \in [n]$, and $b_1, \ldots, b_m \in \mathbb{R}$.

Primal standard form LP:

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\text{maximize} & \sum_{j=1}^{n} c_j x_j & & & \text{(objective function)} \\
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Equivalent Linear Algebraic formulation:

$$
\begin{array}{llcll}
\text{maximize} & c^T x & & & \text{(objective function)} \\
\text{subject to} & Ax & \leq & b & \text{(constraints)} \\
& x & \geq & 0 &
\end{array}
$$

for matrix $A \in \mathbb{R}^{m \times n}$ and column vectors $c, b \in \mathbb{R}^n$.

What does this geometrically mean?

Suppose LP has $n = 2$ variables, i.e. optimize over 2D plane $\mathbb{R}^2$.

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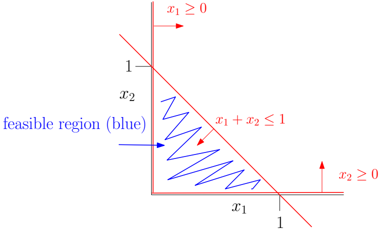

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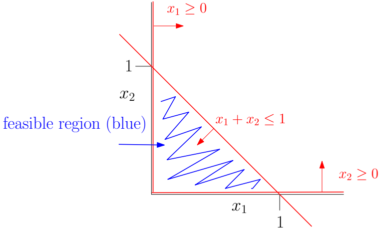

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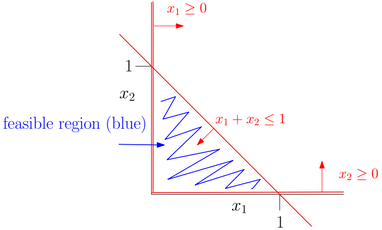

- Each constraint partitions R^2 into pair of *halfspaces*, i.e. "left" and "right" of each "dividing line" (formally, each hyperplane).
- Feasible region is intersection of these halfspaces (formally, convex polyhedron).

What about objective function?

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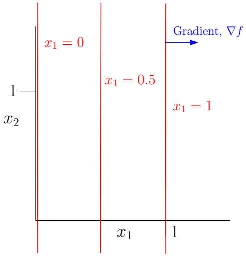
  - Don't know its *offset* from origin.



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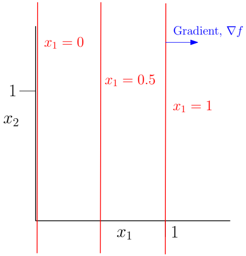

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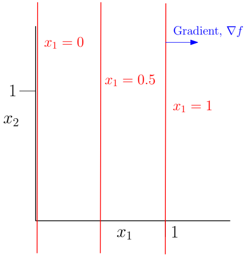

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- Formally, direction of movement given by *gradient* of f,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right).$$

- In this example: $\nabla f = (1, 0)$, hence the blue vector above.

Putting it all together

Optimizing our LP corresponds to (in our example):

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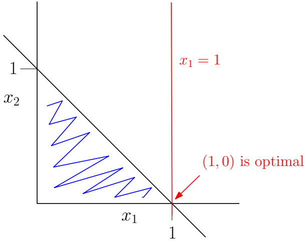

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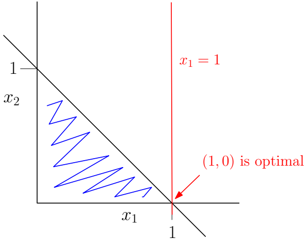

- Sanity check: Convince yourself that $(x_1, x_2) = (1, 0)$ is indeed optimal for:

$$\begin{array}{lrcl} \text{maximize} & x_1 & & \\ \text{subject to} & x_1 + x_2 & \leq & 1 \\ & x_1, x_2 & \geq & 0. \end{array}$$

Boundary case

What if we drop a constraint?

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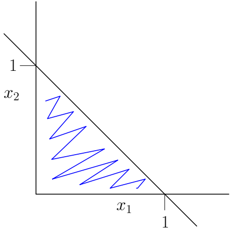

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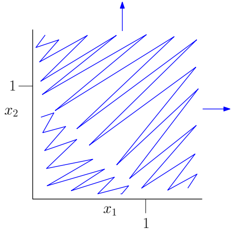

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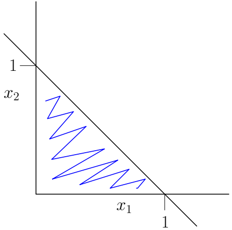

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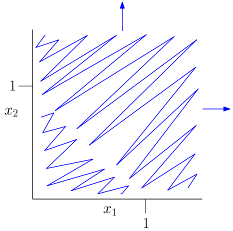

- Can move the vertical objective function line as far right as we like!
- Optimal value is ∞, i.e. the LP is *unbounded*.

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Application 1: Shortest paths

- Recall single-source shortest paths problem:
  - Given graph $G = (V, E)$ with real edge costs, and a source vertex $s \in V$, find smallest weight path to all other $v \in V$.
- Solved by Bellman-Ford in $O(mn)$ time.
- Can also be phrased as LP! Let's consider scaled down problem:

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Claim: The following LP optimally solves SPSP.

For each $v \in V$, introduce variable d_v.

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- Solution: For all $v \in V$, rewrite $d_v = d_{v_1} - d_{v_2}$ with $d_{v_1}, d_{v_2} \geq 0$.

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- Solution:
  - Replace $d_s = 0$ with two constraints:
    1. $d_s \geq 0$
    2. $-d_s \geq 0$

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- The Max Flow problem is, *by definition*, an LP!
- Given a flow network (G, s, t, c) for capacity function $c : E \mapsto \mathbb{R}^+$, the following LP yields max flow value:

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- If we demand *integer* flow, i.e. $f_i(u,v) \in \mathbb{Z}$, then MCF becomes NP-complete.

But there is more black magic to come...

Outline

1. Definitions
2. Applications
3. Duality theory
4. Solving LPs

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  - If x is optimal, can use *duality theory* to prove this.
- Yields powerful method for proving analytic bounds on optimization problems in math proofs. (Where in this course have we used this idea, at least indirectly?)

Intuition

Let's return to our LP example, slightly rewritten below:

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\begin{aligned}
\text{maximize} \quad & x_1 & & & (1)\\
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  - Thus, objective function upper bounded by 1, and $(1, 0)$ is indeed optimal.
- Even better: Can generalize this idea to get tight upper bound on optimal value.
- Idea:
  - To each constraint, assign a "dual" variable y_i.
  - "Do a minimization" over linear combinations of y_i to get upper bound on objective function.
  - This minimization *itself* an LP, called *dual LP*.

I did say there was more black magic to come, no?

Primal-dual pair

LPs come in pairs, known as the primal (left) and dual (right) LP:

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\begin{array}{llll}
\max & \sum_{j=1}^{n} c_j x_j & & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq & b_i & \forall i \in [m] \\
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Q: Can you give dual solution with dual objective function value 1?

Formalization primal vs dual

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Let p^* and d^* denote optimal solutions for P and D, respectively.

Intuitively: Designed D so that d^* yields upper bound on p^*. Let's prove this!

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Theorem (Weak duality)

For any primal feasible $x = \{x_j\}$ and dual feasible $y = \{y_i\}$,

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Proof.

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) \leq \sum_{i=1}^{m} b_i y_i.$$

□

Formalization primal vs dual

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If you can guess primal solution x and dual solution y with matching objective function values $p = d$, then guaranteed x is optimal! (No need to explicitly solve either LP.)

- Q: Is it always true that $p^* = d^*$?

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Returning to Max Flow

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Recall: Max flow is bounded by min capacity across any $s-t$ cut in G.

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Recall: Max flow is bounded by min capacity across any $s-t$ cut in G.

- Claim: This is precisely what the *dual* LP for Max Flow says.
- Unfortunately, dual of our current primal LP is messy.
- Idea: First rewrite LP in an equivalent, but “simpler” way, bringing us into standard form.

Rewriting the primal LP

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- In this new view, each x_p denotes flow along path p.
- Clearly, such flow along any p is limited by the bottleneck edge (u,v) of p.
- Taking dual of this new LP will yield much nicer dual.

Sanity check

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- Does it matter that our new formulation is too big to write down?

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- No, if all you want to do is look at the dual to extract theoretical bounds on primal value. (Our goal.)

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- Objective function value is precisely capacity across S vs T cut.
- Q: Why is this solution feasible?
- A: Each path $p\in\Omega$ takes *some* cut edge to pass from S to T, i.e. $(*)$ satisfied.

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Outline

1. Definitions
2. Applications
3. Duality theory
4. Solving LPs

What if we want to solve an LP?

Observations:

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  - Implies refuting candidate optimal LP values is in co-NP.
- Conclusion: Solving LPs is in NP $\cap$ co-NP.
- But is it also in P?

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- In practice, one uses simplex method or interior-point method.
- Simplex method is, in its current known variants, not poly-time.

Long history

Dates:

- (1827) Fourier proposes method for solving systems of linear inequalities
- (1939-ish) Kantorovich and Koopmans independently study more general LPs. They would later share Nobel prize in Economics (1975).
- (1941) Hitchcock gives solution very similar to simplex method.
- (1947) Dantzig discovers simplex method for solving LPs.
- (1979) Khachiyan discovers poly-time algorithm using ellipsoid method.
- (1984) Karmarkar discovers poly-time algorithm using interior-point method.

Efficiency notes:

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In practice:

- Many LP solvers available.
- Ex: CVX (implemented in Matlab), which can do LPs and a whole lot more.

Have a great break!