

# Fundamental Algorithms

## Chapter 6: Network Flow

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(based on slides of Christian Scheideler)

WS 2019

# Outline

- What is a network flow?
- Ford-Fulkerson algorithm
- Karp-Edmonds algorithms
- Dinic's algorithm
- Goldberg's algorithm
- Variants on network flow

# Foundations

**Definition 6.1:** A **flow network**  $(G,s,t,c)$  consists of a directed graph  $G=(V,E)$ , a **source**  $s \in V$ , a **sink**  $t \in V$ , and a **capacity function**  $c:V \times V \rightarrow \mathbb{R}_{\geq 0}$ , with  $c(u,v) = 0$  if  $(u,v) \notin E$ .

In the following, we assume that  $s \sim_G u \sim_G t$  for all  $u \in V$ , where  $u \sim_G v$  means that there is a directed path from  $u$  to  $v$  in  $G$ . (Otherwise, we can remove  $u$  and all of its edges from  $G$ , because a flow from  $s$  to  $t$  cannot be sent via  $u$ .)

**Definition 6.2:** Let  $(G,s,t,c)$  be a flow network.

- a) A **flow** in  $G$  is a function  $f:V \times V \rightarrow \mathbb{R}$  with the property that
- $f(u, v) \leq c(u, v)$  for all  $u, v \in V$  (capacity constraints)
  - $f(u, v) = -f(v, u)$  for all  $u, v \in V$  (skew symmetry)
  - $\sum_{v \in V} f(u, v) = 0$  for all  $u \in V \setminus \{s, t\}$  (flow conservation)
- b) The **value**  $|f|$  of a network flow  $f$  is defined as
- $$|f| = \sum_{v \in V} f(s, v).$$

# Foundations

A **flow** in  $G$  is a function  $f:V \times V \rightarrow \mathbb{R}$  with the property that

$$f(u, v) \leq c(u, v) \text{ for all } u, v \in V \quad (\text{capacity constraints})$$

$$f(u, v) = -f(v, u) \text{ for all } u, v \in V \quad (\text{skew symmetry})$$

$$\sum_{v \in V} f(u, v) = 0 \text{ for all } u \in V \setminus \{s, t\} \quad (\text{flow conservation})$$

**Remark 6.3:** Let  $f$  be a flow in a flow network  $(G, s, t, c)$ . Then

a)  $f(v, v) = 0$  for all  $v \in V$  (due to skew symmetry).

b)  $\sum_{u \in V} f(u, v) = 0$  for all  $v \in V \setminus \{s, t\}$  (flow conservation & skew symmetry).

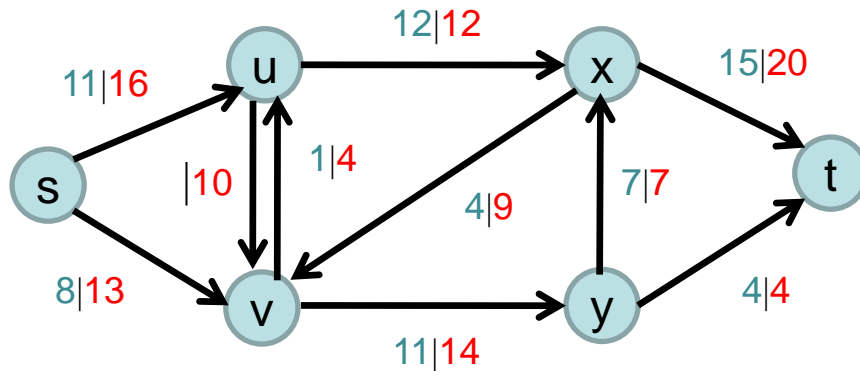
c) For all  $v \in V \setminus \{s, t\}$ ,

$$\sum_{u \in V, f(u, v) > 0} f(u, v) = - \sum_{u \in V, f(u, v) < 0} f(u, v) \quad (\text{flow conserv., skew symmetry})$$

d) A function  $f$  with  $f(u, v) = 0$  for all  $u, v \in V$  is a valid flow.

# Foundations

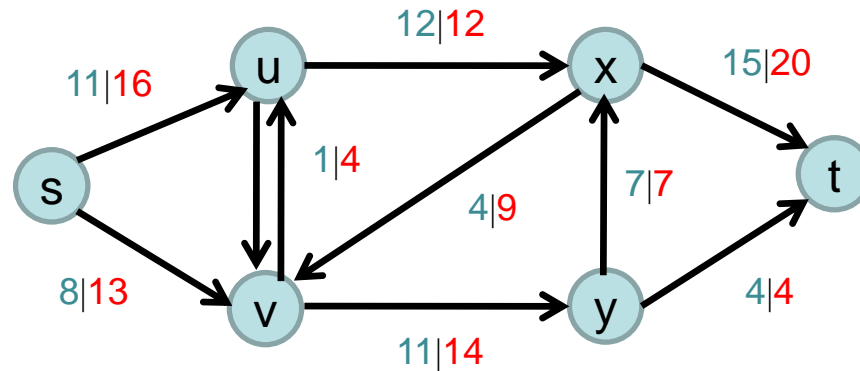
Example of a valid flow:



$f(u, v)|c(u, v)$ ,  $|f| = 19$ ,  
|10 means no flow

- Only positive flows are shown (negative flows are implied by skew symmetry).
- For example,  $f(v, u)=1$ , so  $f(u, v)=-1$ .
- This implies that flow cannot flow at the same time in both directions for a pair  $\{u, v\}$ .

# Foundations



**Claim 6.4:** For any network flow, the outgoing flow of **s** is equal to the incoming flow at **t**.

**Proof:**

- From skew symmetry, we know:

$$\sum_{v \in V} \sum_{w \in V} f(v, w) = \sum_{\{v, w\}} (f(v, w) + f(w, v)) + \sum_{v \in V} f(v, v) = 0$$

- Moreover, it follows from flow conservation:

$$\sum_{v \in V} \sum_{w \in V} f(v, w) = \sum_{w \in V} f(s, w) + \sum_{w \in V} f(t, w) = |f| + \sum_{w \in V} f(t, w)$$

(Recall flow conservation:  $\sum_{v \in V} f(u, v) = 0$  if  $u \in V \setminus \{s, t\}$ )

- Hence, due to skew symmetry:

$$|f| = \sum_{w \in V} f(w, t)$$

# Foundations

Alternative definition of network flows:

**Definition 6.5:** Let  $(G,s,t,c)$  be a flow network. A **flow** in  $G$  is a function  $f : E \rightarrow \mathbb{R}_{\geq 0}$  with

- $0 \leq f(u, v) \leq c(u, v)$  for all  $(u, v) \in E$  (**capacity constraints**)
- $\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$  for all  $u \in V \setminus \{s, t\}$   
(**flow conservation**)

i.e., we drop the skew symmetry constraint.

**Remarks:**

- Definition 6.5 is more intuitive whereas Definition 6.2 is more restrictive and sometimes simplifies the proofs.
- We will use the alternative Definition 6.5 in later parts of this chapter.

## MAXFLOW Problem:

Input: a flow network  $(G, s, t, c)$ .

Output: a flow  $f$  in  $G$  with **maximum** value  $|f|$ .

**Remark 6.6:** A maxflow problem  $(G, s_1, \dots, s_p, t_1, \dots, t_q, c)$  with multiple sources  $s_1, \dots, s_p$  and multiple sinks  $t_1, \dots, t_q$  with the goal to transfer as much flow as possible from the sources to the sinks (i.e., find a flow  $f: V \times V \rightarrow \mathbb{R}$  maximizing  $\sum_{i=1}^p (\sum_{v \in V} f(s_i, v))$ ) can be reduced to the original maxflow problem:

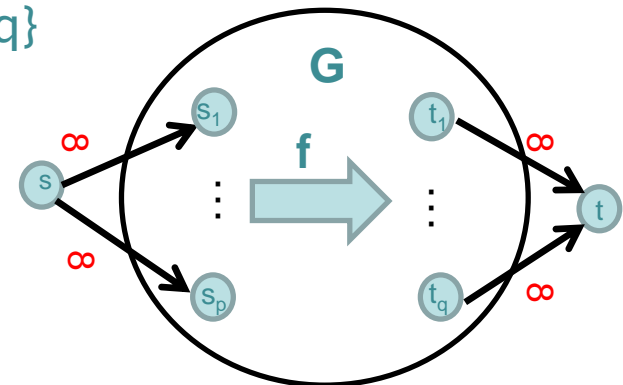
Construct  $G' = (V', E')$  and  $c'$  as follows:

$$V' = V \cup \{s, t\}$$

$$E' = E \cup \{(s, s_i) \mid 1 \leq i \leq p\} \cup \{(t_i, t) \mid 1 \leq i \leq q\}$$

$$c'(u, v) = \begin{cases} c(u, v) & u, v \in V \\ \infty & u = s \text{ or } v = t \end{cases}$$

Then there is a flow  $f$  from  $s_1, \dots, s_p$  to  $t_1, \dots, t_q$  of value  $\phi$  in  $(G, s_1, \dots, s_p, t_1, \dots, t_q, c)$  if and only if there is a flow  $f'$  from  $s$  to  $t$  in  $(G', s, t, c')$  of value  $\phi$  (see the figure).





# Ford-Fulkerson Algorithm

How do we solve the maxflow problem?

Definition 6.7: Let  $(G,s,t,c)$  be a flow network and  $f$  be a flow in  $G$ .

a) For any  $u, v \in V$ , the **residual capacity**  $c_f(u,v)$  is defined as

$$c_f(u,v) = c(u,v) - f(u,v) \geq 0$$

(how much capacity for flow is unused?)

b) The **residual network**  $G_f = (V, E_f)$  is defined as

$$E_f = \{ (u,v) \in V \times V \mid c_f(u,v) > 0 \}$$

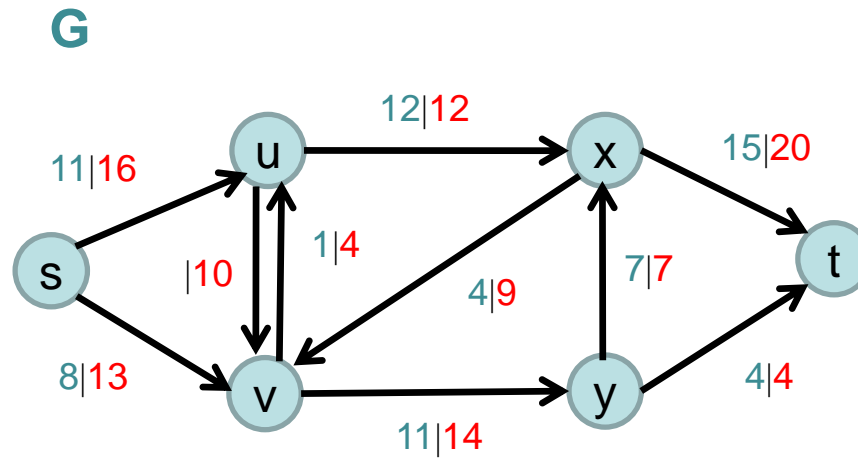
c) A simple path  $P$  from  $s$  to  $t$  in  $G_f$  is called an **augmenting path**.  
The **residual capacity** (or „bottleneck“)  $c_f(P)$  of  $P$  is defined as

$$c_f(P) = \min \{ c_f(u,v) \mid (u,v) \in P \}.$$

# Ford-Fulkerson Algorithm

Example: augmenting path and flow augmentation

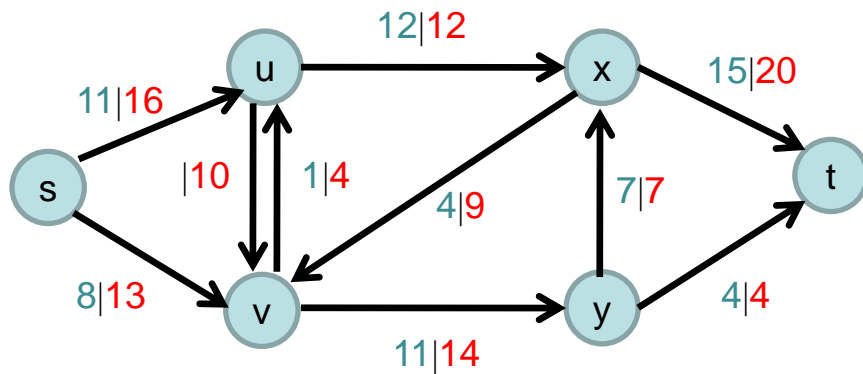
Flow network:



Example: augmenting path and flow augmentation

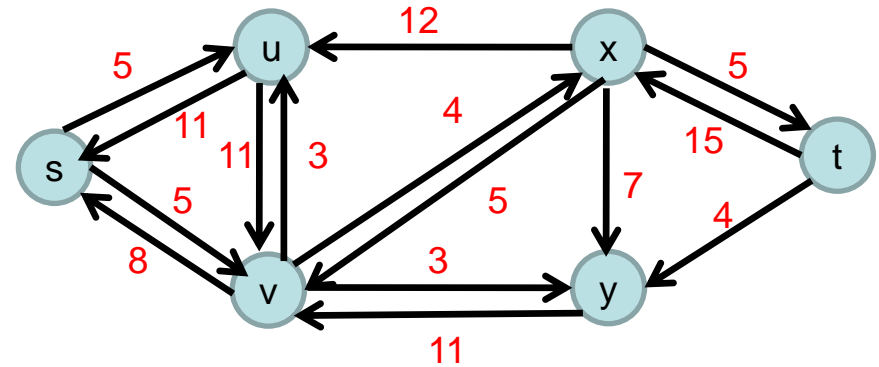
Flow network:

$G$



Residual network:

$G_f$



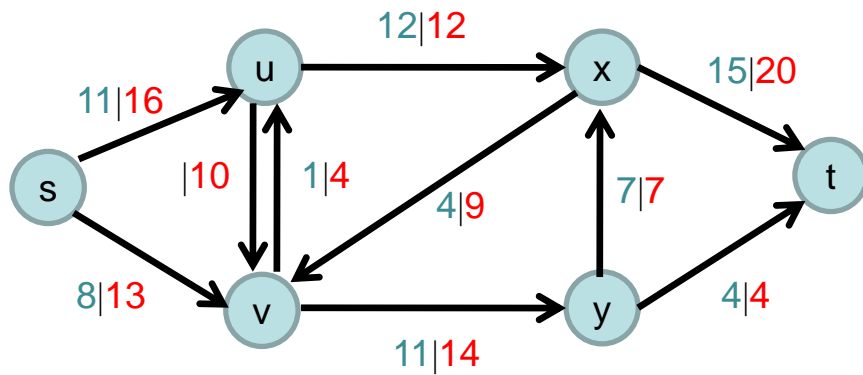
$$c_f(u,v) = c(u,v) - f(u,v)$$

Residual network allows us to both increase or *decrease* flow! (hence many edges are depicted in both directions)

## Example: augmenting path and flow augmentation

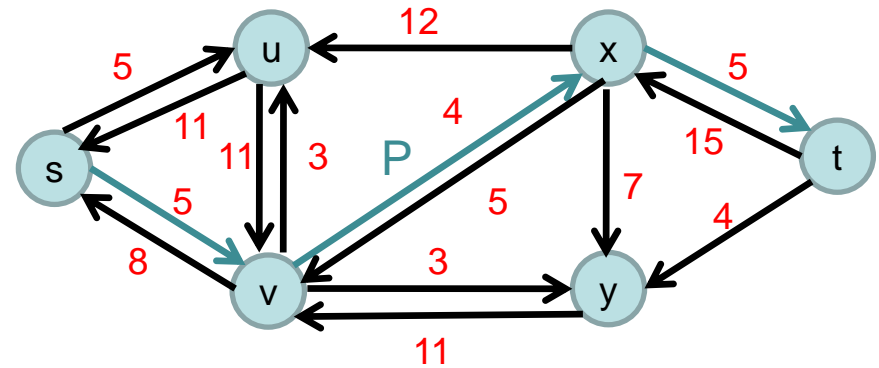
Flow network:

**G**



Residual network with augmenting path:

**G<sub>f</sub>**

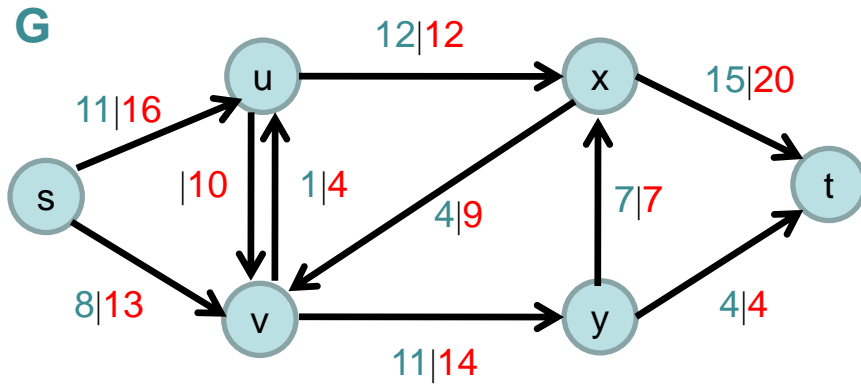


$$c_f(P) = \min \{ c_f(u,v) \mid (u,v) \in P \}$$

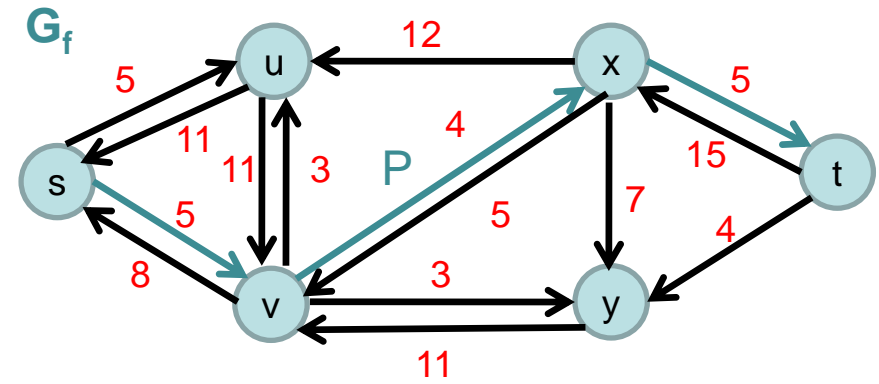
→ residual capacity/bottleneck of path **P**: 4

## Example: augmenting path and flow augmentation

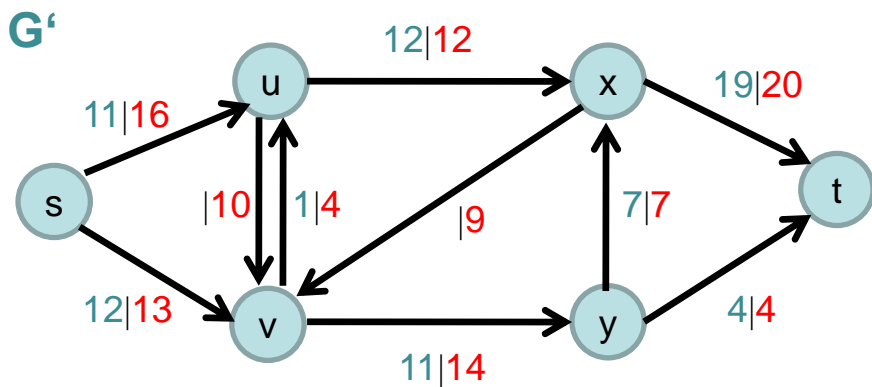
Flow network:



Residual network with augmenting path:



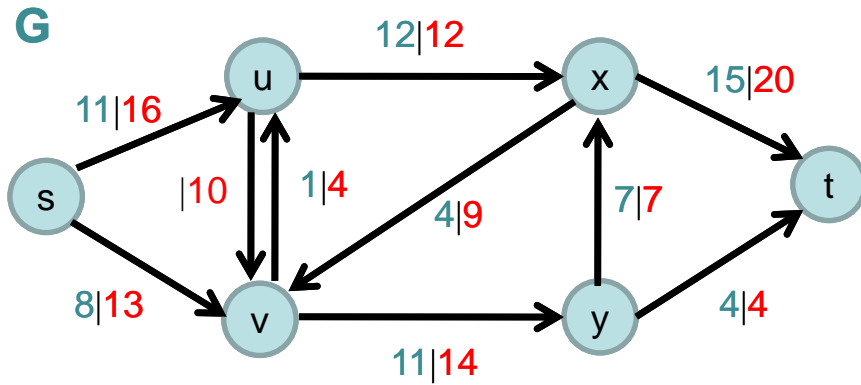
Augmented flow:



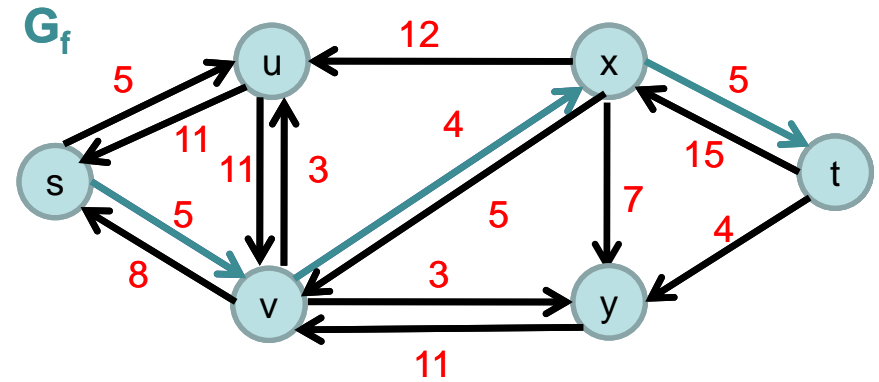
Path  $P$  of residual capacity 4 added to  $G$ :

# Example: augmenting path and flow augmentation

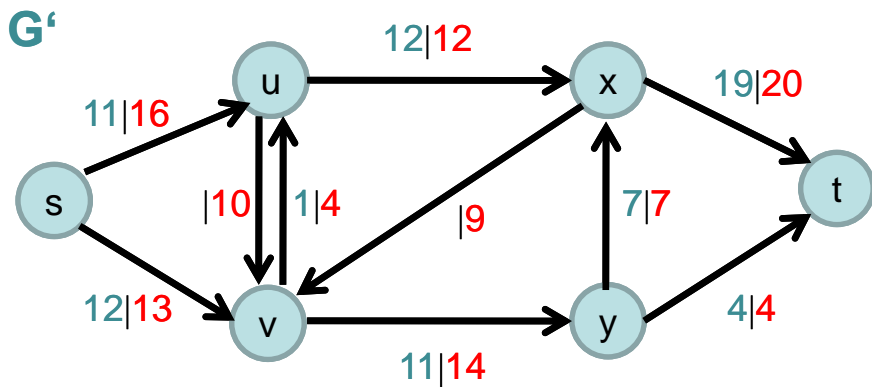
Flow network:



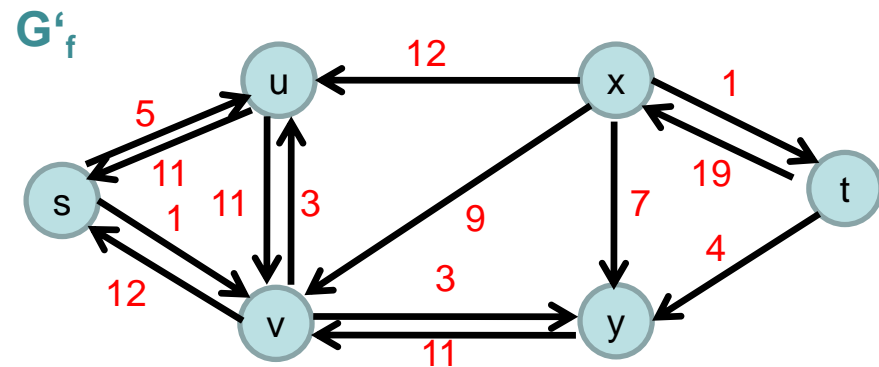
Residual network with augmenting path:



Augmented flow:



New residual network:



# Ford-Fulkerson Algorithm

Let's formalize the idea of augmenting flows with augmenting paths. Need 2 lemmas – first says can safely add „augmenting flow“ to our current flow to get new flow, second shows how to map any augmenting path to a „canonical“ augmenting flow.

**Lemma 6.8:** Let  $(G, s, t, c)$  be a flow network and  $f$  be a flow in  $G$ . Let  $G_f$  be the residual network of  $G$  induced by  $f$ , and let  $f'$  be a flow in  $G_f$ . Then

$$(f + f')(u, v) = f(u, v) + f'(u, v)$$

is a valid flow in  $G$  with value  $|f + f'| = |f| + |f'|$ .

**Proof:**

- Capacity constraints:  
 $f'(u, v) \leq c_f(u, v) = c(u, v) - f(u, v)$  for all  $u, v \in V$ .  
Hence,  $(f + f')(u, v) \leq f(u, v) + c(u, v) - f(u, v) \leq c(u, v)$ .
- Skew symmetry:  
 $f(u, v) = -f(v, u)$  and  $f'(u, v) = -f'(v, u)$  for all  $u, v \in V$ .  
Hence,  $(f + f')(u, v) = -(f + f')(v, u)$  for all  $u, v \in V$ .
- Flow conservation:  
 $\sum_v f(u, v) = 0$  and  $\sum_v f'(u, v) = 0$  for all  $u \in V \setminus \{s, t\}$ .  
Hence,  $\sum_v (f + f')(u, v) = 0$  for all  $u \in V \setminus \{s, t\}$ .

# Ford-Fulkerson Algorithm

How to define a flow for an augmenting path in  $G_f$ ?

**Lemma 6.9:** Let  $(G, s, t, c)$  be a flow network and  $f$  be a flow in  $G$ . Let  $G_f$  be the residual network of  $G$  induced by  $f$  and let  $P$  be an augmenting path in  $G_f$ . Then  $f_P : V \times V \rightarrow \mathbb{R}$  with

$$f_P(u,v) = \begin{cases} c_f(P) & \text{if } (u,v) \text{ belongs to } P \\ -c_f(P) & \text{if } (v,u) \text{ belongs to } P \\ 0 & \text{otherwise} \end{cases} \quad \begin{array}{l} \text{(Recall: } c_f(P) \text{ is the most flow can} \\ \text{hope to push through } G_f, \text{ as per} \\ \text{the bottleneck on path } P.) \end{array}$$

is a valid flow in  $G_f$  with value  $|f_P| = c_f(P) > 0$ .

**Proof:**

Check capacity constraints, skew symmetry and flow conservation.

**Corollary 6.10:** Let  $(G, s, t, c)$  be a flow network and  $f$  be a flow in  $G$ . Let  $G_f$  be the residual network of  $G$  induced by  $f$  and let  $P$  be an augmenting path in  $G_f$ . Let  $f_P$  be defined as in Lemma 6.9. Then  $f' = f + f_P$  is a valid flow in  $G$  with value

$$|f'| = |f + f_P| = |f| + |f_P| > |f|.$$



# Ford-Fulkerson Algorithm

**Idea:** Start with empty flow. Repeatedly find augmenting paths to improve flow.

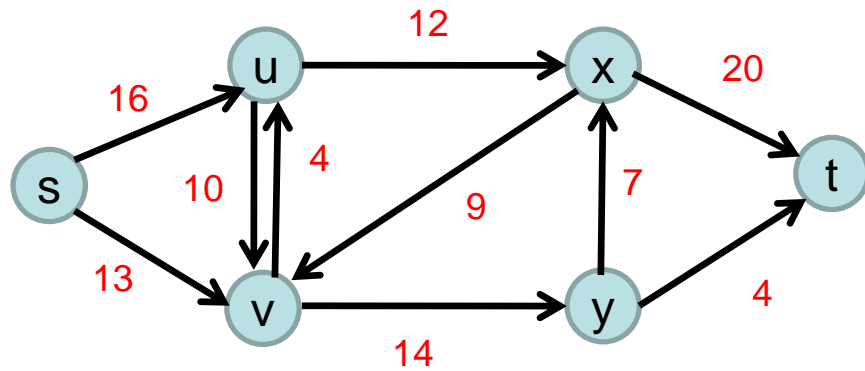
FORDFULKERSON (Flow network  $G = (V, E), s, t, c$ )

```
{  
  for each edge  $(u, v) \in E$   
    {  $f[u, v] := 0; f[v, u] := 0;$  } // initially empty flow  
  
   $G_f :=$  residual network of  $G$  w.r.t.  $f$ ;  
  
  while  $(\exists$  a path  $P$  from  $s$  to  $t$  in  $G_f$ ) //  $P$  is an augmenting path  
  { // compute maximal flow („bottleneck“) along  $P$   
     $c_f(P) := \min \{c_f(u, v) \mid (u, v) \in P\}$ ; //  $c_f(u, v) = c(u, v) - f(u, v)$   
  
    for each edge  $(u, v) \in P$  // update flow along  $P$   
      {  $f[u, v] := f[u, v] + c_f(P); f[v, u] := -f[u, v];$  }  
  
     $G_f :=$  residual network of  $G$  w.r.t.  $f$ ;  
  }  
  output  $f$   
}
```

## Example: Ford-Fulkerson Algorithm

Flow network:

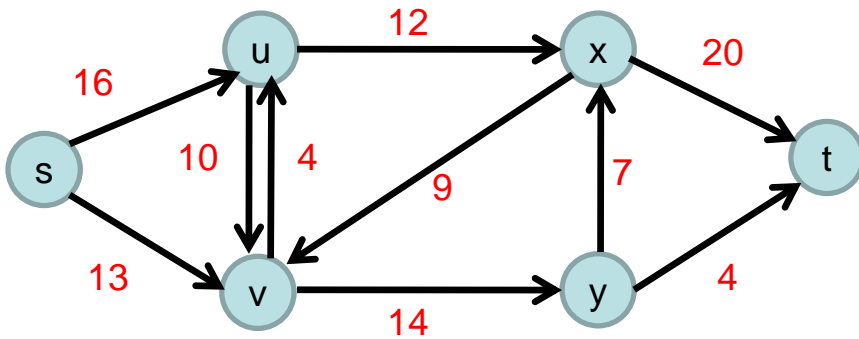
**G**



## Example: Ford-Fulkerson Algorithm

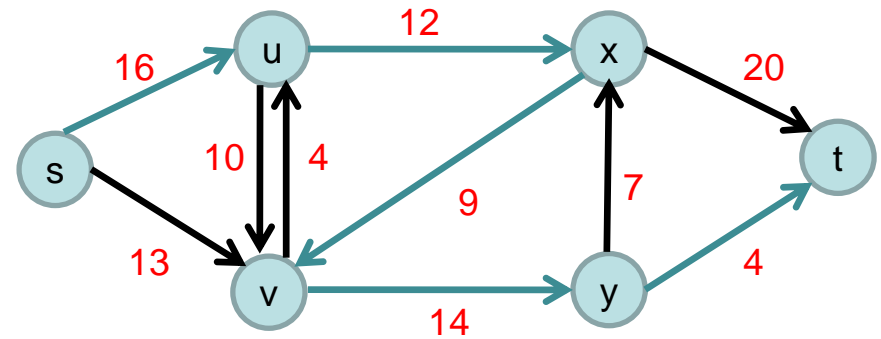
Flow network:

**G**



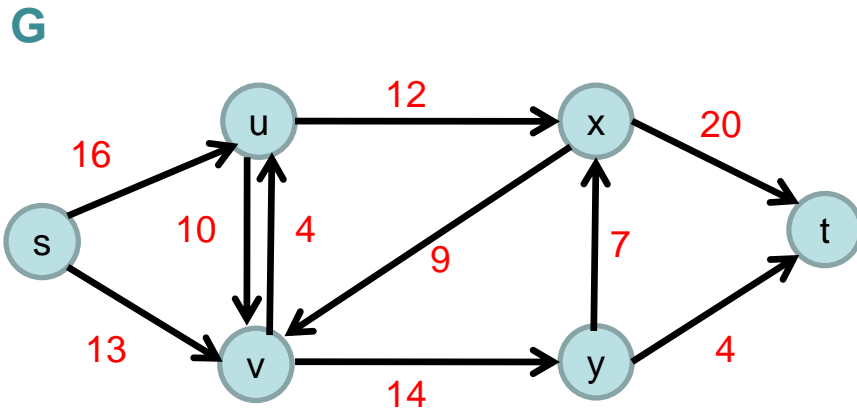
Residual network with augmenting path:

**G<sub>f</sub>**

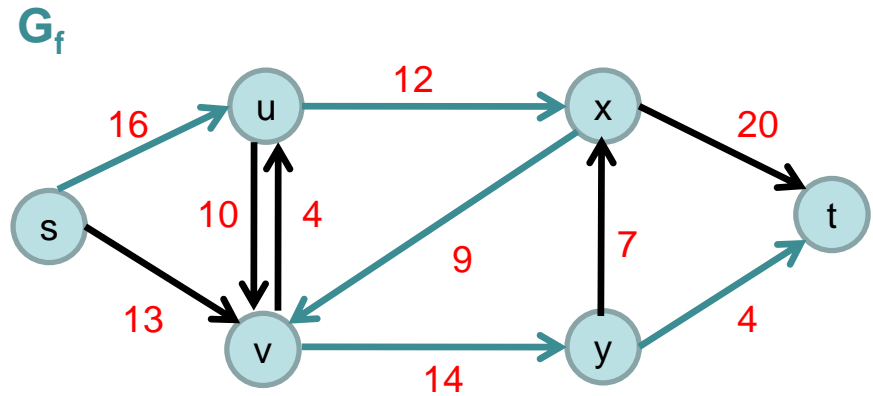


# Example: Ford-Fulkerson Algorithm

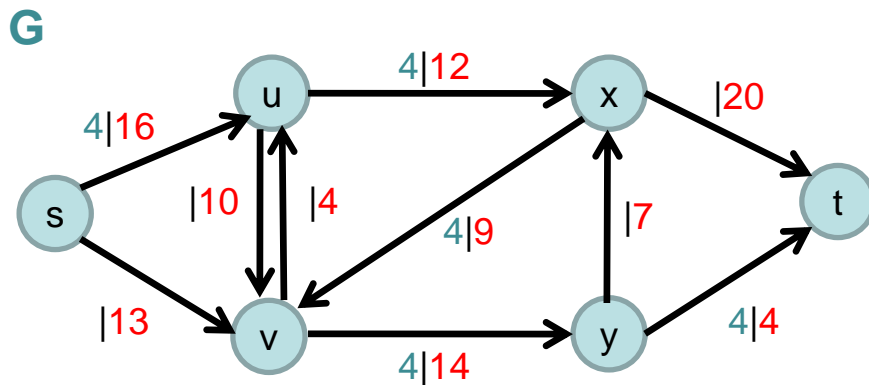
Flow network:



Residual network with augmenting path:



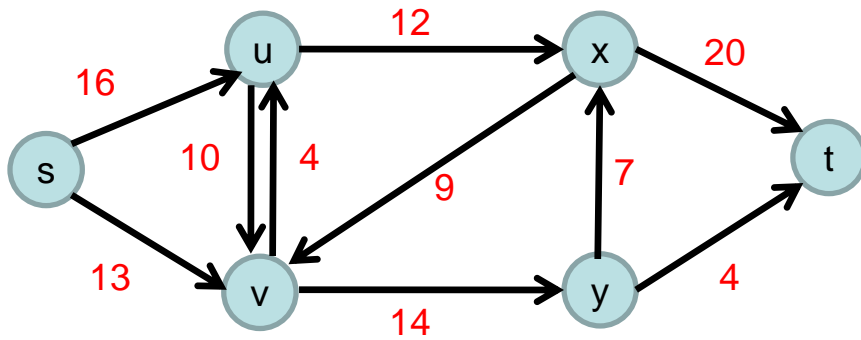
Augmented flow:



# Example: Ford-Fulkerson Algorithm

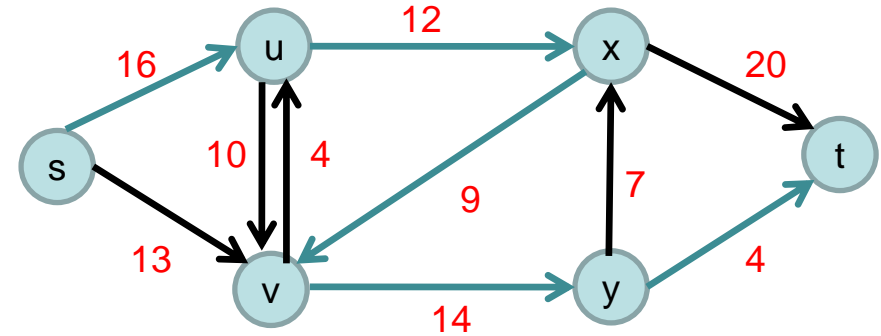
Flow network (only red numbers since no flow initially):

$G$



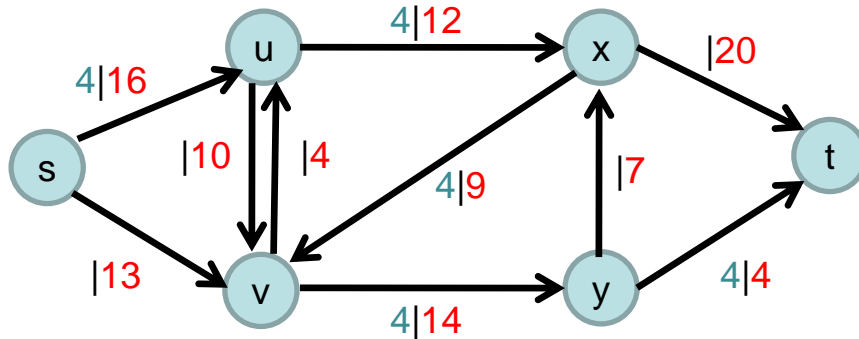
Residual network with augmenting path:

$G_f$



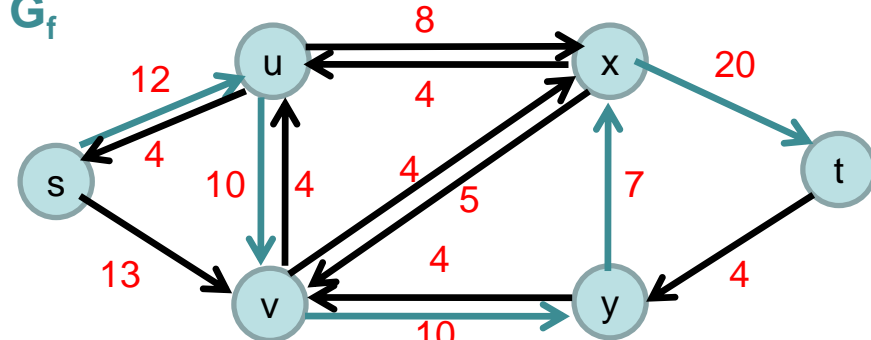
Augmented flow:

$G$



New residual network with augmenting path:

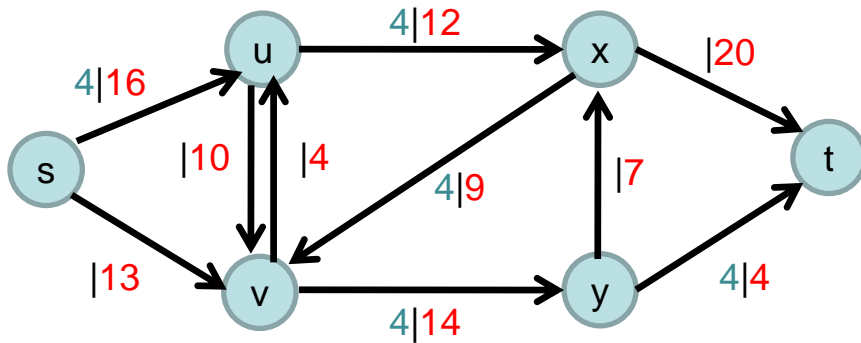
$G_f$



# Example: Ford-Fulkerson Algorithm

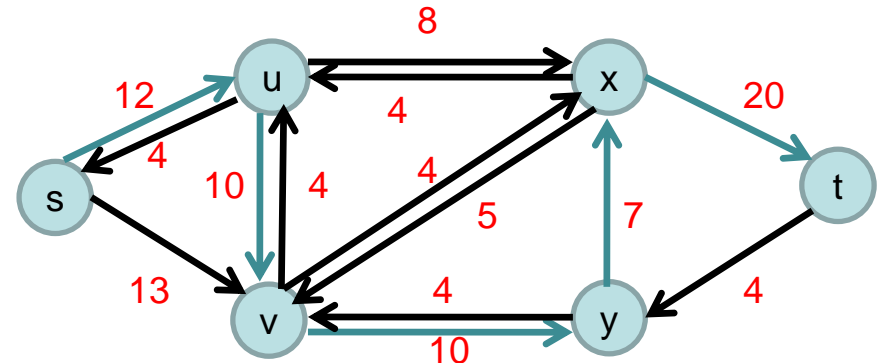
Flow network:

**G**



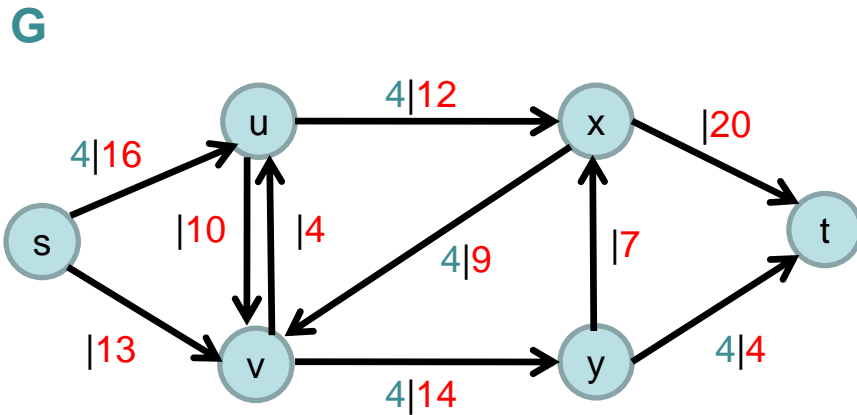
Residual network with augmenting path:

**$G_f$**

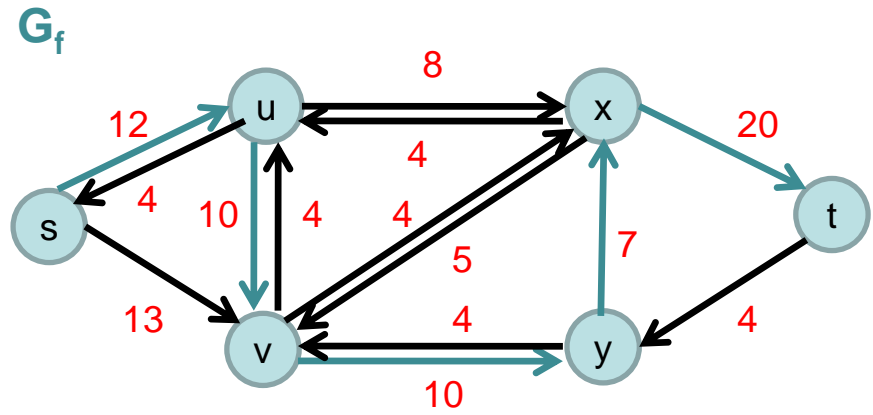


# Example: Ford-Fulkerson Algorithm

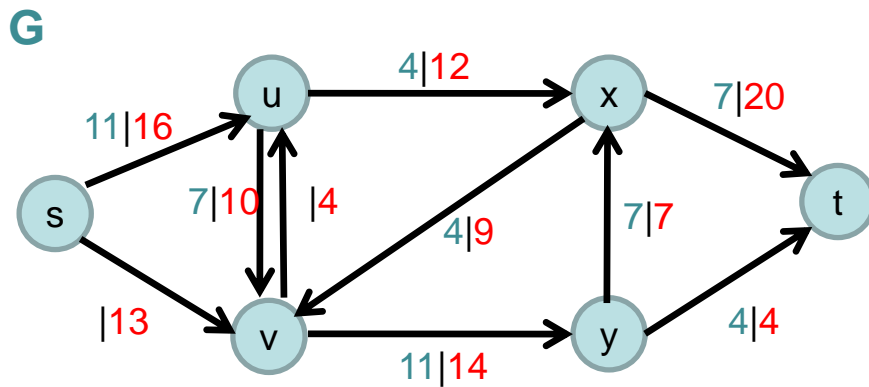
Flow network:



Residual network with augmenting path:

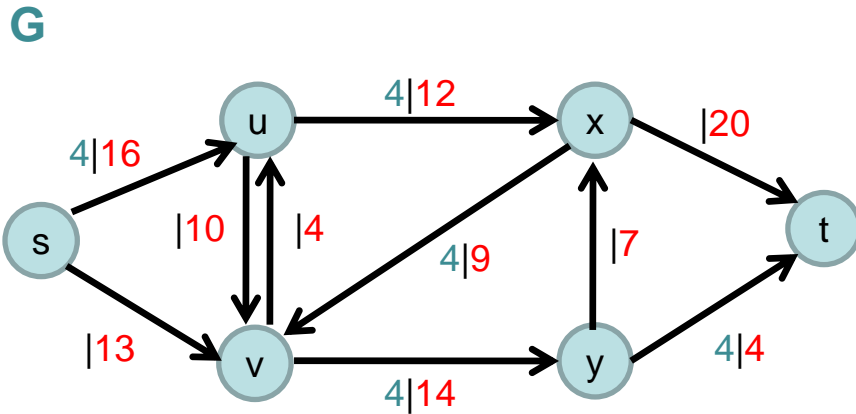


Augmented flow:

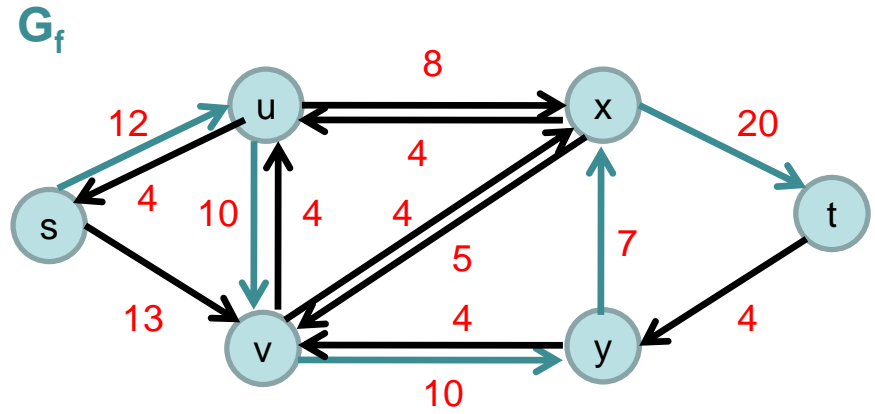


# Example: Ford-Fulkerson Algorithm

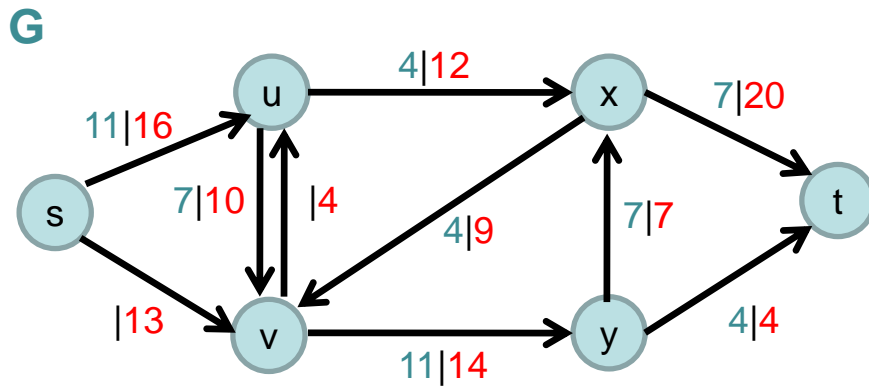
Flow network:



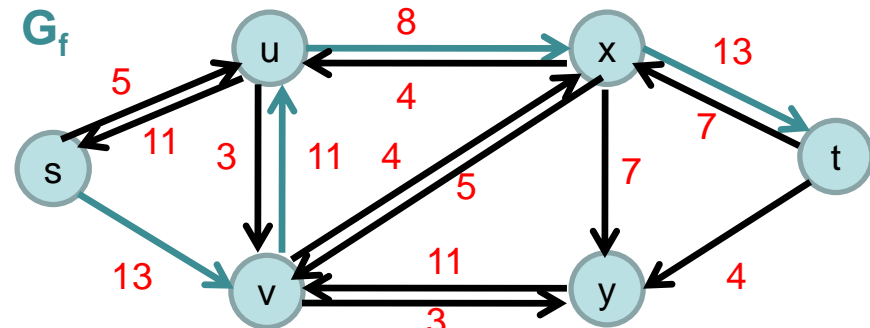
Residual network with augmenting path:



Augmented flow:



New residual network with augmenting path:

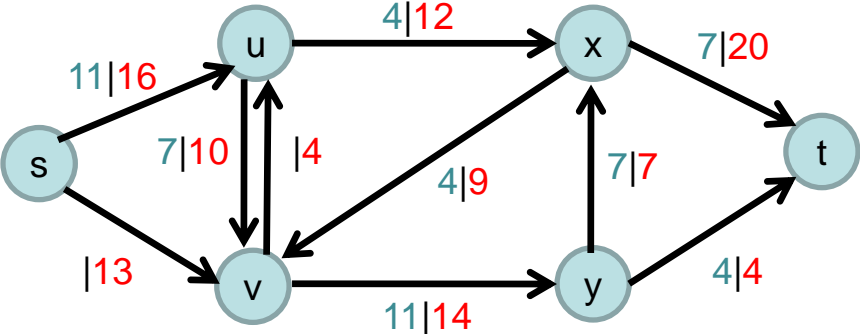




# Example: Ford-Fulkerson Algorithm

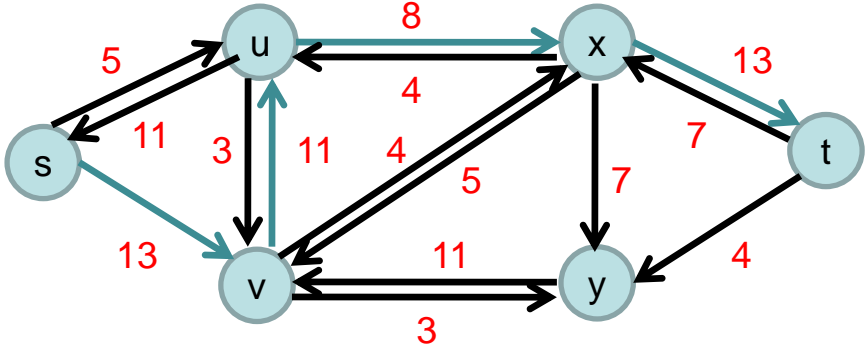
Flow network:

**G**



Residual network with augmenting path:

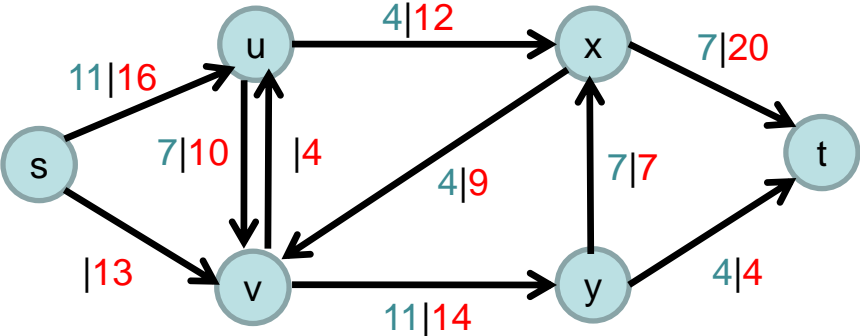
**G<sub>f</sub>**



# Example: Ford-Fulkerson Algorithm

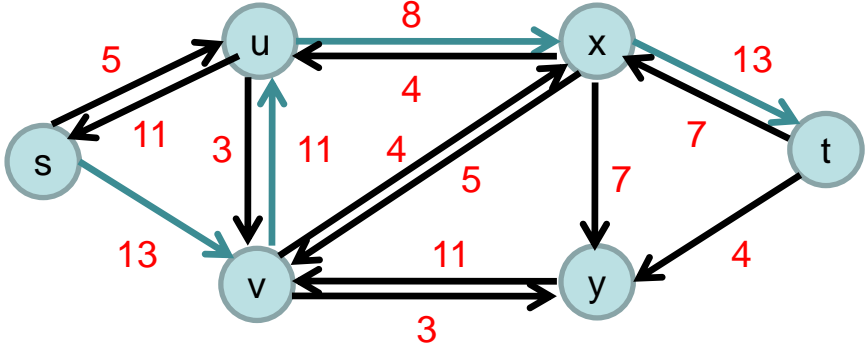
Flow network:

**G**



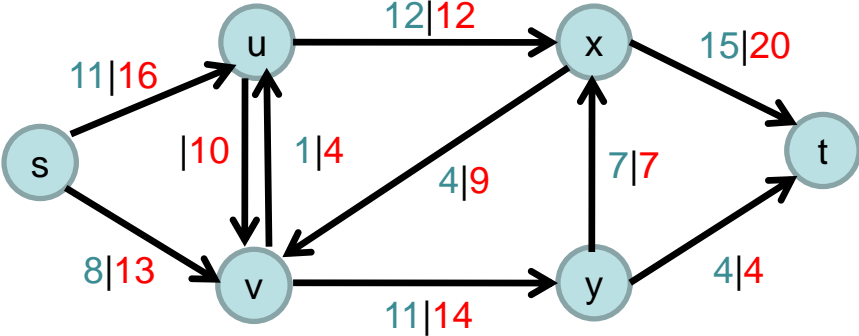
Residual network with augmenting path:

**G<sub>f</sub>**



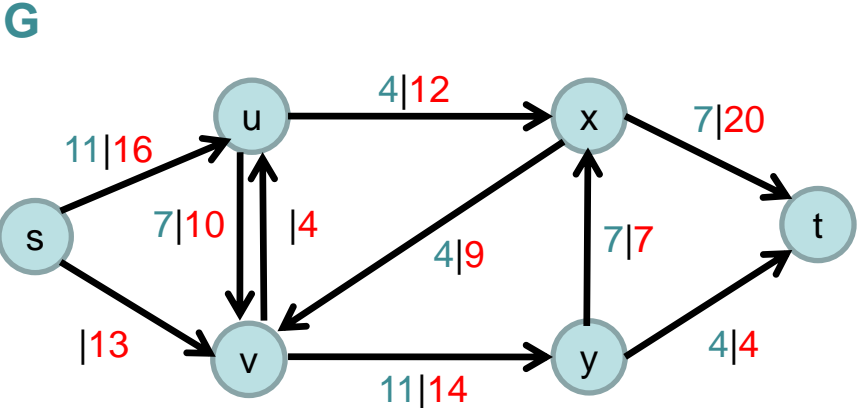
Augmented flow:

**G**

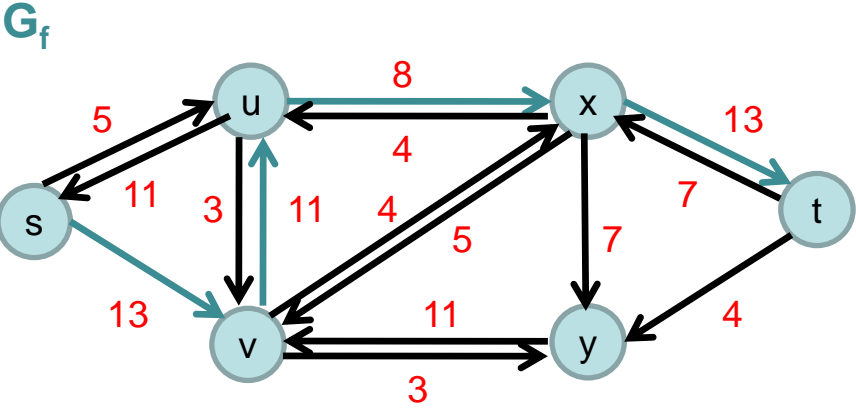


# Example: Ford-Fulkerson Algorithm

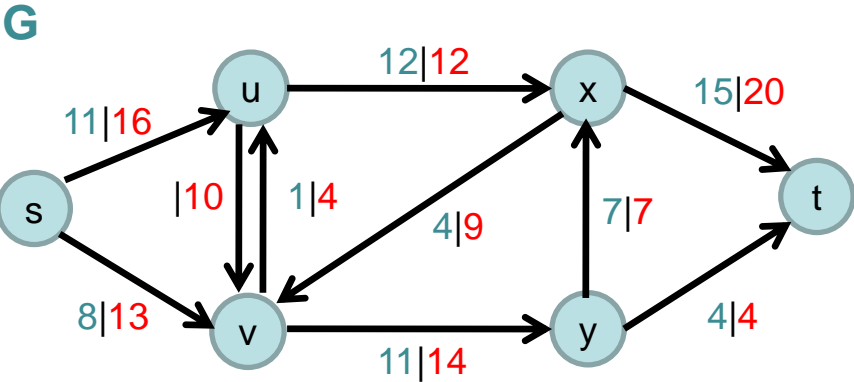
Flow network:



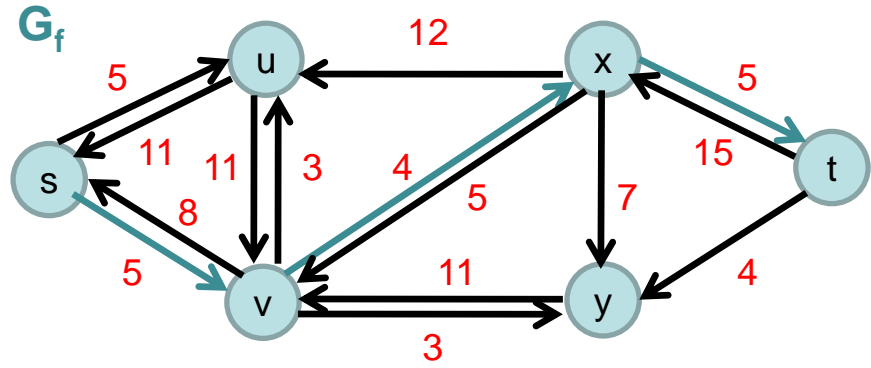
Residual network with augmenting path:



Augmented flow:



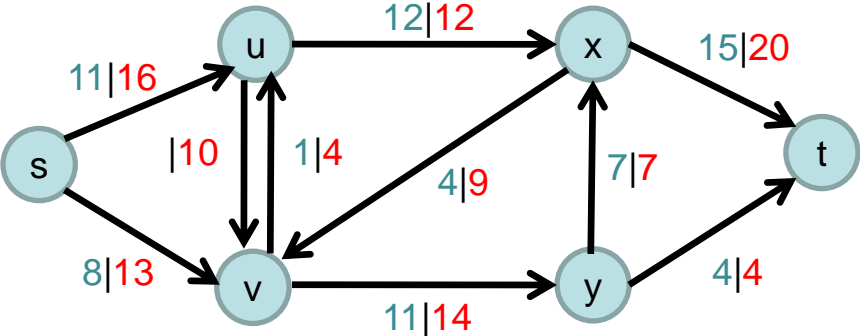
New residual network with augmenting path:



# Example: Ford-Fulkerson Algorithm

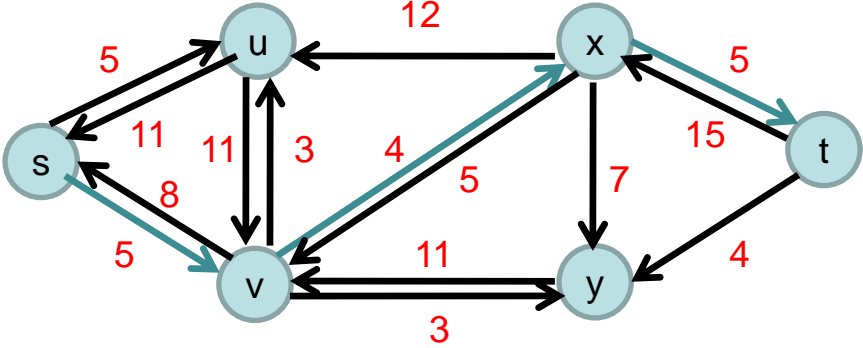
Flow network:

**G**



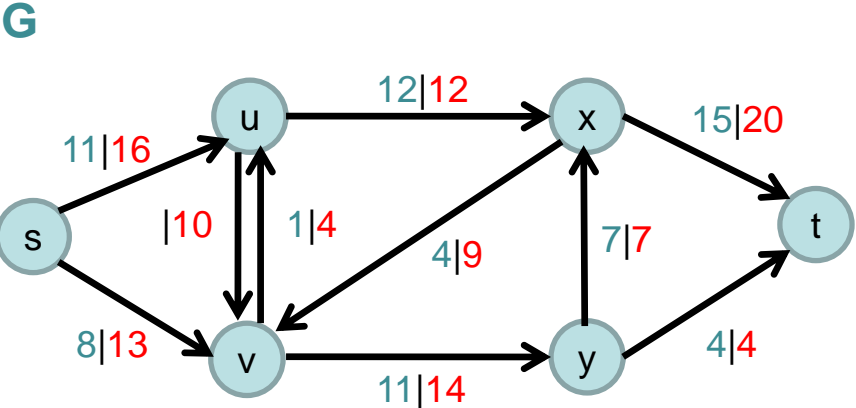
Residual network with augmenting path:

**G<sub>f</sub>**

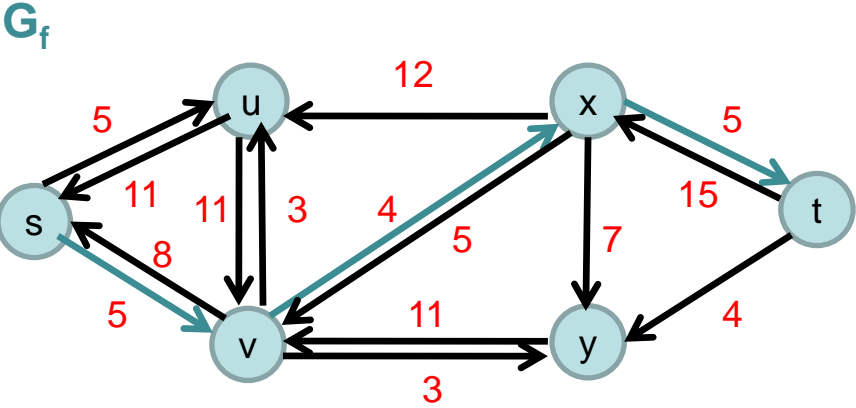


# Example: Ford-Fulkerson Algorithm

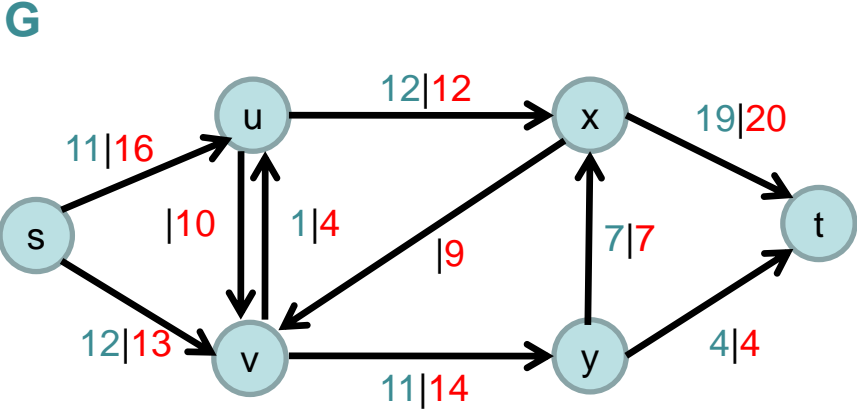
Flow network:



Residual network with augmenting path:



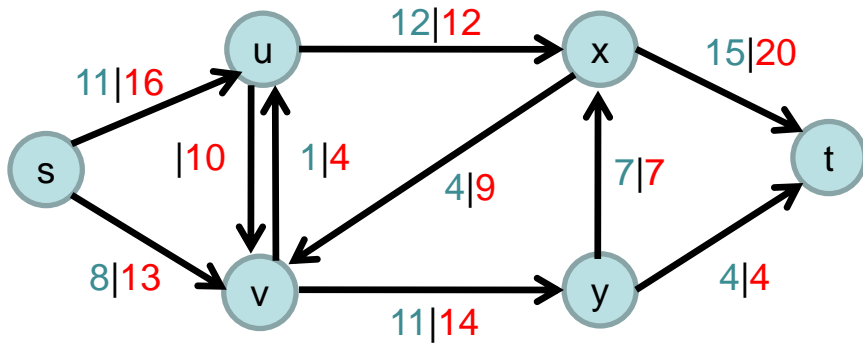
Augmented flow:



# Example: Ford-Fulkerson Algorithm

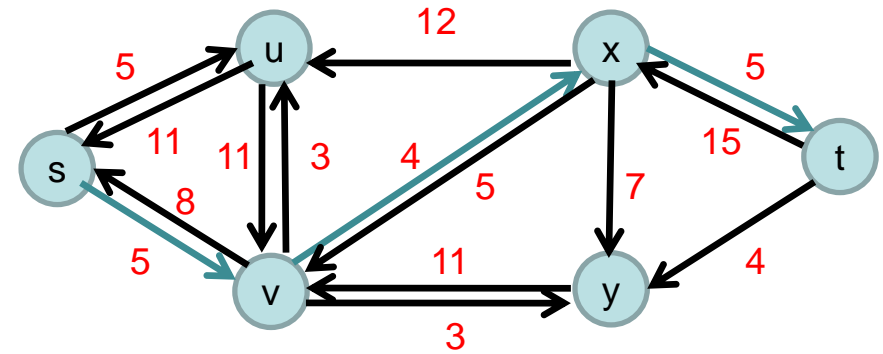
Flow network:

**G**



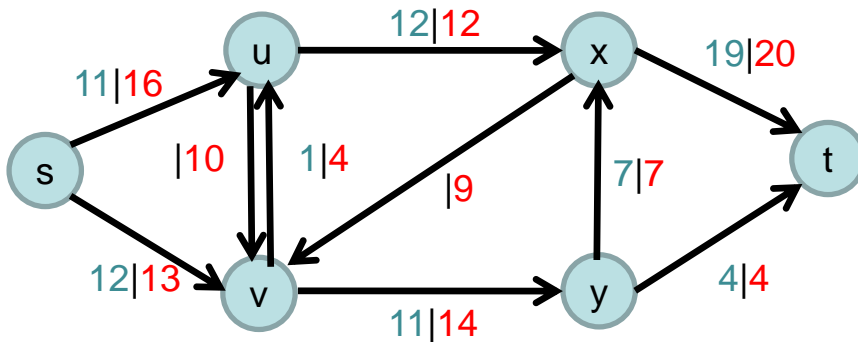
Residual network with augmenting path:

**G<sub>f</sub>**



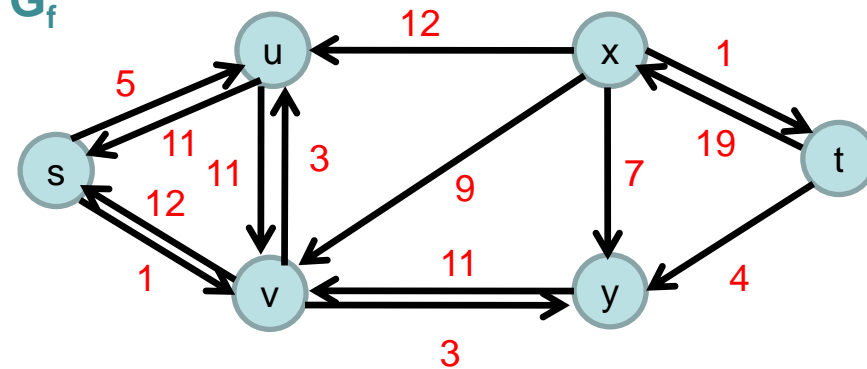
Augmented flow:

**G**



New residual network with no augmenting path:

**G<sub>f</sub>**



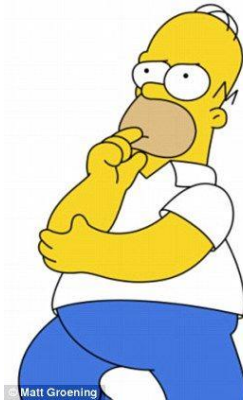
# Question

How do we know that once no more augmenting paths exist, the flow we have is optimal, i.e. a maximum flow?

## Recall:

For matchings, Berge's theorem said that once no more „augmenting paths“ exist, we are guaranteed to have maximum matching.

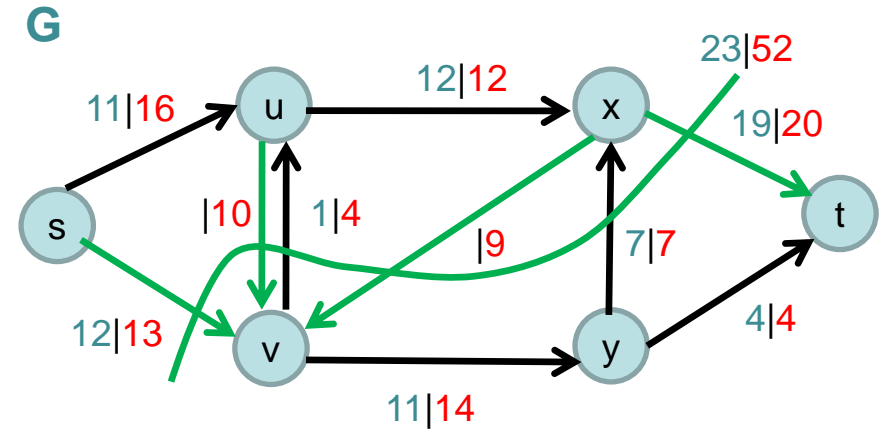
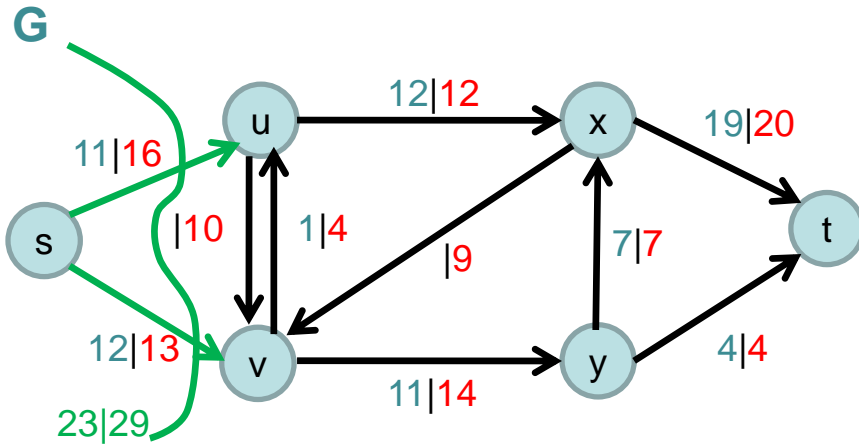
Could an analogous statement hold for flows?



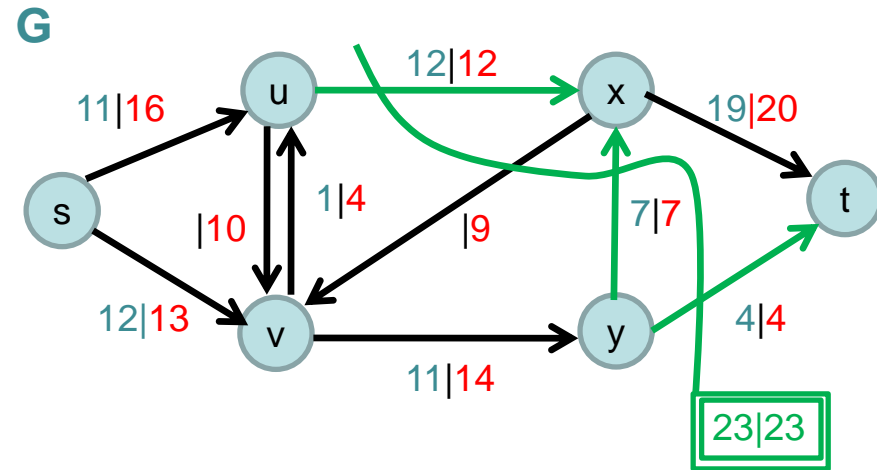
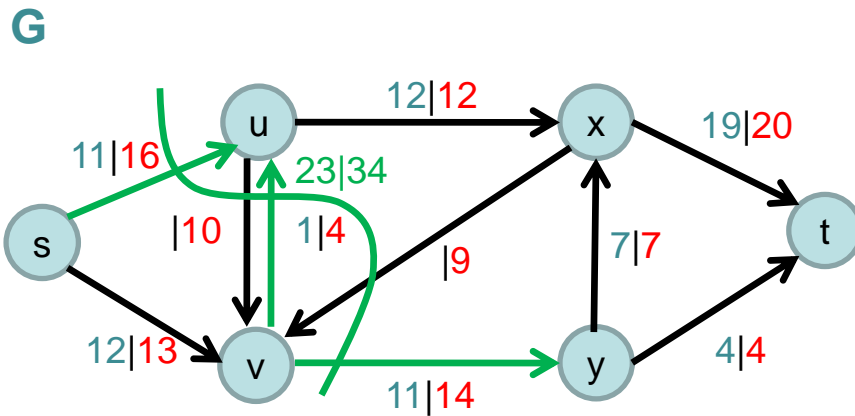
**Tool:** Leverage strong connection between cuts and flows.

# Example: Ford-Fulkerson Algorithm

Cuts versus flows: Think about **net** flow across each cut - *coincidence*?



Augmented flow:

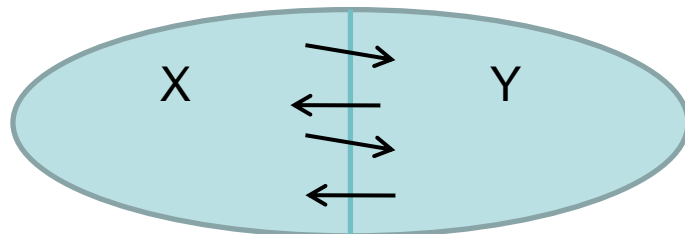




**Correctness:** How do we know that once no more augmenting paths exist, the flow we have is optimal, i.e. a maximum flow?

**Definition 6.11:** Let  $(G,s,t,c)$  be a flow network. For  $X, Y \subset V$  we define

- Flow across cut  $X/Y$ :  $f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$
- Capacity across cut  $X/Y$ :  $c(X, Y) = \sum_{x \in X} \sum_{y \in Y} c(x, y)$
- Shorthand for  $v \in V$ :  $X - v = X \setminus \{v\}$



**Lemma 6.12:** Let  $(G,s,t,c)$  be a flow network and let  $f$  a network flow in  $G$ . Then it holds for all  $X, Y, Z \subseteq V$ :

- $f(X, X) = 0$
- $f(X, Y) = -f(Y, X)$
- If  $X \cap Y = \emptyset$  then  $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$  and  $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$

# Cuts in Flow Networks

**Definition 6.13:** Let  $(G, s, t, c)$  be a flow network and  $f$  be a flow in  $G$ .

- a) A **cut**  $(S, T)$  of  $G$  is a partition of  $V$  into  $S$  and  $T = V \setminus S$  so that  $s \in S$  und  $t \in T$ .
- b) The **flow** across a cut  $(S, T)$  is defined as  $f(S, T)$ .
- c) The **capacity** of a cut  $(S, T)$  is defined as  $c(S, T)$ .

**Remark 6.14:**

- a) The definition of a flow is consistent with the flows that were considered in the previous examples: flows from  $T$  to  $S$  are subtracted:

$$f(S, T) = \sum_{x \in S} \sum_{y \in T} f(x, y) \quad (\text{where } f(x, y) < 0 \text{ if } f(y, x) > 0).$$

- b) The definition of the capacity of a cut is consistent with the capacities that were considered in the previous examples: edges from  $T$  to  $S$  add no capacity to the cut:

$$c(S, T) = \sum_{x \in S} \sum_{y \in T} c(x, y) \quad \text{where } c(x, y) \geq 0.$$

# Cuts in Flow Networks

**Lemma 6.15:** Let  $(G, s, t, c)$  be a flow network and  $f$  be a flow in  $G$ . Let  $(S, T)$  be a cut of  $G$ . Then

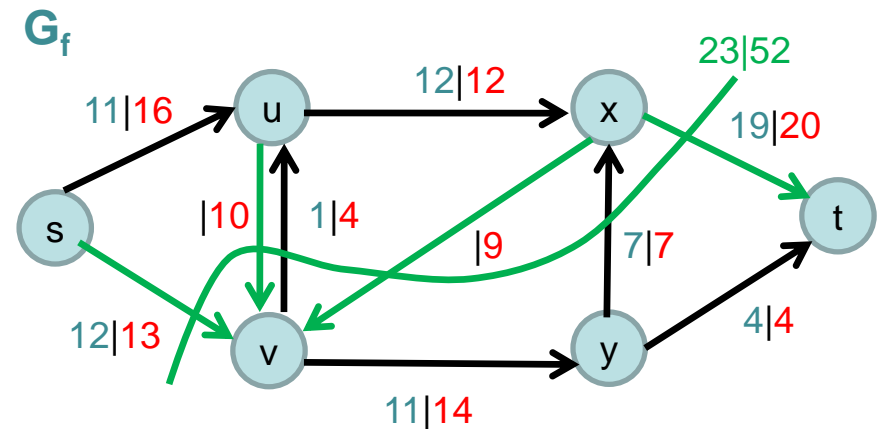
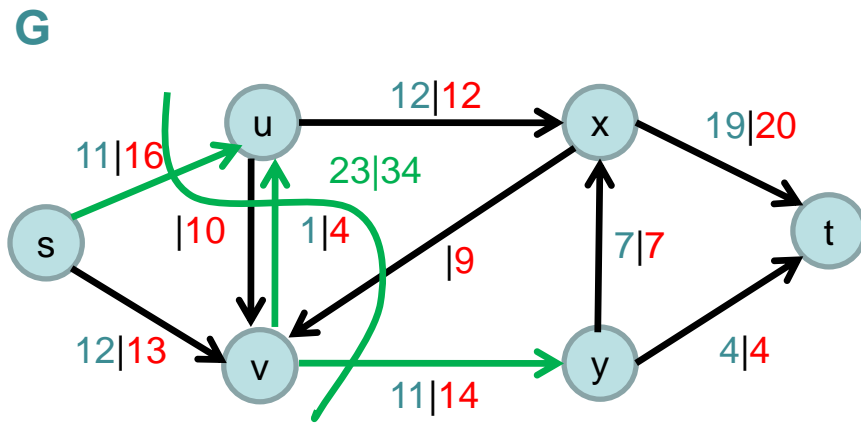
$$f(S, T) = |f|.$$

As a result,

$$|f| = f(s, V - s) = f(V - t, t).$$

**Proof:** Exercise

**Corollary 6.16:** Let  $(G, s, t, c)$  be a flow network. Then the flow value of any flow  $f$  in  $G$  is upper bounded by the capacity of an arbitrary cut in  $G$ .



# Cuts in Flow Networks

Theorem 6.17: (Max-Flow Min-Cut Theorem)

Let  $(G, s, t, c)$  be a flow network and  $f$  be a flow in  $G$ . Then the following statements are equivalent.

- $f$  is a maximum flow in  $G$ .
- The residual network  $G_f$  of  $G$  w.r.t.  $f$  does not contain any augmenting path.
- There exists a cut  $(S, T)$  of  $G$  such that  $|f| = c(S, T)$ .

Proof:

- $a) \Rightarrow b)$ :  $\neg b) \Rightarrow \neg a)$  holds due to Corollary 6.10 and therefore also  $a) \Rightarrow b)$ .
- $b) \Rightarrow c)$ : Let  $S$  be a set of nodes that are reachable from  $s$  in  $G_f$ . (Why is this not all of  $G_f$ ?) Then  $(S, T)$  with  $T = V \setminus S$  is a cut and  $f(S, T) = c(S, T)$  according to the definition of  $G_f$  (Why?). Thus, by Lemma 6.15,  $|f| = c(S, T)$ .
- $c) \Rightarrow a)$ : Follows from Corollary 6.16.

Conclusions:

- The Ford-Fulkerson algorithm is correct, i.e. finds max flow.
- Why (the moral lesson to take home): Max-Flow Min-Cut gives *upper bound* on best flow possible, while FF-alg *constructs* explicit flow matching this bound.

# Cuts in Flow Networks

**Theorem 6.17: (Max-Flow Min-Cut Theorem)**

Let  $(G, s, t, c)$  be a flow network and  $f$  be a flow in  $G$ . Then the following statements are equivalent.

- $f$  is a maximum flow in  $G$ .
- The residual network  $G_f$  of  $G$  w.r.t.  $f$  does not contain any augmenting path.
- There exists a cut  $(S, T)$  of  $G$  such that  $|f| = c(S, T)$ .

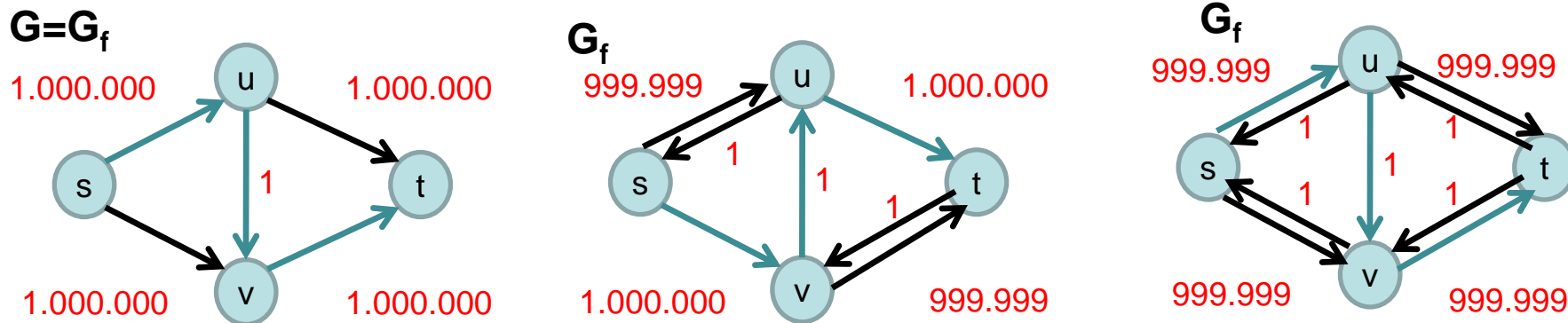
**Proof:**

- $a) \Rightarrow b)$ :  $\neg b) \Rightarrow \neg a)$  holds due to Corollary 6.10 and therefore also  $a) \Rightarrow b)$ .
- $b) \Rightarrow c)$ : Let  $S$  be a set of nodes that are reachable from  $s$  in  $G_f$ . (**Why** is this not all of  $G_f$ ?) Then  $(S, T)$  with  $T = V \setminus S$  is a cut and  $f(S, T) = c(S, T)$  according to the definition of  $G_f$  (**Why?**). Thus, by Lemma 6.15,  $|f| = c(S, T)$ .
- $c) \Rightarrow a)$ : Follows from Corollary 6.16.

**Corollary 6.18:** Let  $(G, s, t, c)$  be a flow network with integer capacities  $c(u, v)$ . Then the Ford-Fulkerson Algorithm computes a maximum flow  $f$  in time  $O(|E| \cdot |f|)$ . (**Why?**)

Remark 6.19:

a) The bound on the runtime of FORDFULKERSON is sharp:

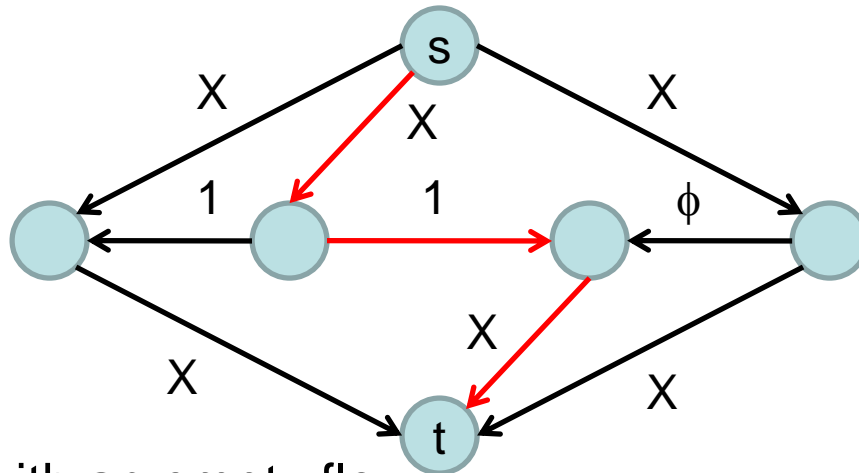


b) If the capacities are rational numbers, then they can be scaled to integer numbers, and FORDFULKERSON can be applied to the scaled network.

c) If the capacities are not rational numbers, then FORDFULKERSON may not terminate, and the flow  $f$  computed by FORDFULKERSON may not converge to the maximum flow.

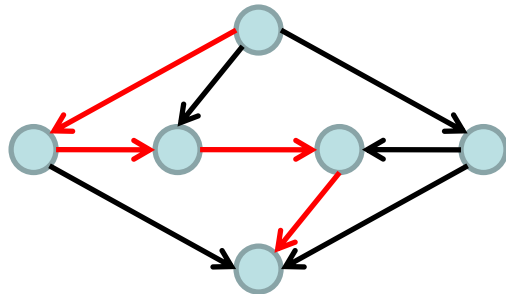
# Problems with irrational Capacities

c): Let  $\phi = (\sqrt{5} - 1)/2 \approx 0,618034$  be chosen so that  $1 - \phi = \phi^2$ . In order to show that the Ford-Fulkerson Algorithm gets stuck, consider the following graph (where  $X \geq 4$ ):

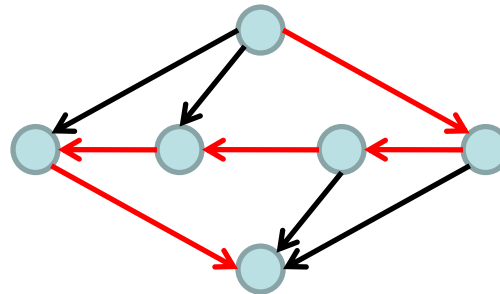


- We start with an empty flow.
- After using the red path, the residual capacities of the horizontal edges are 1, 0 and  $\phi$ , or  $\phi^{k-1}$ , 0 and  $\phi^k$  for  $k=1$ .
- **What is the residual network? (Draw it before going to next slide.)**

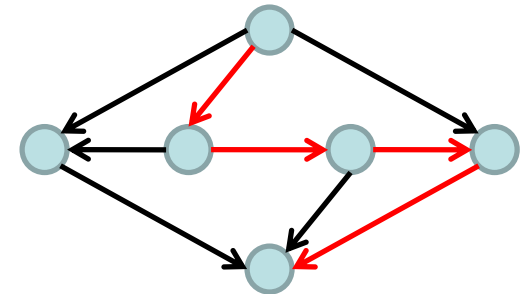
# Problems with irrational Capacities



A



B



C

Suppose that the residual capacities of the horizontal edges are  $\phi^{k-1}$ ,  $0$  and  $\phi^k$  for some odd  $k \in \mathbb{N}$ . Consider the sequence of augmenting paths:

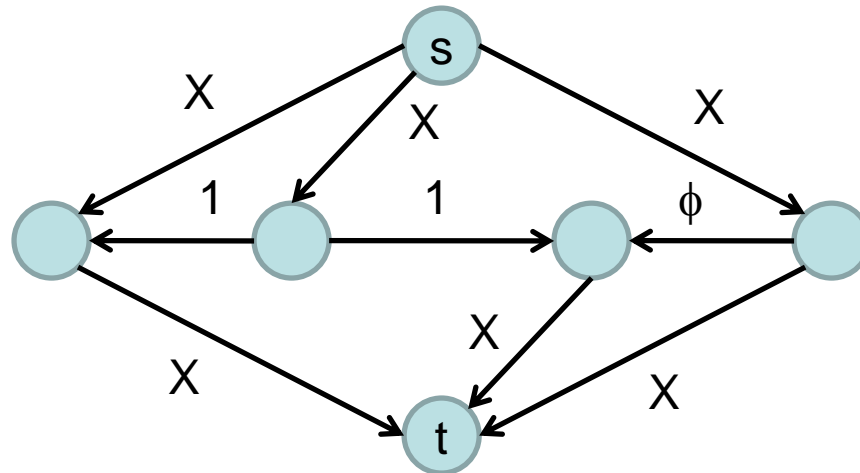
1. Augment along **B**, which adds  $\phi^k$  to the flow (*why?*). The residual capacities are now  $\phi^{k+1}$ ,  $\phi^k$  and  $0$  (*why?*).
2. Augment along **C**, which adds  $\phi^k$  to the flow. The residual capacities are now  $\phi^{k+1}$ ,  $0$  and  $\phi^k$ .
3. Augment along **B**, which adds  $\phi^{k+1}$  to the flow. The residual capacities are now  $0$ ,  $\phi^{k+1}$  and  $\phi^{k+2}$ .
4. Augment along **A**, which adds  $\phi^{k+1}$  to the flow. The residual capacities are now  $\phi^{k+1}$ ,  $0$  and  $\phi^{k+2}$ .



# Problems with irrational Capacities

- That is, after  $4n+1$  augmentations we arrive at residual capacities  $\phi^{2n}$ ,  $0$  and  $\phi^{2n+1}$ .
- As the number of augmentations goes to  $\infty$ , the value of the flow converges to (where does the factor 2 come from below?)  

$$1 + 2\sum_{i \geq 0} \phi^i = 1 + 2/(1-\phi) = 4 + \sqrt{5} < 7$$
 although the maximum flow value is  $2X+1$  (why?).



# Edmonds-Karp Algorithms

**Problem:** the Ford-Fulkerson Algorithm gives too much freedom to the choice of augmenting paths.

In 1972, Edmonds and Karp proposed two heuristics in order to compute maximum flows more efficiently.

**Heuristic 1:** Choose the augmenting path of largest value/bottleneck.

**Heuristic 2:** Choose the shortest augmenting path.

# Edmonds-Karp Algorithms

**Theorem 6.20:** Let  $(G, s, t, c)$  be a flow network with integer capacities  $c(u, v)$ . Then heuristic 1 computes a maximum flow  $f^*$  in time  $O(|E|^2 \cdot \log |E| \cdot \log |f^*|)$ .

**Proof:**

- Let  $f^*$  be a maximum flow in  $G$ .
- Let  $f$  be an arbitrary flow in  $G$  and  $f'$  be a maximum flow in the residual network  $G_f$ . (Initially,  $f$  is empty and therefore  $|f'| = |f^*|$ .)
- Let  $e$  be the bottleneck edge in the augmenting path chosen by heuristic 1. (Recall path chosen greedily to maximize  $c_f(e)$ .)
- $S \subseteq V$ : set of nodes that can be reached from  $s$  along edges in  $G_f$  with residual capacity  $> c_f(e)$ .
- $T = V \setminus S$ : is not empty due to heuristic 1 and the choice of  $e$ . (Why?)
- It holds:  $|f'| \leq c_f(S, T) \leq c_f(e) \cdot |E|$  (why?). So  $c_f(e) \geq |f'|/|E|$ .
- Since  $|f^*| = |f| + |f'|$  (intuition?), the value of  $f$  increases at least by a factor of  $(1 + 1/|E|)$  if  $|f^*| \geq 2|f|$  (why?).

# Edmonds-Karp Algorithms

**Theorem 6.20:** Let  $(G, s, t, c)$  be a flow network with integer capacities  $c(u, v)$ . Then heuristic 1 computes a maximum flow  $f^*$  in time  $O(|E|^2 \cdot \log |E| \cdot \log |f^*|)$ .

**Proof (continued):**

- The value of  $f$  increases by a factor of at least  $(1+1/|E|)$  each round if  $|f^*| \geq 2|f|$ .
- But  $(1+1/|E|)^k \geq |f^*|/2$  if  $k \geq |E| \ln |f^*|$ .
- Therefore, at most  $|E| \ln |f^*|$  augmenting paths suffice to obtain a flow of value at least  $|f^*|/2$ .
- Refining this argument, it takes at most  $|E|$  further augmenting paths to increase the flow value from  $\geq (1-1/2^k)|f^*|$  to  $\geq (1-1/2^{k+1})|f^*|$  for all  $k$ .
- Once  $k = \lfloor \log |f^*| \rfloor + 1$ , we have reached a flow value of  $|f^*|$  since we are only dealing with integer values.
- Time to compute an augmenting path with maximal flow value:  $O(|E| \log |E|)$ . (This is an exercise.)
- Thus, the total runtime is  $O(|E|^2 \cdot \log |E| \cdot \log |f^*|)$ .

# Edmonds-Karp Algorithms

## Analysis of Heuristic 2:

- $G_i$ : residual network after  $i$  augmenting steps, i.e.,  $G_0=G$ .
- For a node  $v$  let  $\text{dist}_i(v)$  be the distance (i.e., the number of edges along a shortest directed path) of  $v$  from  $s$  in  $G_i$ .
- No directed path from  $s$  to  $v$ :  $\text{dist}_i(v)=\infty$ .

# Edmonds-Karp Algorithms

We begin by proving three lemmas:

**Lemma 6.21:** For every node  $v$  with  $\text{dist}_i(v)=\infty$ , also  $\text{dist}_{i+1}(v)=\infty$ .

**Lemma 6.22:** For every node  $v \in V$  it holds that  $\text{dist}_{i+1}(v) \geq \text{dist}_i(v)$ .

**Lemma 6.23:** During the execution of Heuristic 2, every edge  $(u,v)$  can disappear at most  $|V|/2$  times from the residual graph.

# Edmonds-Karp Algorithms

**Lemma 6.21:** For every node  $v$  with  $\text{dist}_i(v)=\infty$ , also  $\text{dist}_{i+1}(v)=\infty$ .

**Proof:**

- Consider an arbitrary node  $v \in V$  with  $\text{dist}_i(v)=\infty$ .
- $U$ : set of nodes that have a directed path to  $v$  in  $G_i$ .
- Then for all nodes  $u \in U$ ,  $\text{dist}_i(u)=\infty$ .
- Suppose that  $\text{dist}_{i+1}(v) \neq \infty$ . Then an augmenting path must have been chosen in round  $i$  that goes through a node in  $U$ . (Why?)
- In this case, there must have been a directed path in  $G_i$  from  $s$  to a node in  $U$ , which contradicts the definition of  $U$ !

# Edmonds-Karp Algorithms

Lemma 6.22: For every node  $v \in V$  it holds that  $\text{dist}_{i+1}(v) \geq \text{dist}_i(v)$ .

Proof:

- $v=s$ : trivial since  $\text{dist}_i(s)=0$  for all  $i$ .
- $v \neq s$ : induction on the distance from  $s$ .
- $p=(s, \dots, u, v)$ : shortest path from  $s$  to  $v$  in  $G_{i+1}$ . (No such path, then we are done according to Lemma 6.21.)
- Since this is a shortest path,  $\text{dist}_{i+1}(u) = \text{dist}_{i+1}(v) - 1$ .
- According to the induction hypothesis,  $\text{dist}_{i+1}(u) \geq \text{dist}_i(u)$ .
- Case 1:  $(u, v)$  was an edge in  $G_i$ . Then  $\text{dist}_i(v) \leq \text{dist}_i(u) + 1$ . Hence,  $\text{dist}_{i+1}(v) = \text{dist}_{i+1}(u) + 1 \geq \text{dist}_i(u) + 1 \geq \text{dist}_i(v)$  (use IH)
- Case 2:  $(u, v)$  was not an edge in  $G_i$ . Then  $(v, u)$  belongs to the  $i$ -th augmenting path. In this case,  $(v, u)$  is on a **shortest path** from  $s$  to  $t$  in  $G_i$  and therefore,  $\text{dist}_i(v) = \text{dist}_i(u) - 1 \leq \text{dist}_i(u) + 1$ , as in Case 1.



# Edmonds-Karp Algorithms

**Lemma 6.23:** During the execution of Heuristic 2, every edge  $(u,v)$  can disappear at most  $|V|/2$  times from the residual graph.

**Proof:**

- Suppose that  $(u,v)$  is in the residual graphs  $G_i$  and  $G_{j+1}$  but not in the residual graphs  $G_{i+1}, \dots, G_j$ .
- Then  $(u,v)$  must be in the  $i$ -th augmenting path, and therefore,  $\text{dist}_i(v) = \text{dist}_i(u) + 1$ .
- Moreover,  $(v,u)$  must be in the  $j$ -th augmenting path, and therefore,  $\text{dist}_j(u) = \text{dist}_j(v) + 1$ .
- Together with Lemma 6.22 it follows that
$$\text{dist}_j(u) = \text{dist}_j(v) + 1 \geq \text{dist}_i(v) + 1 = \text{dist}_i(u) + 2$$
- Since  $|V| - 1$  is an upper bound on the largest finite distance of a node,  $(u,v)$  can disappear at most  $|V|/2$  times.

# Edmonds-Karp Algorithms

Now we are ready to prove a runtime bound for Heuristic 2.

- Since every edge can disappear at most  $|V|/2$  times from the residual network, there are at most  $|E| \cdot |V|/2$  events in which an edge disappears.
- But at least one edge disappears in each iteration, which implies that Heuristic 2 runs for at most  $|E| \cdot |V|/2$  iterations.
- Since a shortest augmenting path can be computed in time  $O(|E|)$  (using breadth first search), we get:

**Theorem 6.24:** Let  $(G, s, t, c)$  be a flow network with integer capacities  $c(u, v)$ . Then Heuristic 2 computes a maximal flow in time  $O(|E|^2 \cdot |V|)$ .

# Dinic's Algorithm

- The runtime of Heuristic 2 does not depend any more on the value of the maximum flow, but it is still too large.
- In the following we will present Dinic's Algorithm, which only needs  $O(|V|^2 |E|)$  time.

**Definition 6.25:** A flow  $f$  in a flow network  $(G, s, t, c)$  is called **blocking** if every path from  $s$  to  $t$  contains at least one saturated edge. An edge  $e$  is called **saturated** if  $f(e)=c(e)$ .

**Remark:** Not every blocking flow is also maximum, but every maximum flow is blocking. (Exercise!)

# Dinic's Algorithm

## Definition 6.26:

- The **level** of node  $v$  is defined  $\text{level}(v) = \delta_f(s, v)$  (number of edges along shortest path in  $G_f$  from  $s$  to  $v$ ).
- The **level graph**  $L_f$  is a subgraph of  $G_f$  that contains all edges  $(u, v)$  with  $(u, v) \in G_f$  and  $\text{level}(u) = \text{level}(v) - 1$ .
  - Roughly,  $L_f$  is  $G_f$  but with „odd cycles removed“.

**Lemma 6.27:**  $L_f$  contains all shortest augmenting paths, can be constructed in  $O(m)$  time using BFS.

- Intuition: To get  $L_f$ , do a BFS on  $G_f$ , and remove precisely those edges which led to the discovery of an odd cycle.

# Dinic's Algorithm

## Dinic's Algorithm:

start with an empty flow  $f$

repeat

    find a blocking flow  $f'$  in level graph  $L_f$

    set  $f := f + f'$

until sink  $t$  is not reachable in  $L_f$

**Note:** Unlike previous algorithms, we do not look for an (augmenting) *path*, but a (blocking) *flow*.

**Lemma 6.28:** Dinic's Algorithm stops after at most  $n-1$  iterations of the repeat-loop.

# Dinic's Algorithm

**Lemma 6.28:** Dinic's Algorithm stops after at most  $n-1$  iterations of the repeat-loop.

**Proof:**

- Consider some fixed iteration  $i$  and let
  - $f$  and  $level$  denote the flow and levels at the beginning and
  - $f'$  and  $level'$  denote the flow and levels at the end of the iteration.
- An edge  $(v,w)$  in  $G_{f'}$  is either
  - an edge in  $G_f$  (if the edge has not been saturated in iteration  $i$ ) or
  - a reverse edge in  $L_f$  (if it was not in  $G_f$ , so  $(w,v)$  used in iteration  $i$ ).
- Thus, for every edge  $(v,w) \in G_{f'}$ ,  $level(w) \leq level(v) + 1$ . (\*) (Why?)  
(Note same statement with  $level'(w) \leq level'(v) + 1$  would be trivial.)
- Consider now a shortest  $s$ - $t$  path  $P$  in  $L_{f'}$ :



- By (\*),  $level'(t) \geq level(t)$ . (Why? Use (\*) to argue  $level(t) \leq |P|$ .)
- If we could strengthen this to  $level'(t) > level(t)$ , our claim is shown...

# Dinic's Algorithm

**Lemma 6.28:** Dinic's Algorithm stops after at most  $n-1$  iterations of the repeat-loop.

**Proof (continued):**

- Suppose that  $level'(t) = level(t)$ . Let  $p$  be any shortest path from  $s$  to  $t$  in  $G_f$ .
- For every edge  $(v,w) \in p$  we know by (\*) that  $level(w) \leq level(v) + 1$ . So it must hold that  $level(w) = level(v) + 1$  (\*\*), since otherwise  $level'(t) > level(t)$ .
- If  $(v,w)$  was not an edge in  $G_f$ , then  $(w,v)$  used by blocking flow on  $L_f$  in iteration  $i$ , implying  $level(w) = level(v) - 1$ . Contradiction with (\*\*).
- Therefore, all edges  $(v,w)$  in  $p$  were also edges in  $G_f$ . By (\*\*),  $p$  was a shortest  $s$ - $t$  path in  $G_f$ .
- Every edge in  $p$  is therefore in  $L_f$ , and none of these was saturated (otherwise they would not be in  $G_f$ !).
- But this contradicts fact that we chose a blocking flow in  $L_f$  in iteration  $i$ .
- Therefore, in each iteration of the repeat-loop, the distance between  $s$  and  $t$  in  $G_f$  increases by at least 1.
- Since a shortest path from  $s$  to  $t$  cannot be longer than  $n-1$  (if such a path exists), the lemma follows.

# Dinic's Algorithm

In unit networks, fewer iterations are needed.

**Definition 6.29:** A **unit network** is a flow network  $(G, s, t, c)$  with integer capacities in which for any node  $v \in V \setminus \{s, t\}$ , precisely one of the following holds:

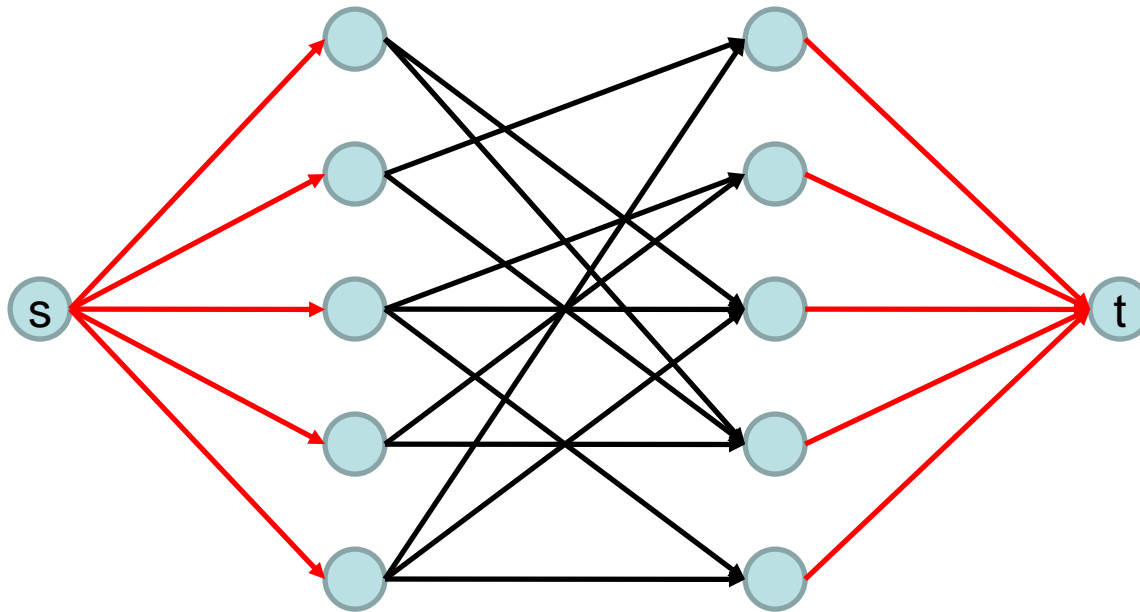
- $v$  has exactly 1 incoming edge, and it has capacity 1.
- $v$  has exactly 1 outgoing edge, and it has capacity 1.

**Remark:** If a node has exactly one incoming edge of capacity 1, it can still have many outgoing edges (and vice versa).



# Dinic's Algorithm

Example of a unit network:



→ : capacity 1

→ : arbitrary integer capacity

# Dinic's Algorithm

**Lemma 6.30:** Suppose we use Def. 6.5's notion of a network flow. Then, in a unit network, Dinic's Algorithm stops after at most  $2\sqrt{n-2}$  iterations.

**Definition 6.5:** Let  $(G,s,t,c)$  be a flow network. A **network flow** in  $G$  is a function  $f : E \rightarrow \mathbb{R}_{\geq 0}$  with

- $0 \leq f(u, v) \leq c(u, v)$  for all  $(u, v) \in E$   
(capacity constraints)
- $\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$  for all  $u \in V \setminus \{s, t\}$   
(flow conservation)

# Dinic's Algorithm

**Lemma 6.30:** Suppose we use Def. 6.5's notion of a network flow. Then, in a unit network, Dinic's Algorithm stops after at most  $2\sqrt{n-2}$  iterations.

**Proof:**

- Let us consider a fixed iteration.
- Let  $f$  be the current flow and  $f^*$  be a maximum flow. (Both have integer flow values.)
- Then  $f^* - f$  is a flow of integer value in  $G_f$ .
- Since  $G$  is a unit network,  $f^*(e) - f(e) \in \{-1, 0, 1\}$  at every edge  $e$ .
- We partition edges  $e$  with  $f^*(e) - f(e) = 1$  into a collection of paths from  $s$  to  $t$ .
- There are exactly  $|f^*| - |f|$  paths from  $s$  to  $t$ . (proof: exercise)
- These paths are node-disjoint (except for  $s$  and  $t$ ).
- Hence, shortest augmenting path has at most  $(n-2)/(|f^*| - |f|) + 1$  nodes.
- But after  $\sqrt{n-2}$  iterations, a shortest augmenting path contains at least  $\sqrt{n-2} + 1$  nodes (according to Lemma 6.28, the distance of  $t$  from  $s$  increases by one with each iteration).
- It holds that  $\sqrt{n-2} + 1 \leq (n-2)/(|f^*| - |f|) + 1 \iff |f^*| - |f| \leq \sqrt{n-2}$ .
- Hence, after at most  $\sqrt{n-2}$  further iterations we obtain a maximum flow.

# Dinic's Algorithm

How can we find a blocking flow?

Repeatedly use DFS:

repeat

    find a path  $p$  from  $s$  to  $t$  in  $L_f$  via DFS and send a flow value of  $c_f(p)$  along  $p$

until there is no augmenting path left in  $L_f$

**Lemma 6.31:** The time needed to compute a blocking flow is  $O(n \cdot m)$ . (Exercise)

**Theorem 6.32:** The runtime of Dinic's Algorithm is  $O(n^2 \cdot m)$ .

# Dinic's Algorithm

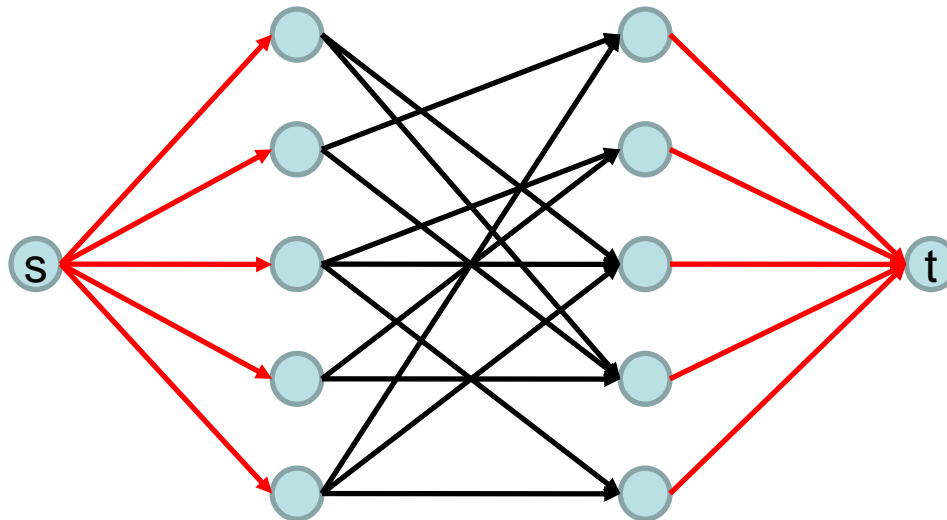
**Theorem 6.33:** The runtime of Dinic's algorithm on unit networks is  $O(\sqrt{n} \cdot m)$ .

**Proof:**

- When searching for a blocking flow, every edge of the unit network has to be visited at most once since it can only lie on at most one augmenting path.
- Thus, a blocking flow can be found in  $O(m)$  time.
- Together with Lemma 6.30 we obtain the runtime bound in the theorem.

# Dinic's Algorithm

Application of Dinic's Algorithm: maximum matching in bipartite graphs.

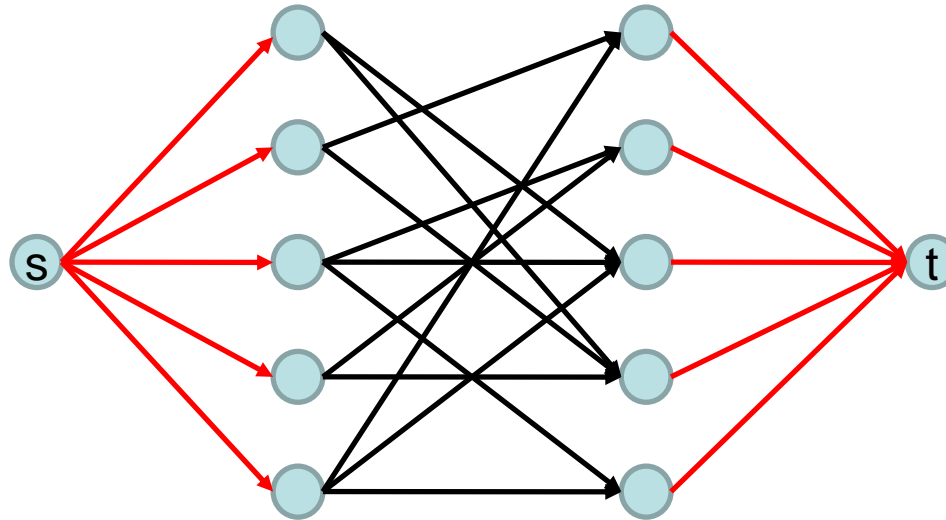


**Theorem 6.34:** Dinic's Algorithm on bipartite graphs  $G=(V,E)$  extended by a source  $s$  and sink  $t$  computes a maximum flow  $f$  in time  $O(\sqrt{n} \cdot m)$ , so that  $|f|$  is the size of a maximum matching in  $G$ .

# Dinic's Algorithm

Proof:

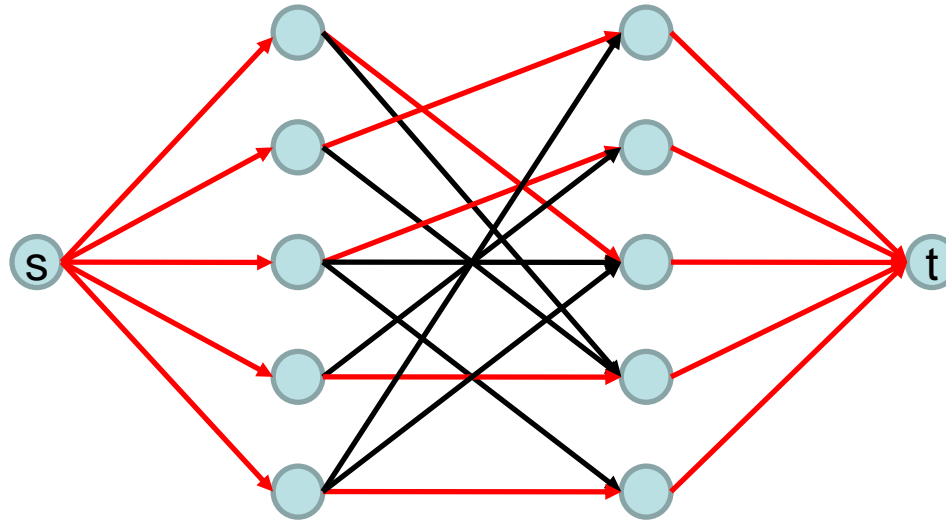
$|f| \geq |M|$ :



- Let  $M$  be a maximum matching in  $G$ .
- Then the flow  $f'$  that uses  $M$  and all edges of  $s$  and  $t$  to  $M$ , is a legal flow of value  $|f'| = |M|$  and therefore,  $|f| \geq |M|$ .

# Dinic's Algorithm

Proof:  
 $|M| \geq |f|$ :



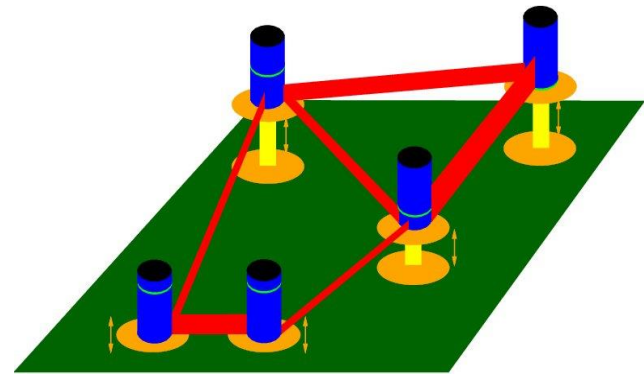
- Let  $f$  be a maximum flow in  $G$  extended by  $s$  and  $t$ .
- Then the set of edges  $M'$  that  $f$  traverses in  $G$  is a matching of size  $|M'| = |f|$  and therefore,  $|M| \geq |f|$ .



# Goldberg's Algorithm

## Intuition:

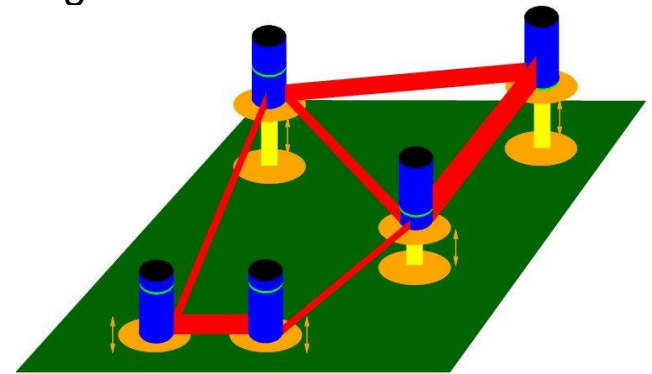
- A flow network can be seen as a **network of liquids**: edges correspond to pipes and nodes correspond to pipe connections.
- Every node has a **reservoir** that can collect an arbitrary amount of liquid.
- Every node, its reservoir, and all of its pipes are arranged on a **platform whose height may increase** during the execution of the algorithm.



# Goldberg's Algorithm

## Intuition:

- The node heights determine how the flow is moved through the network: **flow always flows downhill**.
- Initially, the source  $s$  pumps as much flow as possible into the network ( $= c(s, V - s)$ ).
- If the flow reaches some intermediate node, it is collected in its reservoir. From there it will be sent downhill later.
- If all non-saturated pipes that leave a node  $u$  lead to nodes  $v$  that are above  $u$ , then the height of  $u$  will be increased, i. e., we **lift  $u$** .
- If the total flow that can flow to a sink, reaches it, then the **excess flow in the reservoirs is sent back to the source** by lifting the heights of the intermediate nodes beyond the height of the source.



# Goldberg's Algorithm

**Definition 6.35:** Let  $(G,s,t,c)$  be a flow network. A **preflow** is a function  $f:V \times V \rightarrow \mathbb{R}$  satisfying the following properties:

- $f(u, v) \leq c(u, v)$  for all  $u, v \in V$  (capacity constraints)
- $f(u, v) = -f(v, u)$  for all  $u, v \in V$  (skew symmetry)
- $f(V, u) \geq 0$  for all  $u \in V \setminus \{s\}$  (preflow condition)

(How does the last condition contrast with flow conservation? Why does this support the intuition from the previous slide? Hint: Think about reservoirs.)

- The **excess flow** of a node  $u$  is defined as  $e_f(u) = f(V, u)$ . A node  $u \neq t$  is called **active** if  $e_f(u) > 0$ .
- Goldberg's Algorithm assigns to each node  $v$  a height  $h(v) \in \mathbb{N}_0$ . The height function is called **legal** if  $h(s) = |V|$ ,  $h(t) = 0$ , and for all edges  $(v, w)$  in the residual network  $G_f$ ,  $h(v) \leq h(w) + 1$ .  
(I.e., for all  $(v, w) \in E$  with  $h(v) > h(w) + 1$ ,  $(v, w) \notin E_f$ .)
- An edge  $(v, w)$  in  $G_f$  is called **admissible** if  $h(v) > h(w)$ .  
(Together with the previous condition it follows that  $h(v) = h(w) + 1$ .)

# Goldberg's Algorithm

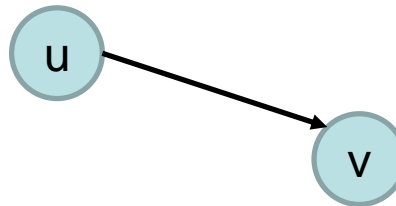
## Basic Operations:

- **Push(u,v)**: push as much flow as possible from **u** to **v**
- **Lift(u)**: lift **u** as much as possible without violating the legality of the height function.

## In pseudocode:

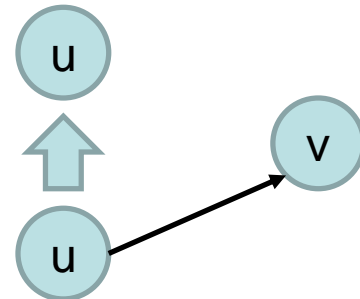
### Push(u,v):

```
 $\delta := \min\{e_f(u), c_f(u,v)\}$   
 $f(u,v) := f(u,v) + \delta$   
 $c_f(u,v) := c_f(u,v) - \delta$   
 $c_f(v,u) := c_f(v,u) + \delta$   
 $e_f(u) := e_f(u) - \delta$   
 $e_f(v) := e_f(v) + \delta$ 
```



### Lift(u):

```
 $h(u) := \min\{h(v) + 1 \mid (u,v) \in E_f\}$ 
```



# Goldberg's Algorithm

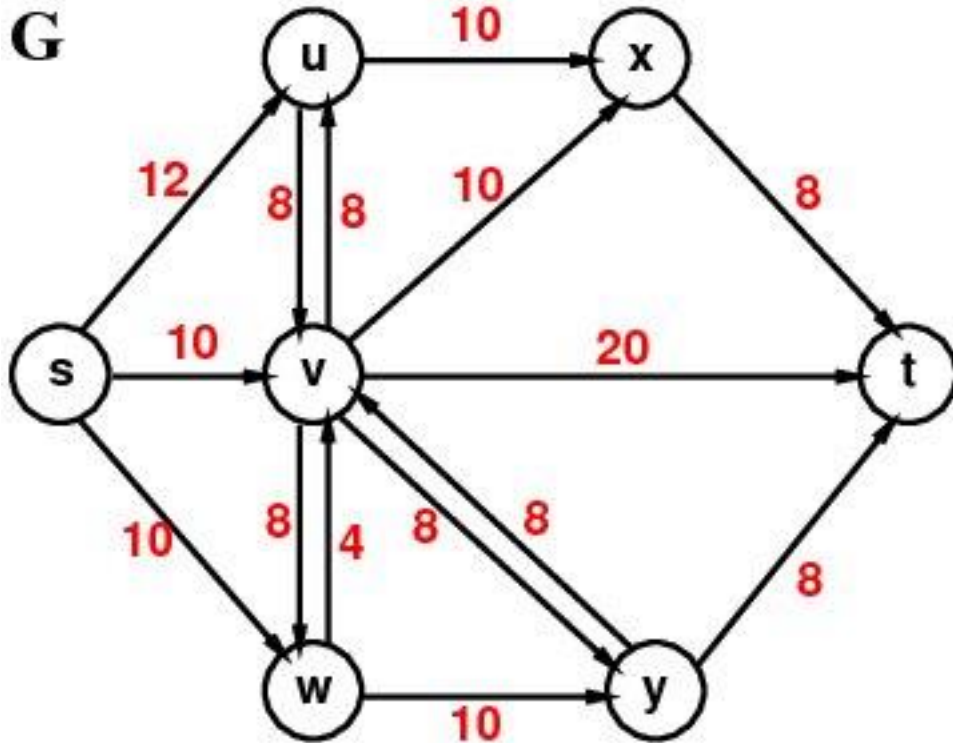
Goldberg's Algorithm works as follows:

Preflow-Push Algorithm:

```
for each  $u \in V \setminus \{s\}$  do  $h(u) := 0$ ;  $e_f(u) := 0$ 
for each  $(u, v) \in E$  do  $f(u, v) := 0$ ;  $f(v, u) := 0$ 
 $h(s) := |V|$ 
for each  $(s, u) \in E$  do
   $f(s, u) := c(s, u)$ ;  $f(u, s) := -f(s, u)$ ;  $e_f(u) := c(s, u)$ 
while (there are active nodes) do
  Pick an active node  $u$  //  $u$  has excess flow
  if (there is an admissible edge  $(u, v)$  ) //  $(u, v)$  has downward slope
    then Push( $u, v$ )
  else Lift( $u$ ) // create admissible edge
```

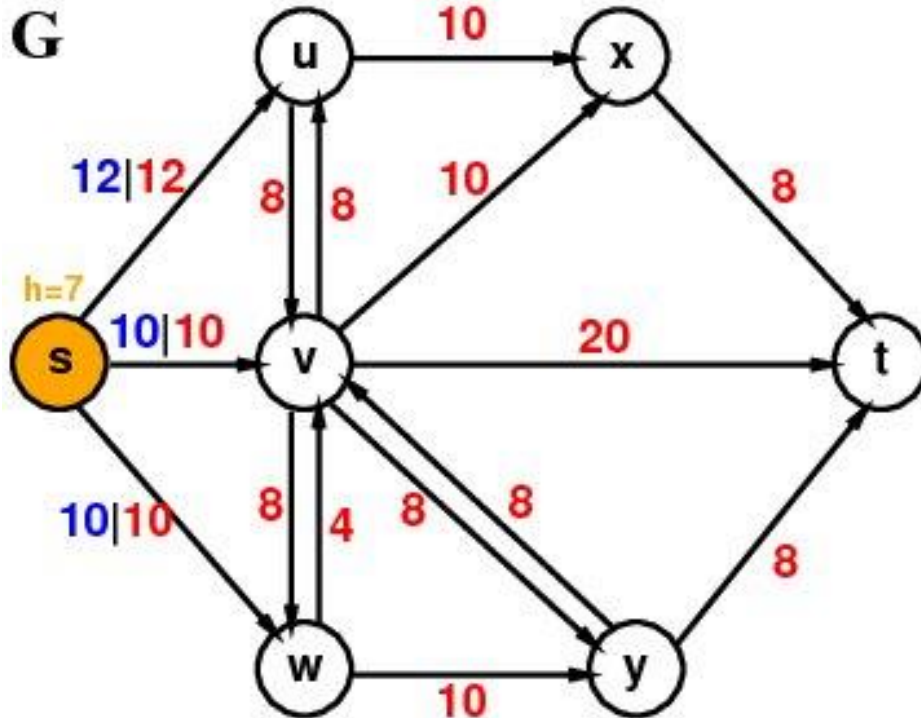
**Question:** Why is the initial height function (ie after the for loop finishes executing) legal?

Example:



Capacities are marked in red

Example:



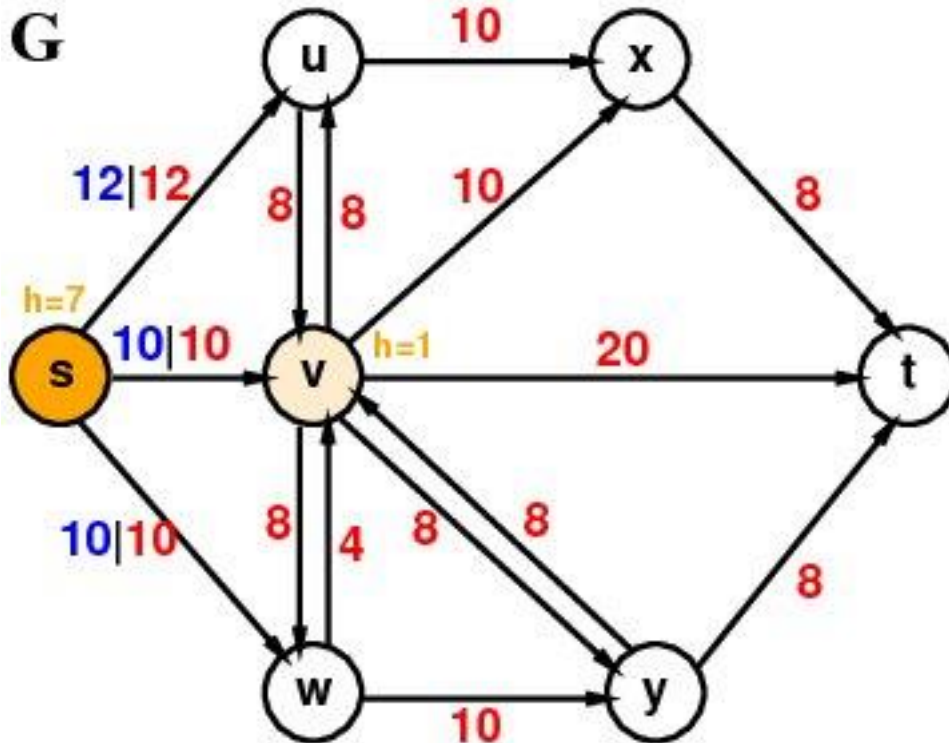
After initialization:

- $s$  is lifted to height 7. The heights of all other nodes are set to 0.
- Every edge from  $s$  is saturated. All other edges have a flow of 0.

No PUSH-operation can currently be executed.

Operations that can be executed are  $LIFT(u)$ ,  $LIFT(v)$  or  $LIFT(w)$ .

Example:



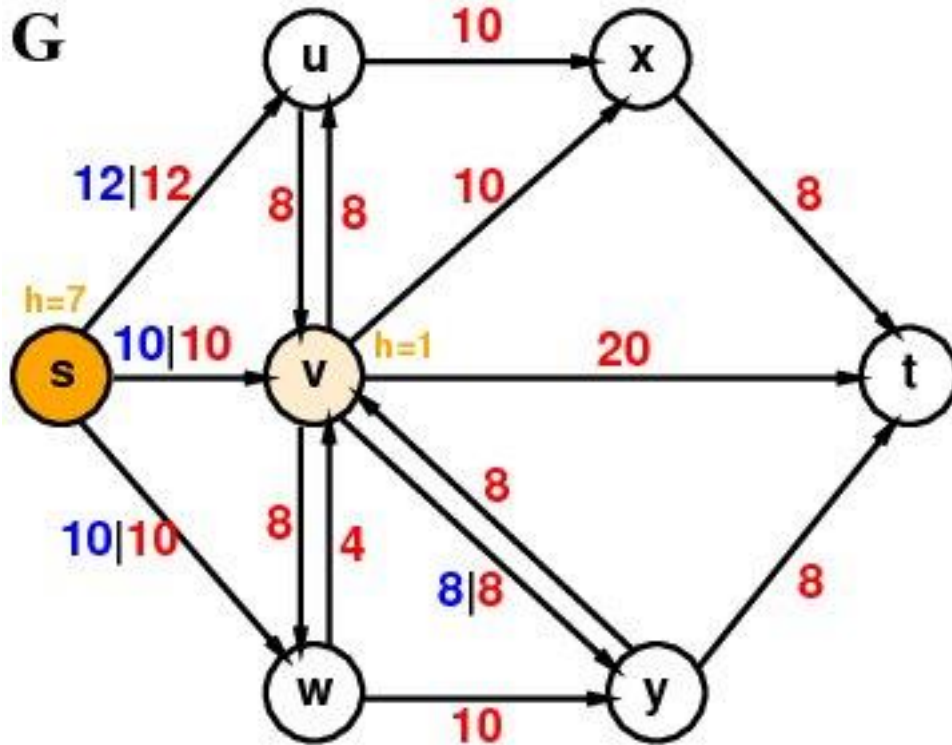
After LIFT(v):

The height  $h(v)$  is set to  
 $1 + \min \{h [u] \mid (v, u) \in E_f\}$   
 $= 1 + 0 = 1.$

Now, operations that can be executed are LIFT(u), LIFT(w) or PUSH(v, u), PUSH(v, w), PUSH(v, x), PUSH(v, y), PUSH(v, t).



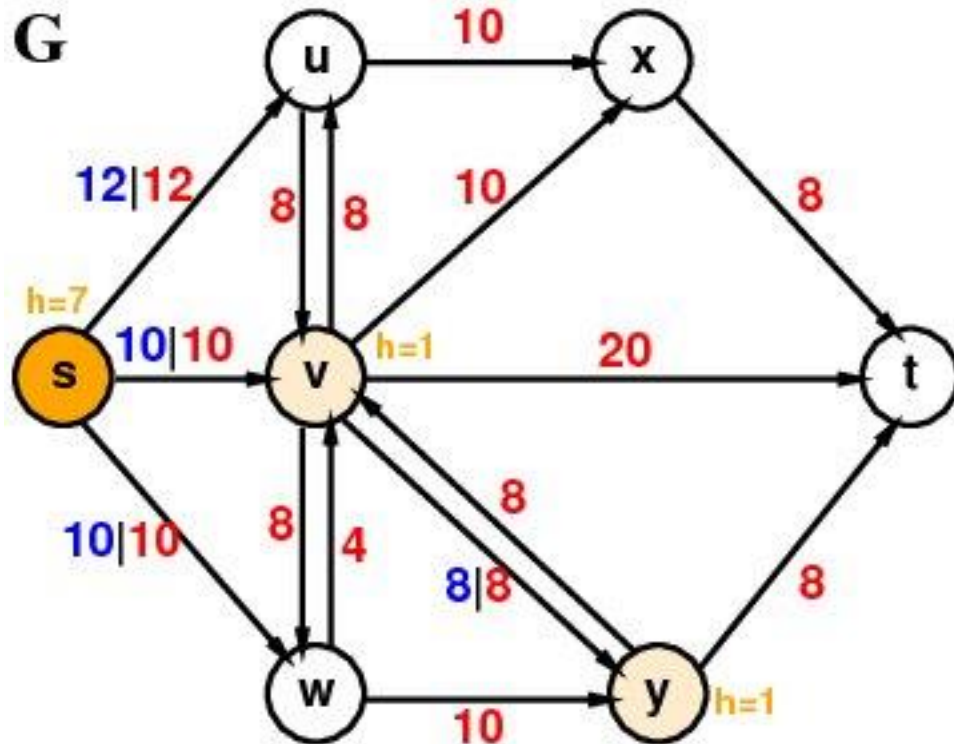
Example:



After  $PUSH(v, y)$ :

Operations that can be executed are  $LIFT(u)$ ,  $LIFT(w)$ ,  $LIFT(y)$  or  $PUSH(v, u)$ ,  $PUSH(v, w)$ ,  $PUSH(v, x)$ ,  $PUSH(v, t)$ .

Example:

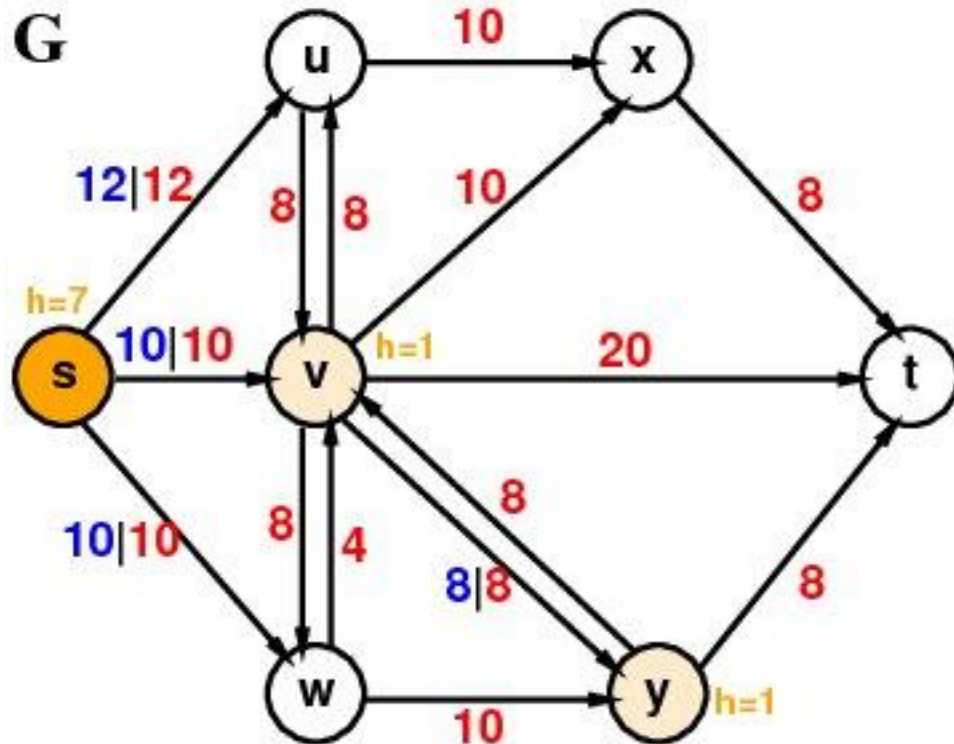


After LIFT(y):

The height  $h(y)$  is set to  
 $1 + \min\{h[u] \mid (y, u) \in E_f\}$   
 $= 1 + 0 = 1.$

Operations that can be executed  
 are LIFT(u), LIFT(w) or  
 PUSH(v, u), PUSH(v, w),  
 PUSH(v, x), PUSH(v, t),  
 PUSH(y, t).

Example:



After  $\text{PUSH}(y, t)$ :

Operations that can be executed are  $\text{LIFT}(u)$ ,  $\text{LIFT}(w)$  or  $\text{PUSH}(v, u)$ ,  $\text{PUSH}(v, w)$ ,  $\text{PUSH}(v, x)$ ,  $\text{PUSH}(v, t)$ .

The algorithm continues to run until no  $\text{PUSH}$  or  $\text{LIFT}$  operation can be executed.

# Goldberg's Algorithm

Let's prove some lemmas!

- 1) Which invariants hold during the execution of the algorithm?
- 2) How do we know when an optimal flow is found?
- 3) What is the runtime of the algorithm?

# Goldberg's Algorithm

**Lemma 6.36:** At any point in time during the execution,  $e_f(s) \leq 0$  and for all nodes  $v \in V \setminus \{s\}$ ,  $e_f(v) \geq 0$ .

**Proof:**

- We perform a complete induction over the number of executed Push and Lift operations.
- Initially, the lemma is obviously true.
- Thus we assume that it is true for some point in time.
- Then a Push operation maintains the property that  $e_f(v) \geq 0$  for all  $v \in V \setminus \{s\}$  due to the choice of  $\delta$ .  
A Lift operation does not change any  $e_f(v)$ .
- Therefore,  $\sum_{u \in V \setminus \{s\}} f(V, u) \geq 0$ . (by definition of  $e_f(v)$ )
- Moreover, we know that for  $f$  it holds
$$\sum_{u \in V} f(V, u) = 0.$$
- Hence,  $e_f(s) = f(V, s) \leq 0$ .

# Goldberg's Algorithm

**Lemma 6.37:** Every Lift( $u$ ) call preserves the legality of the height function and increases  $h(u)$  by at least 1.

**Proof:**

- A Lift operation is only executed for some node  $u$  if there is no admissible edge  $(u,v)$  and therefore  $h(u) \leq h(v)$  for all  $(u,v) \in E_f$ .
- Since the new height is  $h'(u) = \min\{h(v)+1 \mid (u,v) \in E_f\}$ , it follows that  $h'(u) > h(u)$  and  $h'(u) \leq h(v)+1$  for all  $(u,v) \in E_f$ .

# Goldberg's Algorithm

Let's prove some lemmas!

- 1) Which invariants hold during the execution of the algorithm?
- 2) How do we know when an optimal flow is found?
- 3) What is the runtime of the algorithm?

# Goldberg's Algorithm

**Lemma 6.38 (Superoptimality):** For every legal preflow  $f$  and every legal height function  $h$  there is no augmenting path in  $G_f$ .

**Proof:**

- Suppose that there is an augmenting path  $(s=v_1, v_2, \dots, v_l=t)$  in  $G_f$ .
- Since the heights of the nodes can only increase over time (see Lemma 6.37), it holds that  $h(s) \geq n$ , and since  $t$  can never be active,  $h(t)=0$ .
- Hence,  
$$n \leq h(s) \leq h(v_2)+1 \leq h(v_3)+2 \leq \dots \leq h(t)+l-1 = l-1 \leq n-1$$
since the augmenting path is simple and can therefore contain at most  $n$  nodes.
- Thus, we obtain a contradiction. QED.

**Question:** Why are we not done in our optimality analysis? (Hint: Is a legal preflow also a legal flow?)



# Goldberg's Algorithm

**Lemma 6.39 (Optimality):** If there is no active node in  $G_f$ , then the preflow is a maximum flow.

**Proof:**

- By definition of active node, there is no node  $u \in V \setminus \{s, t\}$  with  $e_f(u) > 0$ . Then, it holds for all nodes  $u \in V \setminus \{s, t\}$  according to Lemma 6.36 that  $e_f(u) = 0$ . Hence, the preflow is a legal flow.
- The maximality of the flow follows from Lemma 6.38 and the Maxflow-Mincut Theorem.

# Goldberg's Algorithm

Let's prove some lemmas!

- 1) Which invariants hold during the execution of the algorithm?
- 2) How do we know when an optimal flow is found?
- 3) What is the runtime of the algorithm?  
(And does it even terminate?)

# Goldberg's Algorithm

Lemma 6.40: For every active node  $u$  there is a path in  $G_f$  from  $u$  to  $s$ .

Proof:

- Let  $U$  be the set of nodes that are reachable from  $u$  in  $G_f$ .
- Recall Lemma 6.36 says all nodes other than  $s$  have non-negative excess flow. So if  $s \notin U$ , then all nodes in  $U$  have non-negative excess flow.
- The flow into  $U$  is 0 since if there is edge  $(v,w) \in E$  with  $v \notin U$ ,  $w \in U$ ,  $f(v,w) > 0$ , then  $c_f(w,v) \geq f(v,w) > 0$ . (If there is flow going into  $U$ , it can be cancelled in the opposite direction, so  $G_f$  has an edge leaving  $U$ , contradiction.)
- Hence,
 
$$\begin{aligned}
 0 &= \sum_{v \in V \setminus U} f(v, U) \geq \sum_{v \in V \setminus U} f(v, U) - \sum_{v \in U} f(v, V \setminus U) \quad (\text{by preflow condition}) \\
 &= \sum_{v \in V \setminus U} f(v, U) - \sum_{v \in U} f(v, V \setminus U) + \sum_{v \in U} f(v, U) - \sum_{v \in U} f(v, U) \\
 &= \sum_{v \in U} (f(V, v) - f(v, V)) \quad (\text{note } \sum_{v \in V \setminus U} f(v, U) + \sum_{v \in U} f(v, U) = \sum_{v \in U} (f(V, v))) \\
 &= \sum_{v \in U \setminus \{u\}} (f(V, v) - f(v, V)) + 2e_f(u) > 0 \quad (\text{skew symmetry/preflow condition})
 \end{aligned}$$
- Thus, we arrive at a contradiction, and therefore  $s \in U$ .

# Goldberg's Algorithm

Lemma 6.41: For every active node  $v \in V$ ,  $h(v) \leq 2n-1$ .

Proof:

- Since  $s$  can never be active due to Lemma 6.36 and  $t$  can never be active by definition,  $h(s)=n$  and  $h(t)=0$  at any time.
- Consider an arbitrary active node  $u \in V \setminus \{s, t\}$ .
- According to Lemma 6.40,  $s$  is reachable from  $u$  in  $G_f$ .
- Let  $p=(u=v_1, v_2, \dots, v_l=s)$  be a simple path from  $u$  to  $s$ .
- We know that  $h(s)=n$ . Moreover,  $h(v_i) \leq h(v_{i+1})+1$  for all  $i \in \{1, \dots, l-1\}$ .
- Hence,  $h(u) \leq n+l \leq 2n-1$  because the path cannot contain  $t$  and therefore  $l \leq n-1$ . QED.

Now we are ready to determine the runtime of the Lift and Push operations.

# Goldberg's Algorithm

## LIFT OPERATIONS

**Lemma 6.42:** The total number of Lift operations executed by the algorithm is  $O(n^2)$  and their total runtime is  $O(n \cdot m)$ .

**Proof:**

- Due to Lemmas 6.37 and 6.41 at most  $2n-1$  Lift operations can be applied to any node. Hence, altogether at most  $O(n^2)$  Lift operations are executed by the algorithm.
- The cost of a  $\text{Lift}(v)$  operation is equal to the outgoing degree of a node  $v$  in  $G_f$  because we have to check all nodes reachable from  $v$ .
- Thus, the total runtime of the Lift operations is at most
$$\sum_{v \in V} (2n-1) \cdot \text{deg}(v) = O(n \cdot m)$$
where  $\text{deg}(v)$  denotes the (outgoing) degree of a node  $v$ .

# Goldberg's Algorithm

## PUSH OPERATIONS (2 cases)

A Push operation is **saturation** if  $\delta = c_f(u, v)$ , and otherwise non-saturating. (Recall  $\delta := \min\{e_f(u), c_f(u, v)\}$ .)

**Lemma 6.43:** The total number of saturating Push operations is  $O(n \cdot m)$ .

**Proof:**

- After a saturating Push on  $(u, v)$ , we cannot pump again flow from  $u$  to  $v$  unless  $v$  has performed a Push operation on  $(v, u)$ .
- Since it must hold that  $h(u) \geq h(v) + 1$  at  $\text{Push}(u, v)$  and  $h(v) \geq h(u) + 1$  at  $\text{Push}(v, u)$  and the heights of the nodes are monotonically increasing, the height of  $u$  must have increased by at least 2 for another saturating Push along  $(u, v)$ .
- Hence, there can be at most  $(2n-1)/2$  saturating Push operations via  $(u, v)$ , which results in a total number of at most  $O(n \cdot m)$  saturating Push operations.

# Goldberg's Algorithm

**Lemma 6.44:** The total number of non-saturating Push operations is  $O(n^2m)$ .

**Proof:**

- We use the potential function  $\Phi = \sum_{\text{active } v \in V} h(v)$ .
- Initially,  $\Phi = 0$  since all heights of active nodes are equal to 0. We distinguish between three types of operations that can change  $\Phi$ :
  - **Non-saturating Push along edge  $(v,w)$ :** Then node  $v$  becomes inactive and  $\Phi$  is reduced by  $h(v)$ . On the other hand,  $w$  can now become active, which can increase  $\Phi$  by  $h(w)$ . But since  $h(v) \geq h(w) + 1$ ,  $\Phi$  is always decreased by **at least 1**.
  - **Saturating Push along edge  $(v,w)$ :** This can increase  $\Phi$  by at most  $2n-1$  since in the worst case  $v$  remains active while  $w$  becomes active and  $h(w) \leq 2n-1$ .
  - **Lift operation:** Altogether, the Lift operations increase  $\Phi$  by at most  $(2n-1)n$ .
- Since there are only  $O(n \cdot m)$  saturating Push operations,  $\Phi$  can be increased by at most  $O(n^2m)$  due to these.
- Moreover,  $\Phi$  can be increased by at most  $O(n^2)$  due to Lift operations.
- Altogether,  $\Phi$  can be increased by at most  $O(n^2m)$ . Since  $\Phi \geq 0$  at any time, the number of non-saturating Push operations is at most  $O(n^2m)$ .

# Goldberg's Algorithm

- Since a Push operation only takes constant time:
  - Lemma 6.42: Lift operations  $O(nm)$ .
  - Lemma 6.43: Saturating push operations  $O(nm)$ .
  - Lemma 6.44: Non-saturating push operations  $O(n^2m)$ .
- So total runtime of  $O(n^2m)$  for Goldberg's Algorithm.
- With an improved selection of Push and Lift Operations, this runtime can be improved.

## Rules for the choice of active nodes:

- **FIFO**: The active nodes are organized in a FIFO queue, i.e., new active nodes are added to the back of the queue and active nodes to be processed are taken from the front. With this rule, a runtime of  $O(n^3)$  can be reached.
- **Highest-Label-First**: Always take the active node of largest height. In this case, one can reach a runtime of  $O(\sqrt{m} \cdot n^2)$ .



# Other Variants

- Goldberg, 1985: FIFO PPA:  $O(|V|^3)$ .
- Goldberg, Tarjan, 1986:  
Improved FIFO PPA:  $O(|V| \cdot |E| \cdot \log(|V|^2 \cdot |E|))$ .
- Goldberg, Tarjan, 1986, Cheriyan, Maheshwari 1989:  
Highest Label PPA:  $O(|V|^2 \cdot \sqrt{|E|})$ .
- King, Rao, Tarjan, 1994:  
 $O(|V| \cdot |E| \log_{|E|/(|V| \log |V|)} |V|)$ .
- Orlin, 2013:  
 $O(|V| \cdot |E|)$ .
- Randomized Variants

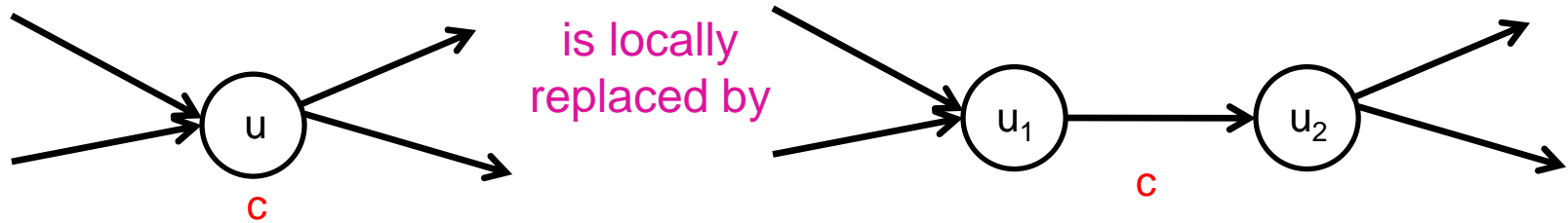
## History of maximum flow algorithms:

$G = (V, E)$  with  $|V| = n$ ,  $|E| = m$ ,  $U$ : value of maximum flow.

	Year	Researcher	Run time
1.	1951	Dantzig	$O(n^2 m U)$
2.	1955	Ford, Fulkerson	$O(n m U)$
3.	1970	Dinitz / Edmonds, Karp	$O(n m^2)$
4.	1970	Dinitz	$O(n^2 m)$
5.	1972	Edmonds, Karp / Dinitz	$O(m^2 \log U)$
6.	1973	Dinitz / Gabow	$O(n m \log U)$
7.	1974	Karzanov	$O(n^3)$
8.	1977	Cherkassky	$O(n^2 \sqrt{m})$
9.	1980	Galil, Naamad	$O(n m \log^2 n)$
10.	1983	Sleator, Tarjan	$O(n m \log n)$
11.	1986	Goldberg, Tarjan	$O(n m \log(n^2/m))$
12.	1987	Ahuja, Orlin	$O(n m + n^2 \log U)$
13.	1987	Ahuja et al.	$O(n m \log(n \sqrt{\log U} / (m + 2)))$
14.	1989	Cheriyān, Hagerup	$E(n m + n^2 \log^2 n)$
15.	1990	Cheriyān et al.	$O(n^3 / \log n)$
16.	1990	Alon	$O(n m + n^{8/3} \log n)$
17.	1992	King et al.	$O(n m + n^{2+\epsilon})$
18.	1993	Philipps, Westbrook	$O(n m (\log_{m/n} n + \log^{2+\epsilon} n))$
19.	1994	King et al.	$O(n m \log_m / (n \log n)^n)$
20.	1997	Goldberg, Rao	$O(m^{3/2} \log(n^2/m) \log U)$ $O(n^{2/3} m \log(n^2/m) \log U)$

# Variants of the Maxflow Problem

... node capacities



... undirected graphs



or make use of skew symmetry to ensure flow in only one direction

# Minimal Cut with minimal Number of Edges

From now on we assume the use of flows following Definition 6.5.

**Definition 6.5:** Let  $(G,s,t,c)$  be a flow network. A **network flow** in  $G$  is a function  $f : E \rightarrow \mathbb{R}_{\geq 0}$  with

- $0 \leq f(u, v) \leq c(u, v)$  for all  $(u, v) \in E$  (capacity constraints)
- $\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$  for all  $u \in V \setminus \{s, t\}$   
(flow conservation)

# Minimal Cut with minimal Number of Edges

From now on we assume the use of flows following Definition 6.5.

Problem MINCUTMINEDGES:

**Input:** flow network  $(G, s, t, c)$  with integer edge capacities

**Output:** minimal cut of  $(G, s, t, c)$  with minimal number of edges  
(among all minimal cuts)

Transform  $(G, s, t, c)$  into a flow network  $(G, s, t, c')$  with  $c'(u, v) = M \cdot c(u, v) + 1$ , where  $M \geq |E| + 1$  is a sufficiently large fixed value.

By Max-Flow Min-Cut theorem, corresponding to optimal solution of the MAXFLOW problem in  $(G, s, t, c')$  is a saturated cut satisfying:

$$c'(S, T) = M \cdot \underbrace{c(S, T)}_{\text{minimal cut in } G} + \underbrace{|\{e \in E \mid e \in S \times T\}|}_{\text{number of edges crossing the cut}}$$

**Question:** Why does  $(S, T)$  yield optimal solution to MinCutMinEdges?

# Circulation

Definition 6.45:

- a) A **circulation network** is a triple  $(G, b, c)$ , where  $G = (V, E)$  is a directed graph and  $b, c : E \rightarrow \mathbb{R}$  are capacity functions with  $b(e) \leq c(e)$  for all  $e \in E$ .
- b) A **circulation**  $f$  is a flow without a source or sink. Formally,  $f : E \rightarrow \mathbb{R}$  with
- $b(e) \leq f(e) \leq c(e)$  for all  $e \in E$  (capacity constraints)
  - $\sum_{(u,v) \in E} f(u,v) = \sum_{(v,w) \in E} f(v,w)$  for all  $v \in V$  (flow conservation)

Problem CIRCULATE:

**Input:** circulation network  $(G, b, c)$

**Output:** circulation  $f$  for  $(G, b, c)$

Redefine the residual network  $G_f = (V, E_f)$  with  $E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}$  and

$$c_f(u,v) = \left[ \underbrace{c(u,v) - f(u,v)}_{\text{residual capacity as before}} + \underbrace{\max\{f(v,u) - b(v,u), 0\}}_{\text{maximal flow that can be removed in opp. direction}} \right] \begin{array}{l} \text{if } f(u,v) < b(u,v) \\ \text{if } f(u,v) \geq b(u,v) \end{array}$$

← increases net flow from u to v!

# Circulation

```

Algorithm CIRCULATE (circulation network  $(G, b, c)$ ) {
   $f(e) = 0$  for all  $e \in E$ ;
  while  $(\exists e \in E : f(e) < b(e))$  {
    choose  $(u, v) \in E$  with  $f(u, v) < b(u, v)$ ;
    if  $(\exists$  path  $P$  from  $v$  to  $u$  in  $G_f)$  {
       $C = (P, (u, v))$  is a cycle with  $(u, v) \in C$ ;

      send  $\delta = \min_{e \in C} c_f(e)$  flow along cycle  $C$ ;
    } else return(NULL);
  } /* while */
  return( $f$ );
} /* CIRCULATE */

```

/\* initialize to trivial non-feasible (why?) flow \*/

/\* flow on edge  $(u, v)$  is too small \*/

/\* notice the new definition of  $G_f$  ! \*/

/\* find cycle along which to push more flow \*/

/\* there is no circulation \*/

Algorithm CIRCULATE terminates. Whenever it outputs a function  $f$ , then  $f$  is a feasible circulation for  $(G, b, c)$ . Whenever the algorithm outputs *NULL*, there is no feasible circulation for  $(G, b, c)$ .

**Lemma 6.46:**  $(G, b, c)$  with  $G = (V, E)$  has a feasible circulation  $\Leftrightarrow$  for every subset  $U \subseteq V$

$$\sum_{(u, v) \in (U, \bar{U})} b(u, v) \leq \sum_{(v, u) \in (\bar{U}, U)} c(v, u)$$

# Circulation

**Lemma 6.46:**  $(G, b, c)$  with  $G = (V, E)$  has a feasible circulation  $\Leftrightarrow$  for every subset  $U \subseteq V$

$$\sum_{(u, v) \in (U, \bar{U})} b(u, v) \leq \sum_{(u, v) \in (\bar{U}, U)} c(v, u)$$

**Proof:**

$\Rightarrow$ : Let  $f$  be a circulation for  $(G, b, c)$ .

- Then it holds for all  $v \in V$

$$\sum_{(u, v) \in E} f(u, v) = \sum_{(v, w) \in E} f(v, w)$$

- Therefore, it holds for all  $U \subseteq V$ :

$$\begin{aligned} \sum_{(u, v) \in (U, \bar{U})} b(u, v) &\leq \sum_{(u, v) \in (U, \bar{U})} f(u, v) \\ &= \sum_{(v, w) \in (\bar{U}, U)} f(v, w) \\ &\leq \sum_{(v, w) \in (\bar{U}, U)} c(v, w) \end{aligned}$$



# Circulation

Proof (continued):

$\Leftarrow$ : suppose there is no circulation  $f$  for  $(G,b,c)$ .

- Then algorithm CIRCULATE outputs NULL.
- That is, there is an edge  $e=(u,v)\in E$  with  $f(e)<b(e)$  so that there is no path in  $G_f$  from  $v$  to  $u$ .
- Define  $U=\{w\in V \mid v\rightsquigarrow w \text{ in } G_f\}$ .
- Then  $v\in U$  and  $u\in\bar{U}$  and for all  $(x,y)\in(U,\bar{U})$ :  $c_f(x,y)=0$ .
- Hence,  $f(e)=c(e)$  for all  $e\in(U,\bar{U})$  and  $f(e)\leq b(e)$  for all  $e\in(\bar{U},U)$ .
- Therefore,

$$\begin{aligned}\sum_{(u,v)\in(\bar{U},U)} b(u,v) &> \sum_{(u,v)\in(\bar{U},U)} f(u,v) \\ &= \sum_{(v,w)\in(U,\bar{U})} f(v,w) \\ &= \sum_{(v,w)\in(U,\bar{U})} c(v,w)\end{aligned}$$

# Lower Bounds for Edge Flows

Definition 6.47:

- a) A **constrained flow network** is a tuple  $(G, s, t, b, c)$ , where  $(G, s, t, c)$  is a flow network and  $b : E \rightarrow \mathbb{R}$  is a function with  $b(e) \leq c(e)$  for all  $e \in E$ .
- b) A **constrained flow**  $f$  for  $(G, s, t, b, c)$  is a feasible flow for  $(G, s, t, c)$  with  $b(e) \leq f(e)$  for all  $e \in E$ .

Problem LOWERBOUNDEDFLOW:

**Input:** constrained flow network  $(G, s, t, b, c)$

**Output:** constrained flow  $f$  for  $(G, s, t, b, c)$  of maximal value  $|f|$

# Lower Bounds for Edge Flows

```
Algorithm BOUNDEDFLOW (constrained flow network  $(G, s, t, b, c)$ ){
  construct circulation network  $(G', b', c')$  with
     $G' = G + (t, s)$ ;
     $b'(e) = b(e)$ ;  $c'(e) = c(e)$  for all  $e \in E$ ;
     $b'(t, s) = 0$ ;  $c'(t, s) = \infty$ ;
   $f' = \text{CIRCULATE}(G', b', c')$ ;
  if  $(f' = \text{NULL})$  return  $(\text{NULL})$ ;
   $f =$  feasible flow for  $(G, s, t, b, c)$  resulting from  $f'$  by deleting edge  $(t, s)$  and its flow;
  solve the MAXFLOW problem for  $(G, s, t, c)$  with
     $f$  as initial flow and
    residual capacities  $c'_f(u, v) = \underbrace{c(u, v) - f(u, v)}_{\text{as before}} + \underbrace{\max\{f(v, u) - b(v, u), 0\}}_{\text{max. flow to be deleted}}$  for all  $(u, v) \in E$ 

  let  $g$  be the maximum flow obtained for  $(G, s, t, c)$ ;
  return( $g$ );
} /* BOUNDEDFLOW */
```

If Algorithm BOUNDEDFLOW outputs *NULL*, then there is no constrained flow for  $(G, s, t, b, c)$ . Otherwise, Algorithm BOUNDEDFLOW outputs a solution  $f$  for the LOWERBOUNDEDFLOW problem.

# Minimal Flow Problem

Problem MINFLOW:

**Input:** constrained flow network  $(G, s, t, b, c)$

**Output:** constrained flow  $f$  for  $(G, s, t, b, c)$  with minimal value  $|f|$

Algorithm MINIMUMFLOW (constrained flow network  $(G, s, t, b, c)$ ) {

$h = \text{BOUNDEDFLOW}(G, s, t, b, c)$ ;

  if ( $h = \text{NULL}$ ) return( $\text{NULL}$ );

  construct flow network  $(G^R, s^R, t^R, c^R)$  with

$G^R = (V, E^R)$ ,  $E^R = \{(v, u) \mid (u, v) \in E\}$ ;

$s^R = t$ ;  $t^R = s$ ;

$c^R(u, v) = c_f(u, v)$  for all  $(u, v) \in E^R$ ,

    where  $c_f()$  represents the redefined residual capacities;

  solve the MAXFLOW problem for  $(G^R, s^R, t^R, c^R)$  and let

$g$  be the maximal flow for  $(G^R, s^R, t^R, c^R)$ ;

$f = h - g$ ;

  return( $f$ );

} /\* MINIMUMFLOW \*/

If Algorithm MINIMUMFLOW outputs  $\text{NULL}$ , then there is no feasible constrained flow for  $(G, s, t, b, c)$ . Otherwise, Algorithm MINIMUMFLOW outputs a solution  $f$  for the MINFLOW problem.

# Flows of minimal Cost

a) A **flow network with edge costs** is tuple  $(G, s, t, c, \gamma)$  where  $G = (V, E)$  is a directed graph,  $c : E \rightarrow \mathbb{N}_0$  defines edge capacities, and  $\gamma : E \rightarrow \mathbb{N}_0$  defines edge costs, i.e.,  $\gamma(e)$  is the cost of sending one unit of flow across  $e$ .

b) The **cost** of a flow  $f$  from  $s$  to  $t$  in  $G$  is defined as

$$\gamma(f) := \sum_{e \in E} \gamma(e) f(e).$$

**Problem MINCOSTMAXFLOW:**

**Input:** flow network  $(G, s, t, c, \gamma)$

**Output:** maximum flow  $f$  with minimal cost  $\gamma(f)$ .

W.l.o.g. we assume that  $G=(V,E)$  is a directed graph with at most one edge for each pair of nodes, i.e., there is no pair of nodes  $u,v \in V$  with  $(u,v), (v,u) \in E$ .

# Flows of minimal Cost

Let  $f$  be a flow.

- **cost** of augmenting path  $p$  for  $f$  (now not necessarily from  $s$  to  $t$ ):  
$$\gamma(p) = \sum_{(u,v) \in p: (u,v) \in E} c_f(p) \cdot \gamma(u,v) - \sum_{(u,v) \in p: (v,u) \in E} c_f(p) \cdot \gamma(v,u)$$
- **augmenting cycle** w.r.t.  $f$ : an augmenting path whose startpoint and endpoint are identical (by definition  $s$  and  $t$  cannot be on such a cycle; **why?**).

**Lemma 6.49:** For every augmenting cycle  $p$  for a flow  $f$  there is a flow  $f'$  with  $|f'| = |f|$  and

$$\gamma(f') = \gamma(f) + \gamma(p) .$$

**Proof:**

- Let  $f' = f + p$ . Then  $f'$  is still a feasible flow. Also, since  $p$  does not go through  $s$ , it cannot change flow value, i.e.  $|f'| = |f|$ .
- Furthermore, the cost of  $f'$  satisfies:

$$\begin{aligned} \gamma(f') &= \gamma(f) + \sum_{(u,v) \in p: (u,v) \in E} c_f(p) \cdot \gamma(u,v) - \sum_{(u,v) \in p: (v,u) \in E} c_f(p) \cdot \gamma(v,u) \\ &= \gamma(f) + \gamma(p) \end{aligned}$$

# Flows of minimal Cost

**Theorem 6.50:** A flow  $f$  has minimal cost among all flows of value  $|f|$  if and only if there is no augmenting cycle of negative cost for  $f$ .

**Proof:**

$\Rightarrow$ : Follows from Lemma 6.49.

$\Leftarrow$ : Let  $f$  be a flow that does not have a minimal cost. Let  $g$  be a flow of minimal cost and  $|g|=|f|$ .

- Consider network  $G_{g-f} = (V, E_{g-f})$  with  $E_{g-f} = E_{g-f}^+ \cup E_{g-f}^-$ , where  
 $E_{g-f}^+ = \{(u,v) \mid g(u,v) > f(u,v)\}$  and  
 $E_{g-f}^- = \{(v,u) \mid g(u,v) < f(u,v)\}$  (note: no equality allowed!)
- Let the edge capacities be defined as  $c_{g-f} = |g(e) - f(e)|$ .
- Then one can show that for all nodes  $v \in V$  that

$$\sum_{u \in V} c_{g-f}(u,v) - \sum_{w \in V} c_{g-f}(v,w) = 0 \quad (*) \text{ „flow conservation“}$$

# Flows of minimal Cost

Proof (continued):

- Let  $c_{\min} = \min\{c_{g-f}(u,v) \mid (u,v) \in E_{g-f}\}$ . (Recall  $c_{g-f} = |g(e) - f(e)|$ , and by definition of  $E_{g-f} = E_{g-f}^+ \cup E_{g-f}^-$ ,  $c_{\min} > 0$ .)
- If we start at any  $v \in V$  with at least one neighbor in  $G_{g-f}$ , then there is always an edge  $(v,w)$  with  $c_{g-f}(v,w) \geq c_{\min}$ , and every node not visited „can be left“ due to „flow conservation“ (\*).
- Since the number of nodes is limited by  $|V|$ , there must be an augmenting cycle in  $G_{g-f}$ .
- When removing this cycle, the flow conservation and capacity constraints are still satisfied.
- Thus,  $G_{g-f}$  can be decomposed into a set of augmenting cycles. When applying these cycles to  $f$ , we obtain  $g$ .
- Since  $\gamma(g) < \gamma(f)$ , at least one of these cycles must have negative cost.



# Flows of minimal Cost

**ALGORITHM:** If have integer edge capacities and edge costs bounded by constant  $C$ , can use the following polynomial time algorithm to compute a maximum flow with minimal cost:

- Use Ford-Fulkerson to compute a maximum flow. Runtime:  $O(C \cdot n \cdot m)$ . (Recall Ford-Fulkerson requires  $O(|E| \cdot |f|)$  time, for  $f$  an optimal flow.)
- Search for augmenting cycles of negative cost in  $G_f$  until no such cycles can be found. A negative augmenting cycle can be found via Bellman-Ford in  $O(n \cdot m)$  time (why?). Every such cycle has an integer capacity of  $>0$  and integer cost  $<0$ . I.e., the cost of the maximal flow reduces by at least 1 for each such augmenting cycle. The total runtime for this part is therefore  $O((C^2 \cdot m) \cdot (n \cdot m)) = O(C^2 \cdot n \cdot m^2)$ . (Q: Why is  $C^2 \cdot m$  an upper bound on max cost for any flow? Recall cost of a flow is  $\gamma(f) := \sum_{e \in E} \gamma(e) f(e)$ .)

More advanced techniques can also compute a maximal flow of minimal cost for arbitrary capacities and cost values (see, e.g., the book by Ahuja, Magnanti und Orlin: Network Flows).

# Flows of minimal Cost

## Definition 6.48:

- a) A flow network with edge costs is a tuple  $(G, c, \gamma, \delta)$  where  $G = (V, E)$  is a directed graph,  $c : E \rightarrow \mathbb{N}_0$  defines the edge capacities,  $\delta : V \rightarrow \mathbb{Z}$  defines the **node demands**,  $\sum_{v \in V} \delta(v) = 0$ , and  $\gamma : E \rightarrow \mathbb{N}_0$  defines the edge costs, i.e.,  $\gamma(e)$  is the cost to send one unit of flow across  $e$ .  
Note that  $\delta(v) > 0$  means a **node that consumes flow** while  $\delta(v) < 0$  means a **node that produces flow**.
- b) A flow  $f$  of minimal cost for  $(G, c, \gamma, \delta)$  is a function  $f : E \rightarrow \mathbb{N}_0$  with
1.  $f(e) \leq c(e)$  for all  $e \in E$  (capacity constraints)
  2.  $\sum_{(u,v) \in E} f(u, v) - \sum_{(v,w) \in E} f(v, w) = \delta(v)$  for all  $v \in V$ ,  
which minimizes the cost  $\gamma(f) := \sum_{e \in E} \gamma(e) f(e)$ . (flow conservation)

Problem MINCOSTFLOW:

**Input:** flow network with edge costs  $(G, c, \gamma, \delta)$

**Output:** flow  $f$  of minimal cost for  $(G, c, \gamma, \delta)$

Solution possible by combining LOWERBOUNDEDFLOW problems with the MINCOSTMAXFLOW problem.