Fundamental Algorithms

Chapter 5: Matchings

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(based on slides of Christian Scheideler)

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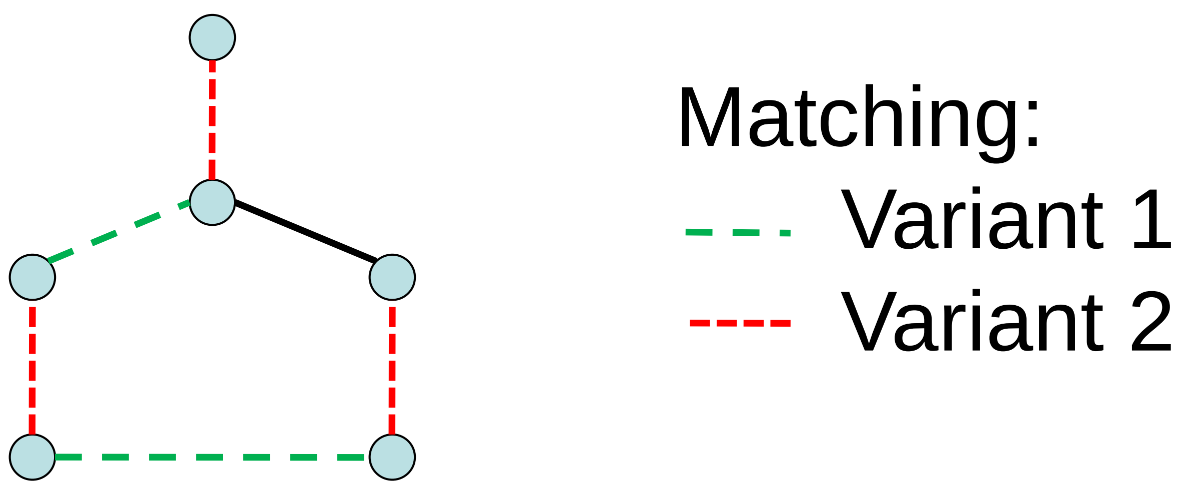
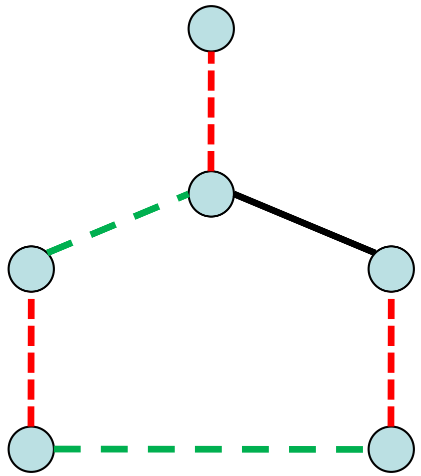
Basic Notation

Definition 5.1: Let G=(V,E) be an undirected graph. A matching M in G is a subset of E in which no two edges share a common node.



Basic Notation

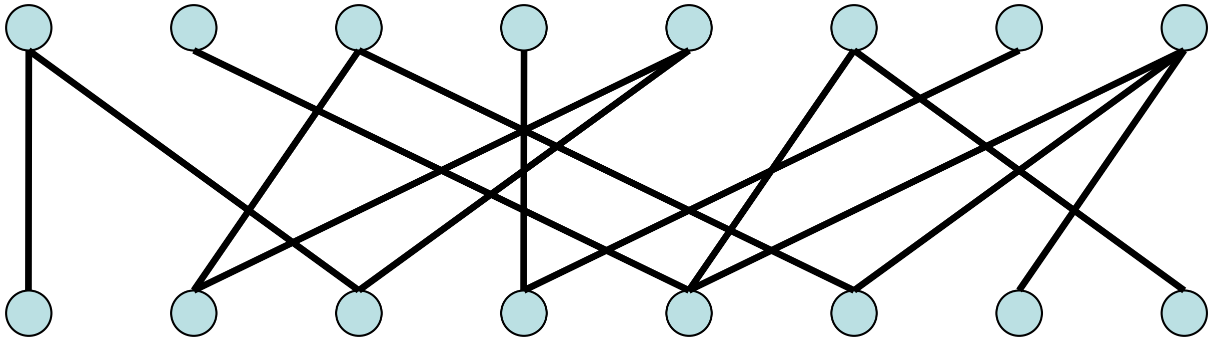
Definition 5.2:



- A matching M in G=(V,E) is called perfect if |M|=|V|/2.
- A matching M is called a maximum matching if there is no matching M´ in G with |M´|>|M| (example: red edges)
- A matching M is called maximal if it is maximal w.r.t. „⊆“, i.e., it cannot be extended (example: green edges)

Basic Notation

Definition 5.3: Let $G=(V,E)$ be an undirected graph. If V can be partitioned into two non-empty subsets V_1 and V_2 (i.e., $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$) so that $E \subseteq V_1 \times V_2$, then G is called bipartite (in this case, G may also be defined as $G=(V_1,V_2,E)$).



Foundations

Theorem 5.4: A graph $G=(V,E)$ has a perfect matching if and only if $|V|$ is even and there is no $S \subseteq V$ so that the subgraph induced by $V \setminus S$ contains more than $|S|$ connected components (CC) of odd size.

Proof:

„$\Rightarrow$“: (only direction we prove here)

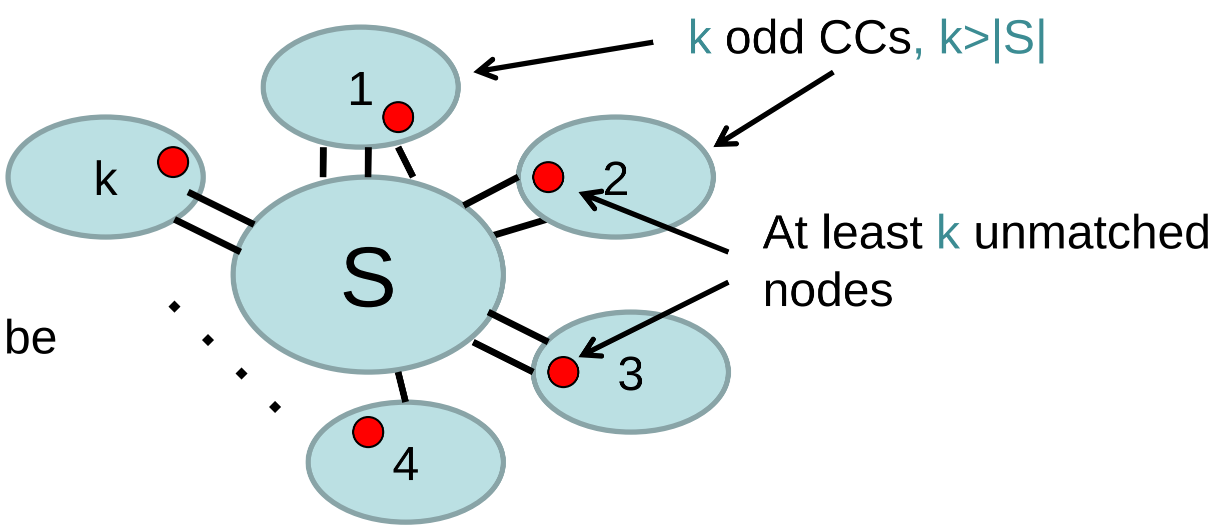
- $|V|$ is odd: certainly, no perfect matching possible
- Assume there is an $S \subseteq V$ so that the subgraph induced by $V \setminus S$ contains more than $|S|$ connected components of odd size

Foundations

Theorem 5.4: A graph G=(V,E) has a perfect matching if and only if |V| is even and there is no S⊆V so that the subgraph induced by V\S contains more than |S| connected components (CC) of odd size.

Proof:

„⇒“:



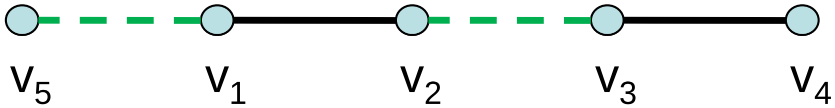
Not all ● can be matched by S.

Foundations

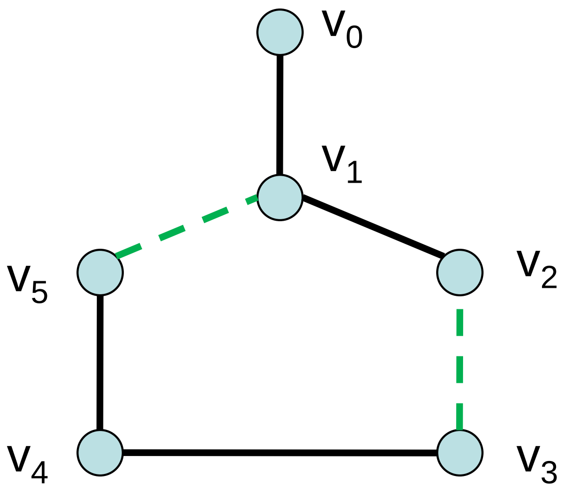
Definition 5.5: A simple path (cycle) $v_0,v_1,\ldots,v_k$ is called alternating w.r.t. a matching M if the edges $\{v_i,v_{i+1}\}$ are alternately in M and not in M.

even length:

green edges: in M



odd length:

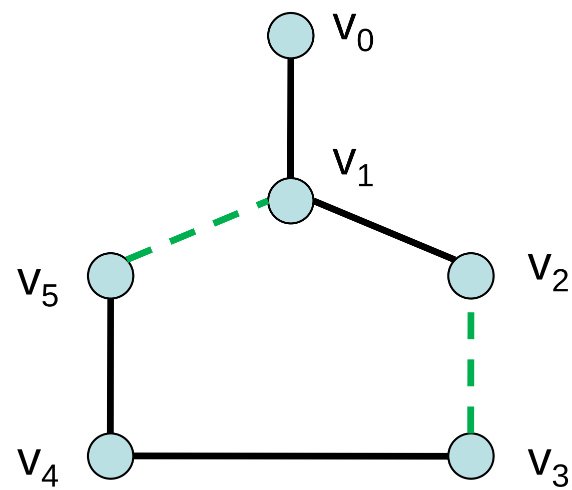


Foundations

Definition 5.6: An alternating path w.r.t. a matching M is called augmenting if it contains unmatched nodes at both ends and does not form a cycle.

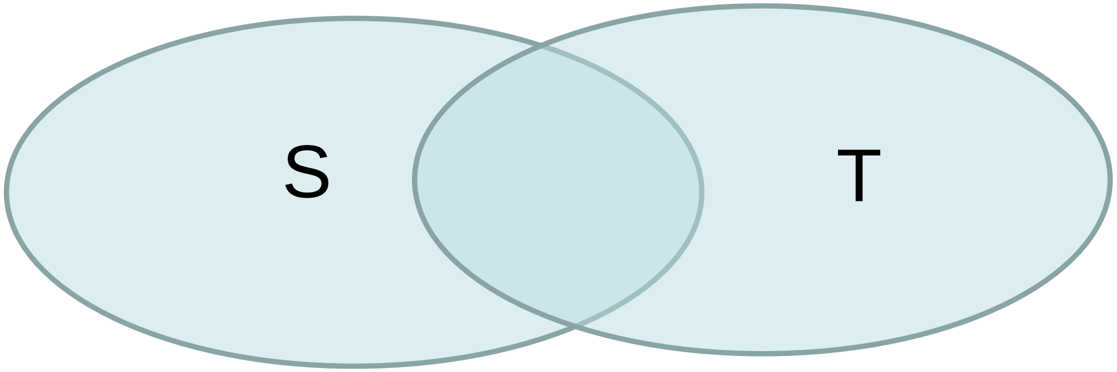
not augmenting (v_1 matched):

augmenting:



Foundations

Definition 5.7: Let S and T be two sets. Then S⊖T denotes the symmetric difference of S and T, i.e., $S \ominus T = (S \setminus T) \cup (T \setminus S)$.



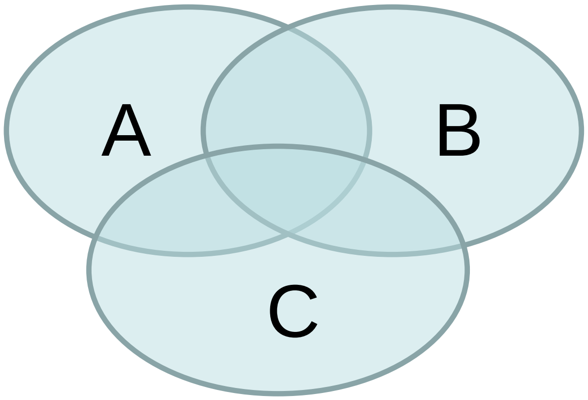
S⊖T: all elements in S and T not in S∩T

Foundations

Definition 5.7: Let S and T be two sets. Then $S \ominus T$ denotes the symmetric difference of S and T, i.e., $S \ominus T = (S \setminus T) \cup (T \setminus S)$.

Rules: for all sets A,B,C,

- $A \ominus A = \emptyset$
- $A \ominus B = B \ominus A$
- $(A \ominus B) \ominus C = A \ominus (B \ominus C)$



Foundations

Definition 5.7: Let S and T be two sets. Then S⊖T denotes the symmetric difference of S and T, i.e., S⊖T = (S\T)∪(T\S).

Lemma 5.8: Let M be a matching and P be an augmenting path w.r.t. M. Then also M⊖P is a matching, and it holds that |M⊖P| = |M|+1.

Proof:

change w.r.t. augmenting path P:

Foundations

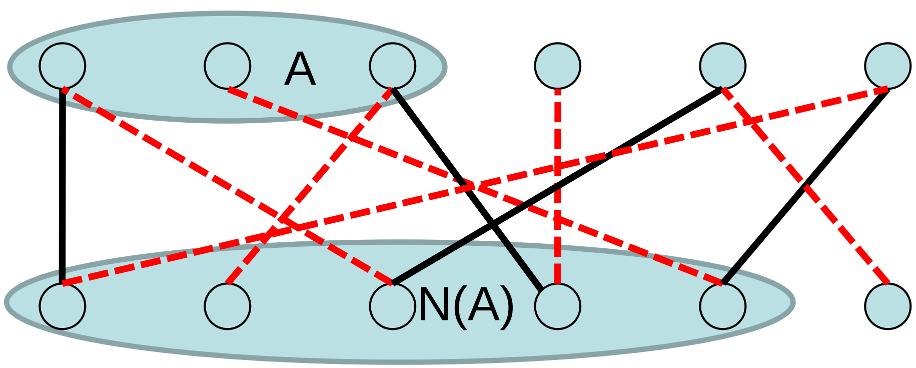
Theorem 5.9: (Hall´s Theorem)

Let $G=(U,V,E)$ be a bipartite graph. G contains a matching of cardinality $|U|$ if and only if:

$$\forall A \subseteq U: |N(A)| \geq |A|$$

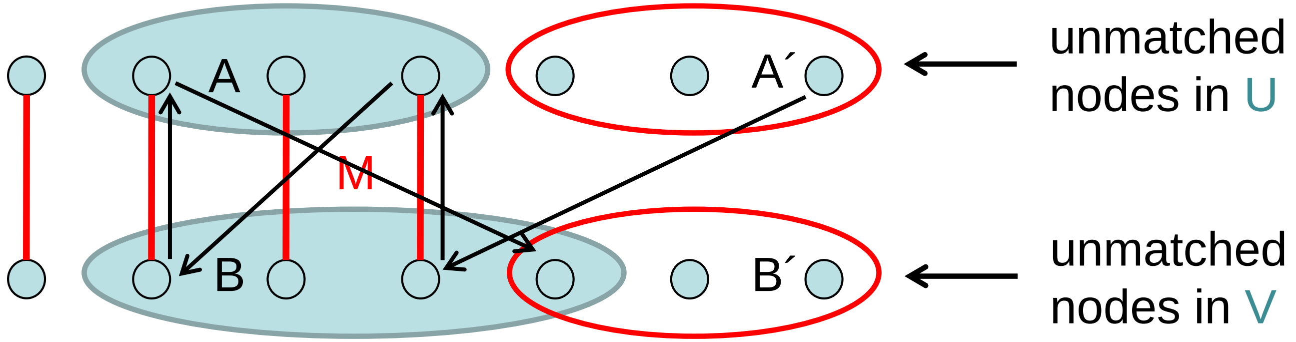
Proof:

„$\Rightarrow$“: clear due to matching edges



Foundations

Proof: „⇐" = If maximum matching has cardinality <|U| then ∃A⊆U: |N(A)|<|A|.
Let M be a maximum matching in G with |M|<|U|.

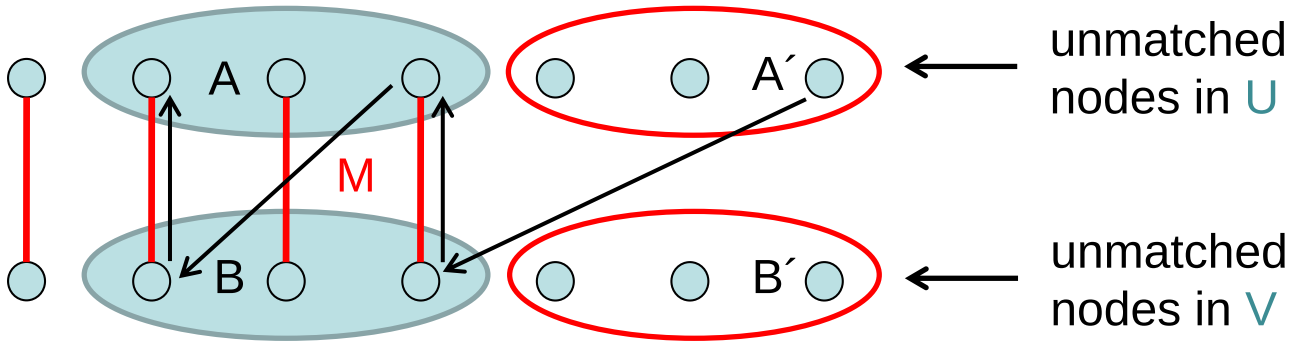


Define A⊆U: nodes reachable via alternating paths starting in A´
Define B⊆V: nodes reachable via alternating paths starting in A´
Observations:

- A∩A´=∅ because a node in U can only be reached by an alternating path from A´ if it has an edge in M
- B∩B´=∅ because if B∩B´≠∅ then there is an augmenting path (see picture), so M is not maximum, leading to a contradiction!

Foundations

Proof: (Max matching has size $<|U|$ then $\exists A \subseteq U: |N(A)|<|A|$.)
„$\Leftarrow$“: Let M be a maximum matching in G with $|M|<|U|$.



$A \cap A´=\emptyset$ and $B \cap B´=\emptyset$:

- $|A|=|B|$ since $A=\{ u \in U \mid \exists v \in B$ with $\{u,v\} \in M \}$
- $N(A´) \subseteq B$ and $N(A) \subseteq B$ because otherwise B would be extendible
- Hence, $|N(A \cup A´)| \leq |B|=|A|<|A \cup A´|$ since $|A´|>0$. Done!

Foundations

Alternative proof for „$\Leftarrow$“ via construction of an augmenting path:

- Suppose that $\forall A \subseteq U: |N(A)| \geq |A|$.
- Let M be a matching in G with $|M|<|U|$, and let $u_0 \in U$ be an unmatched node.
- Since $|N(\{u_0\})| \geq 1$, u_0 has a neighbor $v_1 \in V$. If v_1 is unmatched, we are done because we have already found an augmenting path.
- Otherwise let $u_1 \in U$ be the node matched with v_1. Since $u_1 \notin \{u_0\}$ and $|N(\{u_0,u_1\})| \geq 2$, there is a node $v_2 \notin \{v_1\}$ that is adjacent to u_0 or u_1. If v_2 is unmatched, we are done because we have already found an augmenting path.
- Otherwise, let $u_2 \in U$ be the node matched with v_2. Since $u_2 \notin \{u_0,u_1\}$ and $|N(\{u_0,u_1,u_2\})| \geq 3$, there is a node $v_3 \notin \{v_1, v_2\}$ that is adjacent to a node in $\{u_0,u_1,u_2\}$. If v_3 is unmatched, then we are done, otherwise we continue as above.
- Since $|M|<|V|$ and $|V|<\infty$, we finally have to get to an unmatched node v_k, and we can increase the matching.

Battle plan for maximum matching algorithms

1. Prove Berge‘s theorem, which says a matching M is maximum iff it has no augmenting paths. Thus, reduced to repeatedly finding augmenting paths.
2. (Easier) Show how to find augmenting paths in bipartite graphs via alternating DFS. Yields O(n(n+m)) time for max matching.
3. (Harder) Hopcroft-Karp algorithm for max matching in bipartite graphs in O($\sqrt{n}(n+m)$) time.
4. (Harder) Edmond‘s algorithm for finding augmenting paths in general graphs. Runtime O(n(n+m)) for max matching.

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Foundations

Theorem 5.11: (Berge's theorem)
A matching in an arbitrary graph is a maximum matching if and only if there is no augmenting path for that matching.

Proof:

„$\Rightarrow$“ direction:

- Suppose that there is an augmenting path P for some matching M.
- Then it follows from Lemma 5.8 that $|M \ominus P| = |M|+1$, which implies that M cannot be a maximum matching.

Foundations

Theorem 5.11: (Berge's theorem)
A matching in an arbitrary graph is a maximum matching if and only if there is no augmenting path for that matching.

Proof:

„$\Leftarrow$" direction: Follows from lemma below.

Lemma 5.12: Suppose M is a non-maximum matching, and let N be a matching in G with $|N|>|M|$. Then $N \ominus M$ contains at least $|N|-|M|$ node-disjoint augmenting paths w.r.t. M.

Foundations

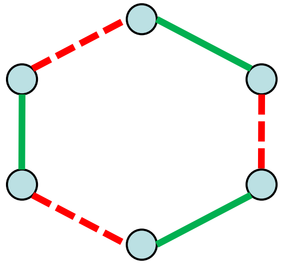
Lemma 5.12: Let M and N be matchings in G, and let $|N|>|M|$. Then $N \ominus M$ contains at least $|N|-|M|$ node-disjoint augmenting paths w.r.t. M.

Proof:

The degree of a node in $(V, N \ominus M)$ is at most 2 (why?). Thus, connected components of $(V,N \ominus M)$ are either

- isolated nodes (where green $\subseteq E(N)$, red $\subseteq E(M)$),

- simple cycles (of even length), or



- alternating paths (not necessarily augmenting!)

Foundations

Proof of Lemma 5.12:

- Let $C_1,\dots,C_k$ be the connected components in $(V, N \ominus M)$.
- Then since $C_i \ominus C_j = C_i \cup C_j$ for node-disjoint C_i and C_j,

$$M \ominus \underbrace{C_1 \ominus \dots \ominus C_k}_{N \ominus M} = N$$

- Note that the C_i´s are node-disjoint, so they can be applied independently to M via Lemma 5.8.
- It is easy to check that if C_i is a simple cycle or an alternating path that is not augmenting, then $|M \ominus C_i| \leq |M|$.
- Hence, only those C_i´s that are augmenting paths w.r.t. M can increase the matching, and this by exactly 1.
- Therefore, there must be at least $|N|-|M|$ C_i´s (why?) that are augmenting (and node-disjoint) paths w.r.t. M.

Foundations

Berge´s theorem implies the following algorithm for computing a maximum matching:

$M := \emptyset$
while $\exists$ augmenting P w.r.t. M do
    $M := M \ominus P$
output M

Runtime:

- The while-loop is executed at most n times.
- The search for an augmenting path can be done in $O(n+m)$ time in general graphs, as we will see later (Edmond‘s algorithm).

Therefore, a runtime of $O(n\cdot(n+m))$ is possible.

Matching in Bipartite Graphs

Berge´s theorem implies the following algorithm for computing a maximum matching:

$M := \emptyset$
while $\exists$augmenting P w.r.t. M do
    $M := M \ominus P$
output M

Easier first step:

- In a bipartite graph $G=(U,V,E)$ it suffices to search for augmenting paths starting from unmatched nodes in U because every augmenting path must have one unmatched node in U and one in V.
- In bipartite graphs we can use an alternating DFS approach to find augmenting paths (since there are no cycles in such graphs).

Battle plan for maximum matching algorithms

1. Prove Berge‘s theorem, which says a matching M is maximum iff it has no augmenting paths. Thus, reduced to repeatedly finding augmenting paths.
2. (Easier) Show how to find augmenting paths in bipartite graphs via alternating DFS. Yields O(n(n+m)) time for max matching.
3. (Harder) Hopcroft-Karp algorithm for max matching in bipartite graphs in O($\sqrt{n}(n+m)$) time.
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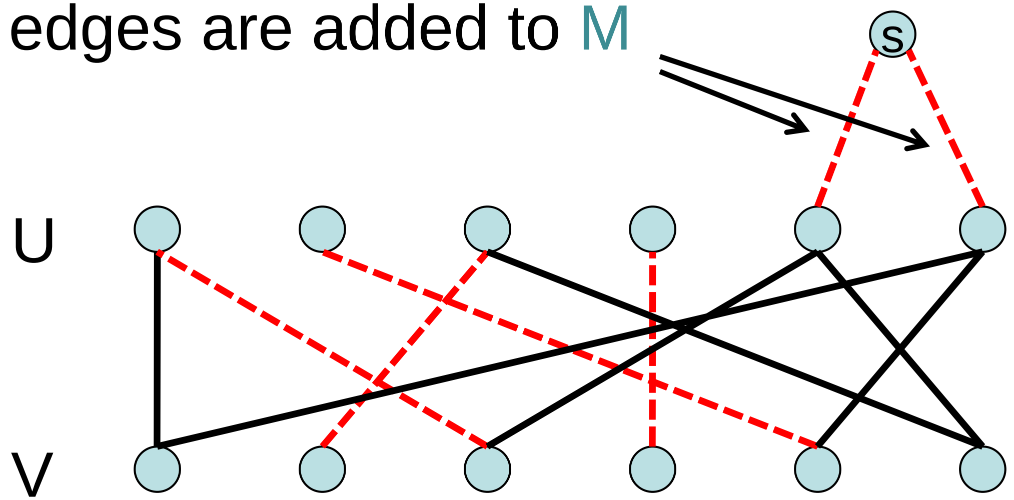
Matching in Bipartite Graphs

Simplification for alternating DFS in bipartite graphs:

artificial source s with edges to all unmatched nodes in U

Artificial edges are added to M



Note: Adding s has effect of running DFS from all unmatched nodes in U in parallel – hence O(m+n) time per augmenting path found

Matching in Bipartite Graphs

- E(u): edge set of node u

```
Procedure AlternatingBipartiteDFS(s: Node, M: Matching)
  d = <∞,…, ∞>: Array [1..n] of IN
  parent = <⊥,…,⊥>: Array [1..n] of Node
  d[key(s)]:=0              // s has distance 0 to itself
  parent[key(s)]:=s         // s is its own parent
  q:=<s>: Stack of Node
  while q ≠<> do            // as long as q is not empty
    u:= q.pop()             // process nodes according to LIFO rule
    if (d[key(u)] is even) then A:=M else A:=E\M
    if A∩E(u)=∅ and (d[key(u)] is even) then   // u unmatched?
      return augmenting path (via parent[])
    else
      foreach {u,v}∈A∩E(u) do
        if parent(key(v))=⊥ then  // v not visited so far?
          q.push(v)               // add v to q
          d[key(v)]:=d[key(u)]+1
          parent[key(v)]:=u
```

Matching in Bipartite Graphs

Correctness of AlternatingBipartiteDFS:

- Suppose that there is an augmenting path $p=(s,u_1,v_1,u_2,v_2,\ldots,v_k)$ w.r.t. M but AlternatingBipartiteDFS does not find any.
- Let w be the last node in p that was explored by the algorithm. Certainly, $w \neq v_k$ because otherwise the algorithm would have found an augmenting path.
- Suppose that $w=v_i$ for some $i<k$. Then the algorithm would have also explored u_{i+1} via the matching edge, leading to a contradiction.
- So suppose that $w=u_i$ for some $i<k$. Then the algorithm would have also explored v_i via a non-matching edge, also leading to a contradiction.

Battle plan for maximum matching algorithms

1. Prove Berge‘s theorem, which says a matching M is maximum iff it has no augmenting paths. Thus, reduced to repeatedly finding augmenting paths.
2. (Easier) Show how to find augmenting paths in bipartite graphs via alternating DFS. Yields O(n(n+m)) time for max matching.
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4. (Harder) Edmond‘s algorithm for finding augmenting paths in general graphs. Runtime O(n(n+m)) for max matching.

Foundations

Next, we will study the following *refined* approach:

$M := \emptyset$

while $\exists$ augmenting path P w.r.t. M do

- determine a shortest augmenting path P w.r.t. M
- $M := M \ominus P$

output M

In the following let

- P_i: augmenting path found in round i
- M_i: matching at the end of round i

Shortest augmenting Paths

Lemma 5.13: Let M be a matching of cardinality r and let s be the maximum cardinality of a matching in $G=(V,E)$, $s>r$. Then there is an augmenting path w.r.t. M of length $\leq 2\lfloor r/(s-r)\rfloor+1$.

Proof:

- Let N be a maximum matching in G, i.e., $|N|=s$.
- By Lemma 5.12, $N \ominus M$ contains $\geq s-r$ augmenting paths w.r.t. M, which are node-disjoint and therefore also edge-disjoint.
- At least one of these paths contains $\leq\lfloor r/(s-r)\rfloor$ edges from M.

Shortest augmenting Paths

Lemma 5.14: Let s be the maximum cardinality of a matching in $G=(V,E)$. Then the sequence $|P_1|, |P_2|, \ldots$ of shortest augmenting paths computed by the refined algorithm contains at most $2\sqrt{s}+1$ different values.

Proof:

- Let $r := \lfloor s-\sqrt{s} \rfloor$. By construction, $|M_i|=i$ (why?), and therefore $|M_r|=r$. From Lemma 5.13 it follows that

$$|P_r| \le 2\left\lfloor \frac{\lfloor s-\sqrt{s}\rfloor}{s - \lfloor s-\sqrt{s}\rfloor} \right\rfloor +1 \le 2\lfloor s/\sqrt{s}\rfloor+1 \le 2\lfloor\sqrt{s}\rfloor+1$$

- Thus, for $i \le r$, $|P_i|$ is one of the odd (why?) numbers in $[1, 2\sqrt{s}+1]$, and therefore one of $\lfloor\sqrt{s}\rfloor+1$ odd numbers.
- $P_{r+1},\ldots,P_s$ contribute at most $s-r<\sqrt{s}+1$ additional lengths.

Shortest augmenting Paths

Lemma 5.14: Let s be the maximum cardinality of a matching in $G=(V,E)$. Then the sequence $|P_1|, |P_2|, \ldots$ of shortest augmenting paths computed by the refined algorithm contains at most $2\sqrt{s}+1$ different values.

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- Thus, for $i \le r$, $|P_i|$ is one of the odd (why?) numbers in $[1, 2\sqrt{s}+1]$, and therefore one of $\lfloor\sqrt{s}\rfloor+1$ odd numbers.

Note – above we used the fact that $|P_1| \le |P_2| \le \ldots$!

Shortest augmenting Paths

Lemma 5.15: Let P be a shortest augmenting path w.r.t. M and P' be an augmenting path w.r.t $M \ominus P$. Then it holds that:

$$|P'| \geq |P| + 2|P \cap P'|$$

Proof:

- Let $N = M \ominus P \ominus P'$, so $|N| = |M| + 2$.
- By Lemma 5.12, $M \ominus N$ contains at least 2 node-disjoint augmenting paths w.r.t. M, called P_1 and P_2.
- It holds: $|M \ominus N| = |P \ominus P'| = |(P \setminus P') \cup (P' \setminus P)|$
  $= |P| + |P'| - 2|P \cap P'|$
  and $|M \ominus N| \geq |P_1| + |P_2| \geq 2|P|$ (by def. of P)
- Therefore, $|P| + |P'| - 2|P \cap P'| \geq 2|P|$
  $\Rightarrow \quad |P'| \geq 2|P| - |P| + 2|P \cap P'|$

Shortest augmenting Paths

Recall our refined matching algorithm:

$M := \emptyset$

while $\exists$augmenting path w.r.t. M do
- determine a shortest augmenting path P w.r.t. M
- $M := M \ominus P$

output M

- Let $P_1, P_2, \ldots$ be the sequence of shortest augmenting paths constructed by the algorithm.
- Lemma 5.15: $|P_{i+1}| \geq |P_i|$ for all i.

Shortest augmenting Paths

Lemma 5.16: For every sequence P_1, P_2,... of shortest augmenting paths it holds for all P_i and P_j with $|P_i|=|P_j|$ that P_i and P_j are node-disjoint.

Proof:

- Suppose that there is a sequence $(P_k)_{k\geq 1}$ with $|P_i|=|P_j|$ for some $j>i$ so that P_i and P_j are not node-disjoint, where $j-i$ is minimal.
- Then the paths $P_i,\ldots,P_{j-1}$ resp. $P_{i+1},\ldots,P_j$ are node-disjoint (why?).
- By the latter, P_j is an augmenting path w.r.t. the matching M after the augmentations by $P_1,\ldots,P_i$.
- From Lemma 5.15 it follows that $|P_j|\geq|P_i|+2|P_i\cap P_j|$, and since $|P_i|=|P_j|$, P_i and P_j must be edge-disjoint.
- The matching edges created by P_i are still in $M\ominus P_{i+1}\ominus P_{i+2}\ominus\ldots\ominus P_{j-1}$ because $P_i,\ldots,P_{j-1}$ are node-disjoint.
- Since P_j has a node in common with P_i, P_j has to have an edge (namely, a matching edge) in common with P_i as well.
- However, this cannot be, so P_i and P_j must be node-disjoint.

Shortest augmenting Paths

Hopcroft-Karp Algorithm:

$M := \emptyset$

while $\exists$ augmenting path w.r.t. M do
- $l :=$ length of shortest augmenting path w.r.t. M
- determine w.r.t. „$\subseteq$" a maximal set of node-disjoint augmenting paths $Q_1,\dots,Q_k$ w.r.t. M that have length l
- $M := M \ominus Q_1 \ominus \dots \ominus Q_k$

Corollary 5.17: The while-loop above is executed at most $O(\sqrt{n})$ times.

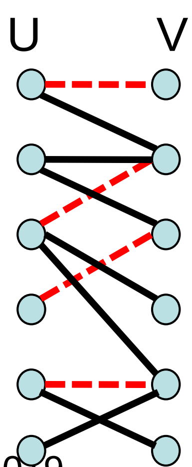
Proof: follows from Lemmas 5.14-5.16. (Why?)

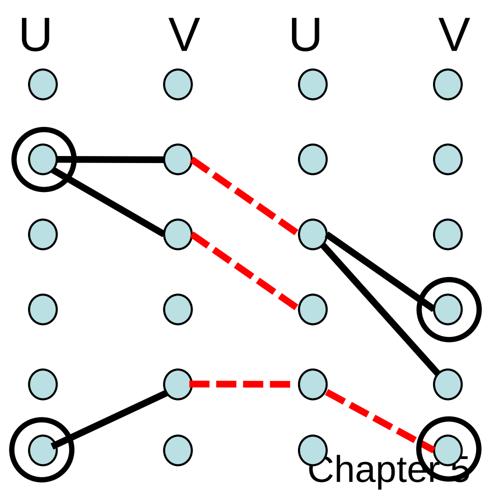
Shortest augmenting Paths

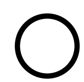
Question: How can we quickly find a set of shortest augmenting paths w.r.t. matching M?

Graph G bipartite, i.e., G=(U,V,E):

- Determining the shortest length l: „parallel alternating BFS“, starting with all unmatched nodes in U, until an unmatched node is found in V





: unmatched node

here: l=3

Note: Each edge visited at most once!

Shortest augmenting Paths

- s: artificial node (see Slide 21) – recall this lets us simulate a parallel search from all unmatched notes in U
- E(u): edge set of node u

```
Procedure AlternatingBipartiteBFS(s: Node, M: Matching)
  d = <∞,…, ∞>: Array [1..n] of IN
  parent = <⊥,…,⊥>: Array [1..n] of Node
  d[key(s)]:=0            // s has distance 0 to itself
  parent[key(s)]:=s       // s is its own parent
  q:=<s>: Queue of Node
  while q ≠<> do          // as long as node is not empty
    u:= q.dequeue()       // process nodes according to FIFO rule
    if (d[key(u)] is even) then A:=M else A:=E\M
    if A∩E(u)=∅ and (d[key(u)] is even) then
      augmenting path (via parent[]), stop
    else
      foreach {u,v}∈A∩E(u) do
        if parent(key(v))=⊥ then  // v not visited so far?
          q.enqueue(v)  // add v to the queue q
          d[key(v)]:=d[key(u)]+1
          parent[key(v)]:=u
```

Shortest augmenting Paths

Graph G bipartite, i.e., G=(U,V,E):

- Step 1: Determine the shortest length l.
  - Run alternating BFS, started with all unmatched nodes in U, until an unmatched node is found in V or all nodes have been found.
  - Store the BFS-depth of each node.
- Step 2: Determine maximal set of shortest augmenting paths.
  - Initially, nodes are unmarked. Perform, in sequence, from each unmatched node in U an alternating DFS along unmarked nodes of increasing BFS-depth (i.e. BFS-depth increases by 1 with each step along path) up to depth l until we have found an augmenting path Q_i or all edges have been explored.
  - For every found path Q_i, all nodes in Q_i are marked and we continue to execute DFS from another unmatched node in U.
  - Every node at which DFS backtracks (i.e., no augmenting path was found) will be marked.
  - Note: As before, we add an artificial start vertex s to run the search above, hence the search is really just one big (modified) DFS run

Since every node and edge is only processed once in the BFS and DFS, the runtime is $O(n+m)$.

Shortest augmenting Paths

Correctness of the algorithm for determining a maximal set of shortest augmenting paths (here called refined AlternatingBipartiteDFS):

- Suppose that there is an augmenting path $p=(u_1,v_1,u_2,v_2,\ldots,v_{2k+1})$ w.r.t. M of length $l=2k+1$ that is not discovered by the refined AlternatingBipartiteDFS algorithm.
- This can only happen if the nodes of p do not have a consecutive BFS-depth.
- Suppose w.l.o.g. that BFS-depth(v_i) $\neq$ BFS-depth(u_i)+1 for some i.
- Case 1: BFS-depth(v_i) > BFS-depth(u_i)+1. Then the alternating BFS algorithm would not have worked correctly because it should have reached v_i from u_i, so that cannot happen.
- Case 2: BFS-depth(v_i) < BFS-depth(u_i)+1. Then it is possible to construct an augmenting path of length less than l (go along the shortest alternating path from an unmatched node u to v_i instead of using p to reach v_i), also contradicting our assumption that the alternating BFS algorithm works correctly.

Shortest augmenting Paths

Corollary 5.18: In bipartite graphs, a maximum matching can be computed in $O(\sqrt{n}\,(n+m))$ time.

Is this also possible for arbitrary graphs?

Yes, but it´s much more complicated:

- Vijay V. Vazirani. A theory of alternating paths and blossoms for proving correctness of the $O(\sqrt{V}\,E)$ general graph maximum matching algorithm. Combinatorica 14(1), pp. 71-109 (1994).

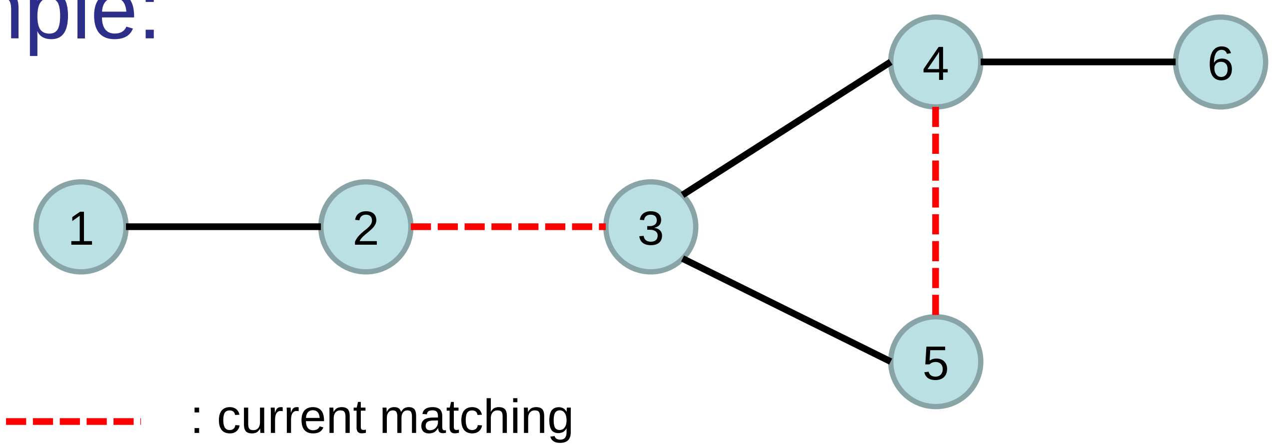
Battle plan for maximum matching algorithms

1. Prove Berge‘s theorem, which says a matching M is maximum iff it has no augmenting paths. Thus, reduced to repeatedly finding augmenting paths.
2. (Easier) Show how to find augmenting paths in bipartite graphs via alternating DFS. Yields O(n(n+m)) time for max matching.
3. (Harder) Hopcroft-Karp algorithm for max matching in bipartite graphs in O($\sqrt{n}(n+m)$) time.
4. (Harder) Edmond‘s algorithm for finding augmenting paths in general graphs. Runtime O(n(n+m)) for max matching.

Matching in arbitrary Graphs

Problem: BFS no longer works for finding augmenting paths in non-bipartite graphs!

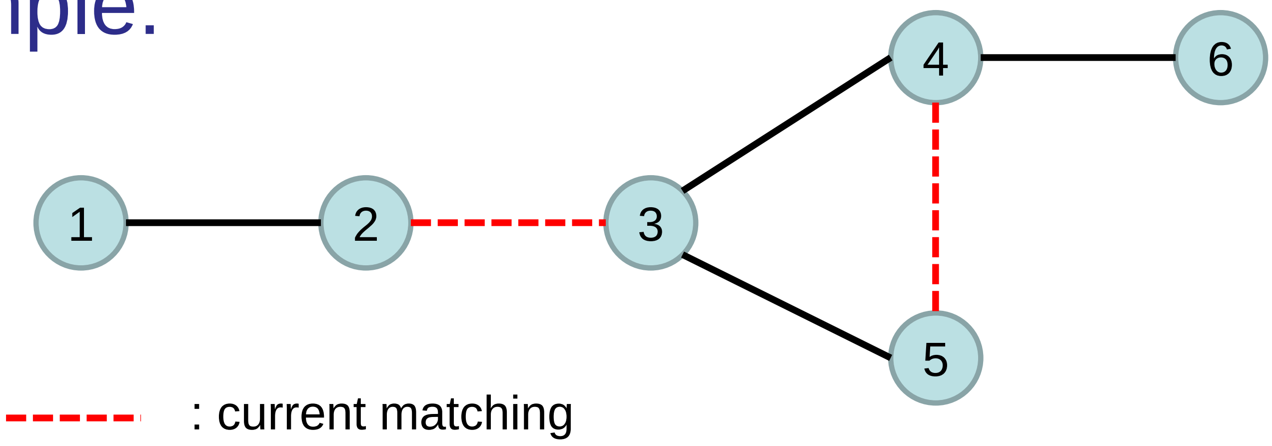
Example:



Matching in arbitrary Graphs

Alternating BFS from 1 via node 4:
misses augmenting path 1-2-3-5-4-6 since 4 has already been visited

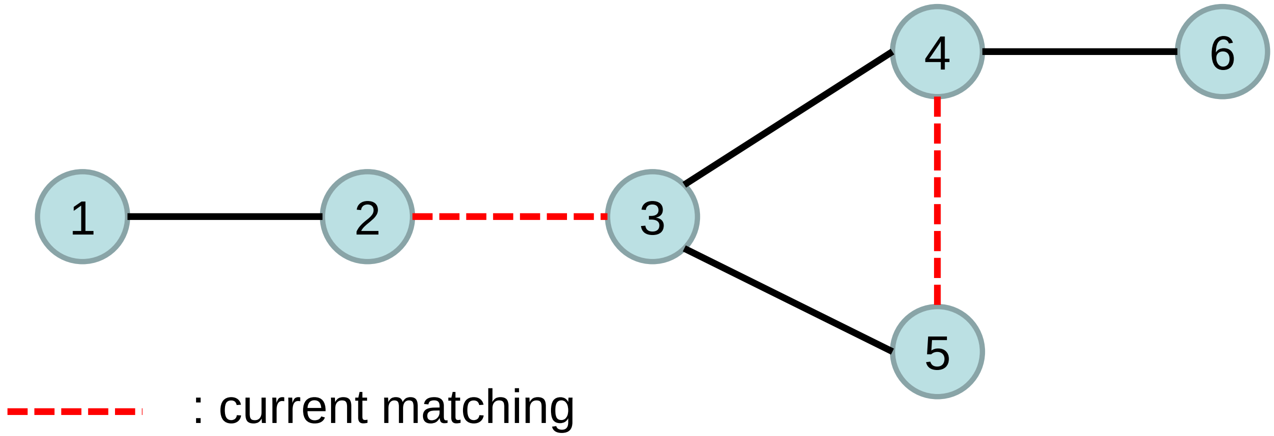
Example:



Matching in arbitrary Graphs

Obvious question: What differentiates a bipartite graph from a non-bipartite graph?

Answer: A graph is non-bipartite iff it has an odd cycle.

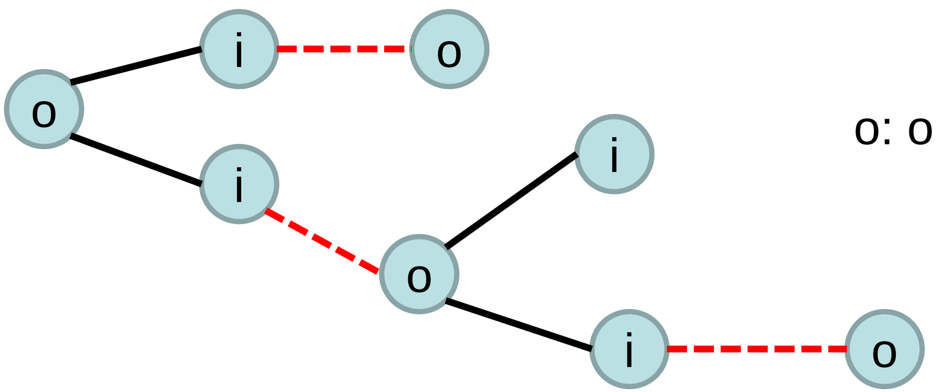


Conclusion: Need a way to deal with odd cycles

Matching in arbitrary Graphs

Basic starting point of Edmonds´ Algorithm:

- Build a tree of alternating paths via alternating BFS.
- The root and all nodes of even distance from the root are the outer nodes.
- The other nodes are the inner nodes.

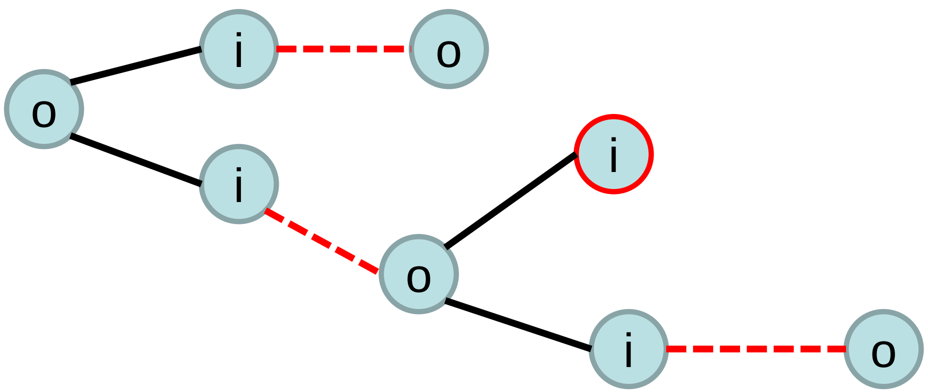


o: outer, i: inner

Matching in arbitrary Graphs

Basic starting point of Edmonds‘ Algorithm:

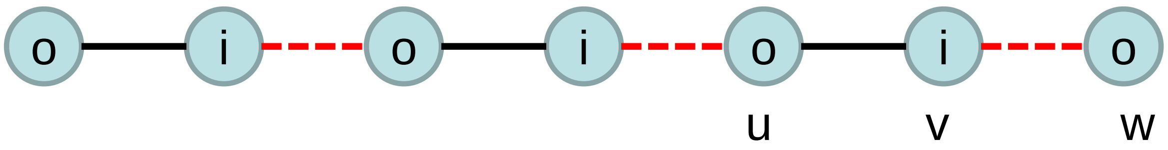
- Build a tree of alternating paths via alternating BFS.
- The root and all nodes of even distance from the root are the outer nodes.
- The other nodes are the inner nodes.
- If the search ends in an unmatched inner node, then there is an augmenting path to that node, as one can easily check.



Matching in arbitrary Graphs

Basic starting point of Edmonds´ Algorithm:

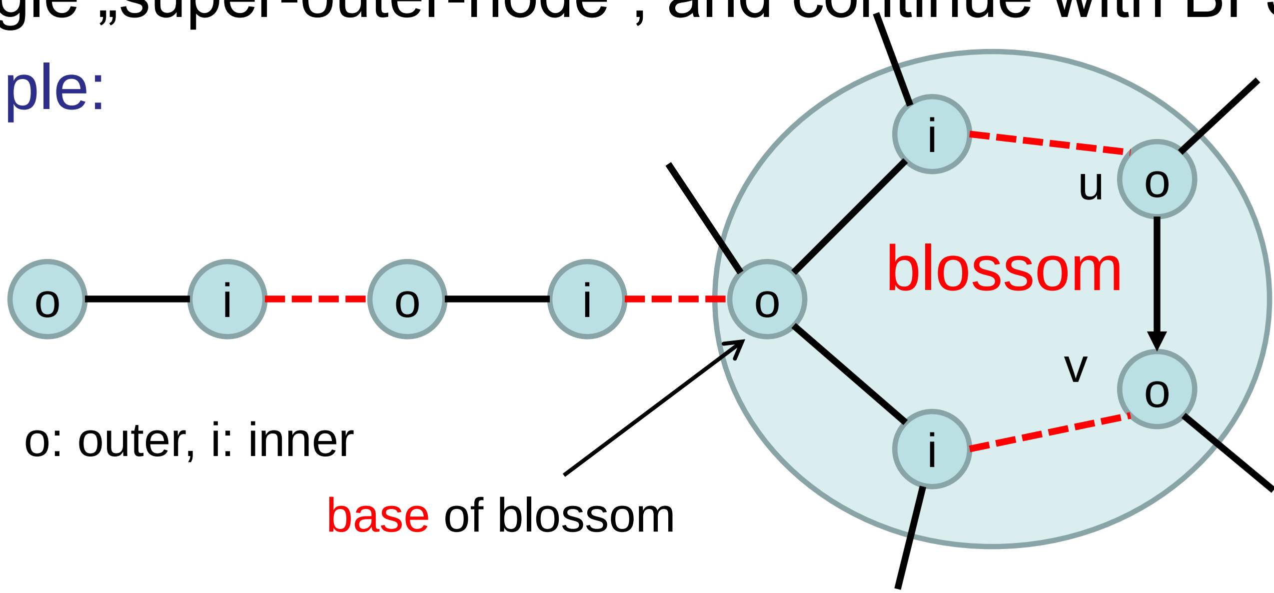
- Build a tree of alternating paths via alternating BFS.
- The root and all nodes of even distance from the root are the outer nodes.
- The other nodes are the inner nodes.
- If the search ends in an unmatched inner node, then there is an augmenting path to that node, as one can easily check.
- If the BFS is currently at an outer node u, then all unmatched edges {u,v} for some node v that is not already in the tree are added to the tree. Such a node v is then an inner node. If v is not matched, we have found an augmenting path. Otherwise, if w is not already in the tree, we also add the unique matching edge {v,w} to the tree and declare w an outer node.



Matching in arbitrary Graphs

- If for some outer (resp., inner) node u an edge {u,v} is found where v is already an outer (resp., inner) node, then we have an odd cycle, called a blossom.
- Key idea: Don‘t deal with the cycle now – just treat it as a single „super-outer-node“, and continue with BFS.

Example:



Matching in arbitrary Graphs

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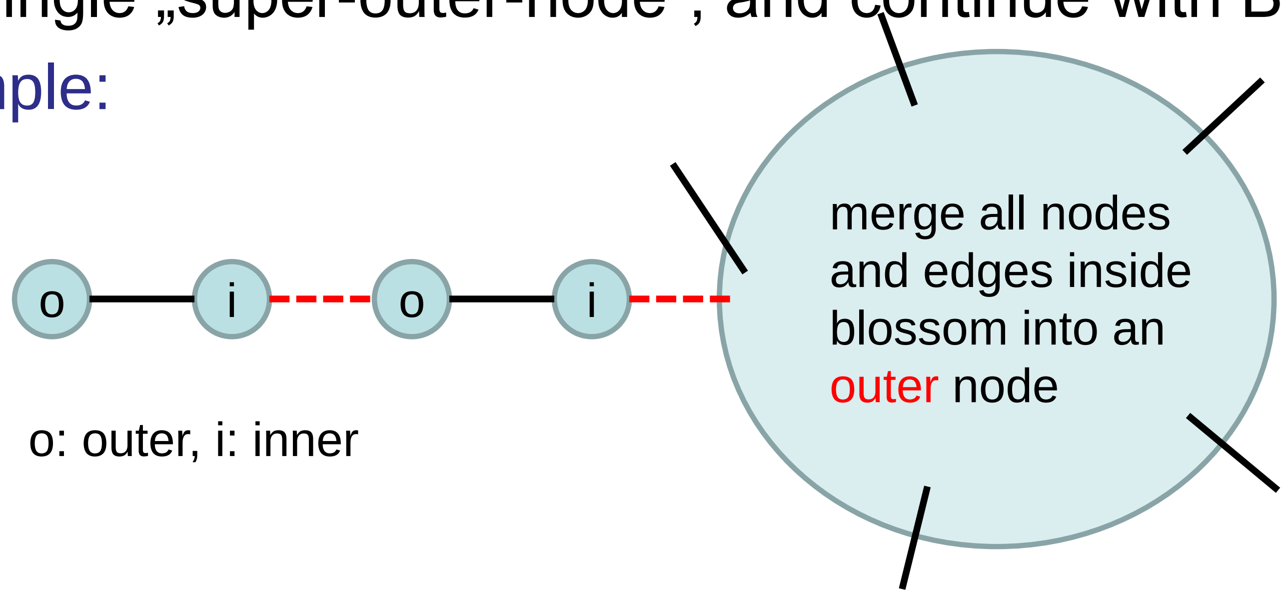

Matching in arbitrary Graphs

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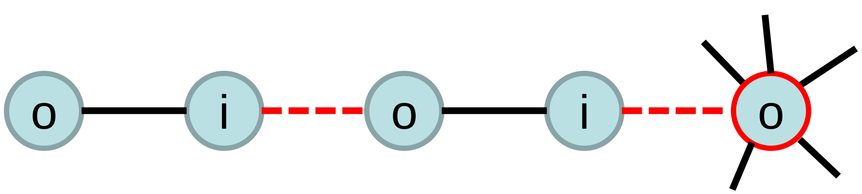


o: outer, i: inner

resulting graph: contracted graph

Matching in arbitrary Graphs

Ok, so via alternating BFS over nodes and supernodes, we find an alternating path. Now what?

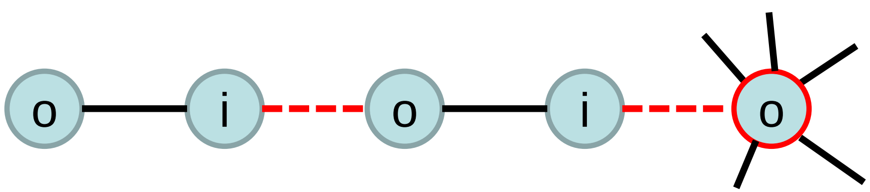
Key question: What does this mean about the existence of an alternating path in the original graph G (i.e. before supernodes were introduced)?

Matching in arbitrary Graphs

Lemma 5.19: The contracted graph G´ (i.e. with supernodes) has an augmenting path iff original graph G has an augmenting path.

Proof sketch: Let P be the augmenting path in G´.

Claim: for each contracted node, there is an internal alternating path from its base to any of its edges, starting with a non-matched and ending with a matched edge

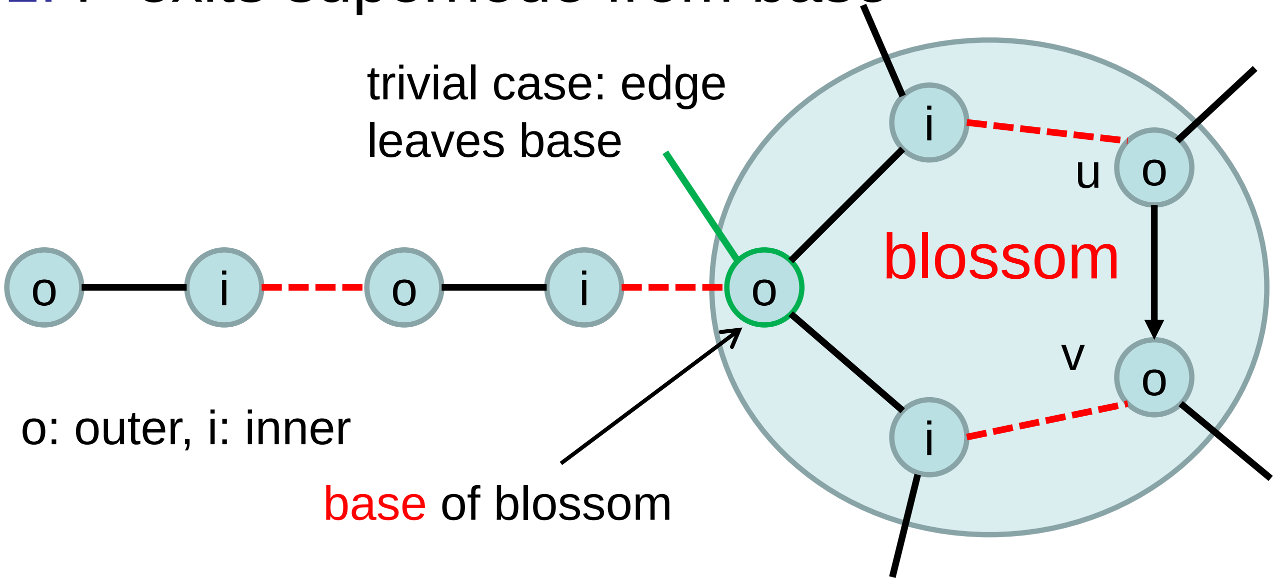


o: outer, i: inner

Matching in arbitrary Graphs

Invariant: for each contracted node, there is an internal alternating path from its base to any of its edges, starting with a non-matched and ending with a matched edge.

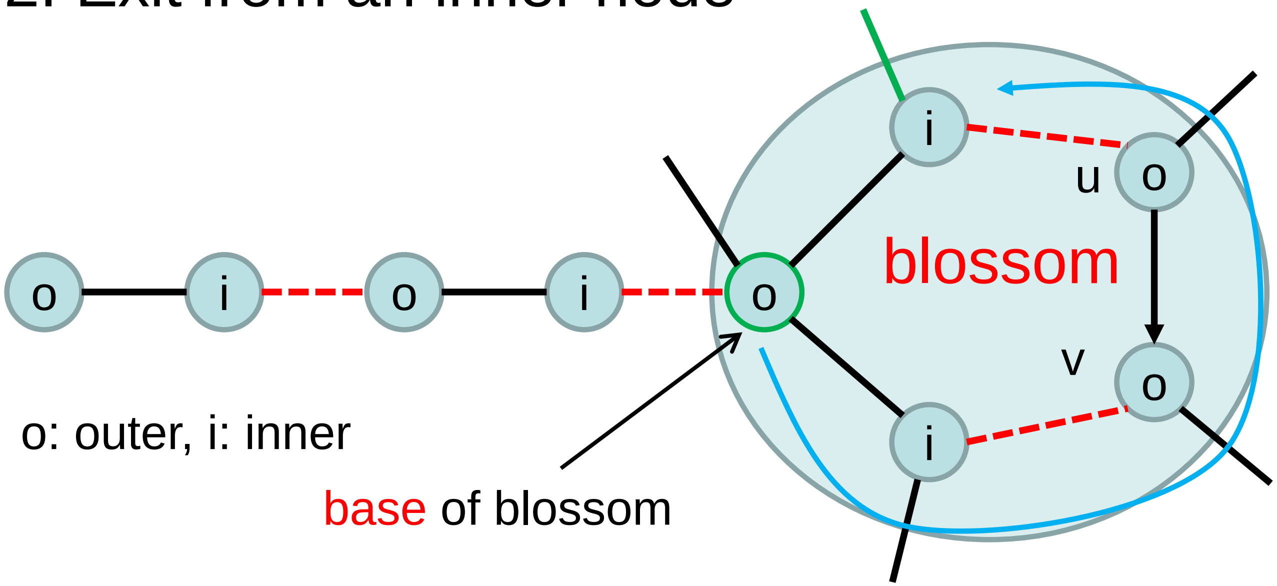
Case 1: P exits supernode from base



Matching in arbitrary Graphs

Invariant: for each contracted node, there is an internal alternating path from its base to any of its edges, starting with a non-matched and ending with a matched edge.

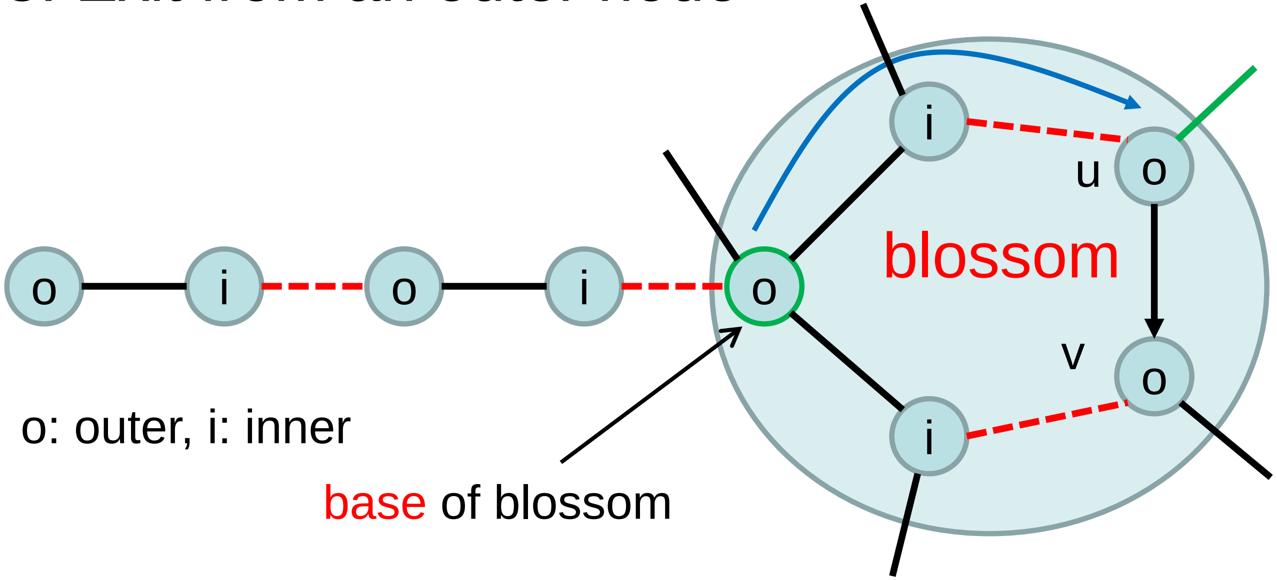
Case 2: Exit from an inner node



Matching in arbitrary Graphs

Invariant: for each contracted node, there is an internal alternating path from its base to any of its edges, starting with a non-matched and ending with a matched edge.

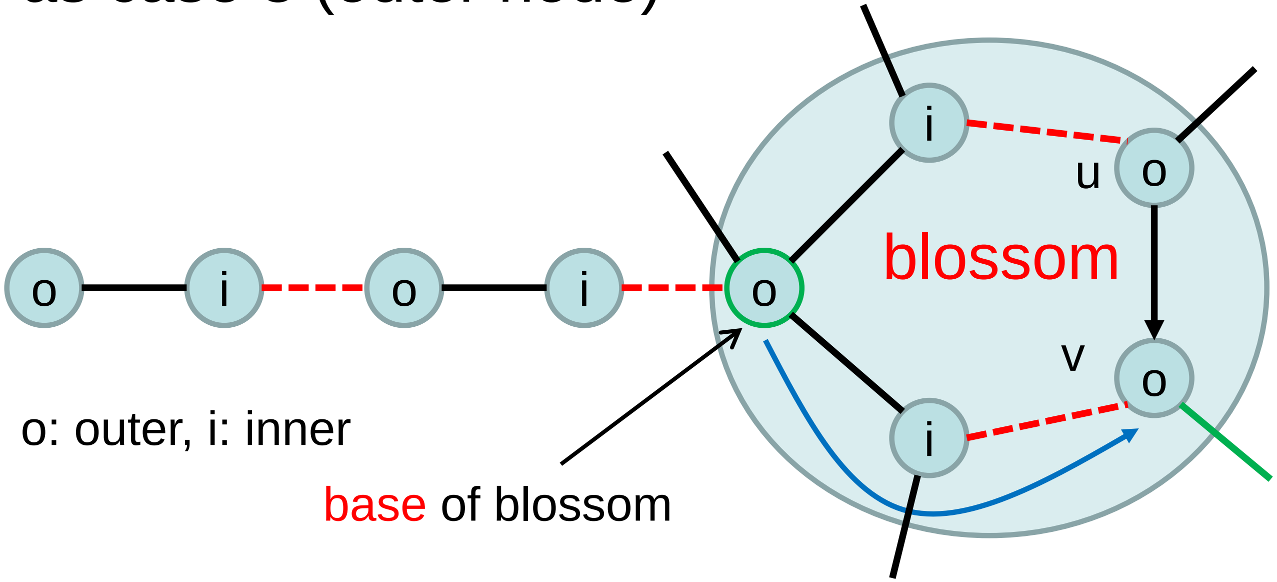
Case 3: Exit from an outer node



Matching in arbitrary Graphs

Invariant: for each contracted node, there is an internal alternating path from its base to any of its edges, starting with a non-matched and ending with a matched edge.

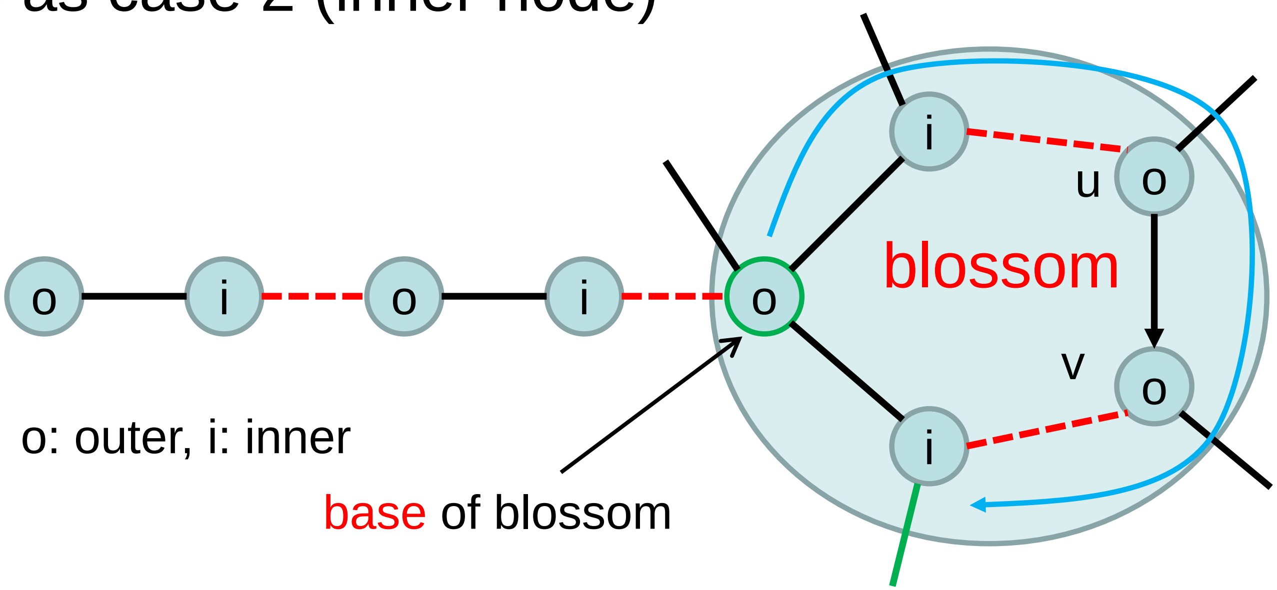
Same as case 3 (outer node)



Matching in arbitrary Graphs

Invariant: for each contracted node, there is an internal alternating path from its base to any of its edges, starting with a non-matched and ending with a matched edge.

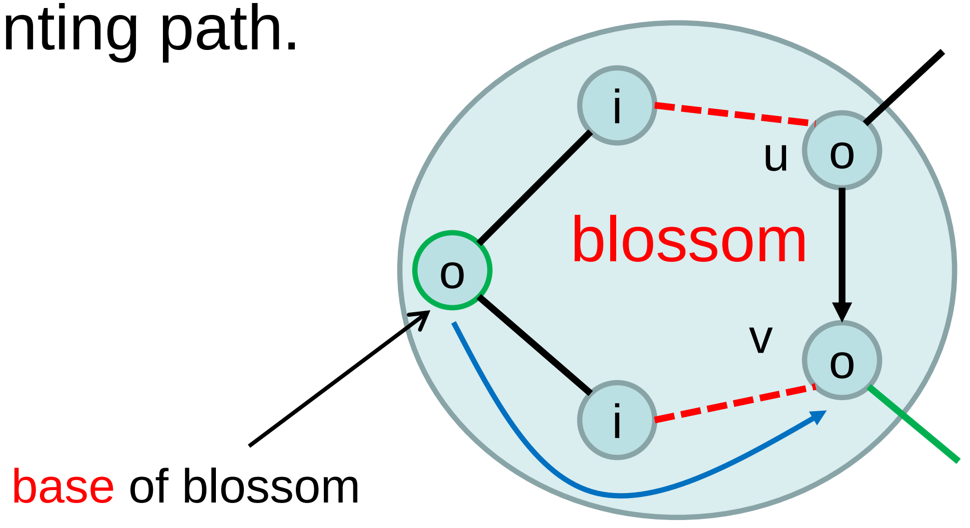
Same as case 2 (inner node)



Matching in arbitrary Graphs

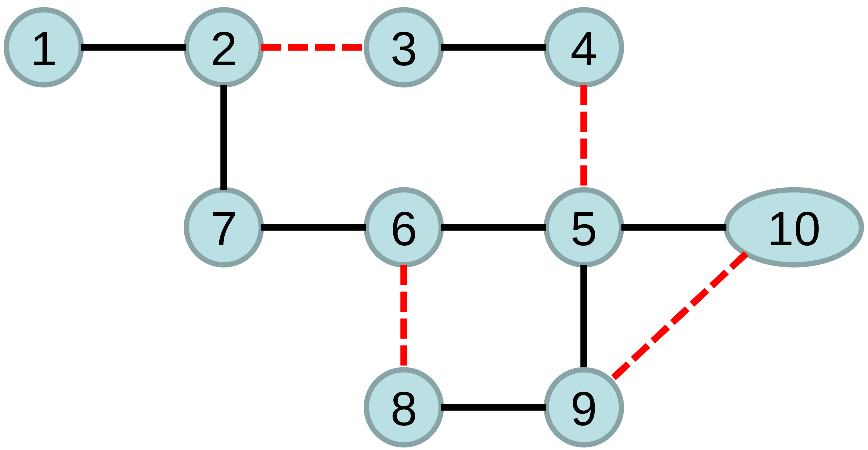
Invariant: for each contracted node, there is an internal alternating path from its base to any of its edges, starting with a non-matched and ending with a matched edge.

Note: Base of blossom can also be starting point of augmenting path.

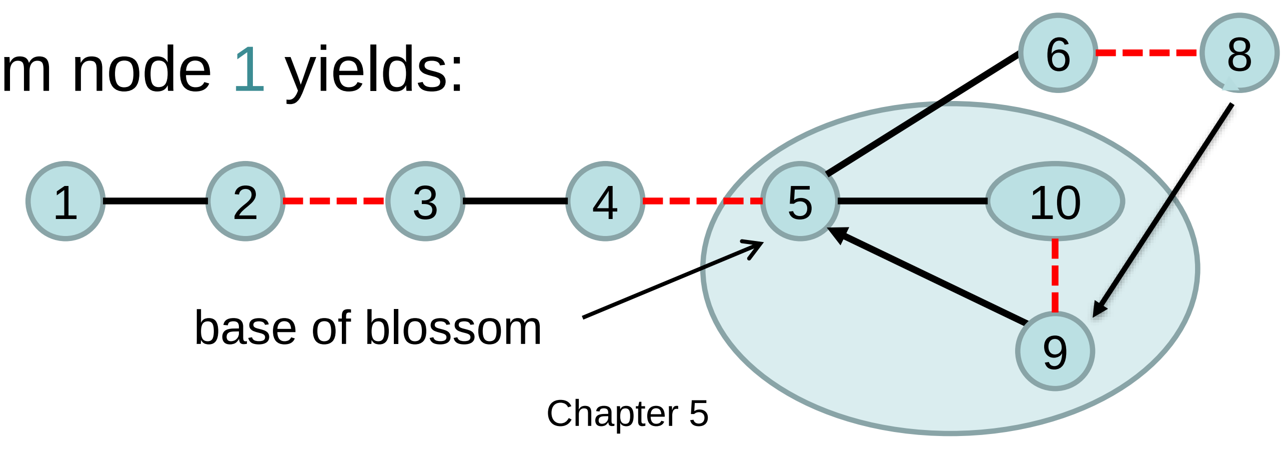


Matching in arbitrary Graphs

Example:

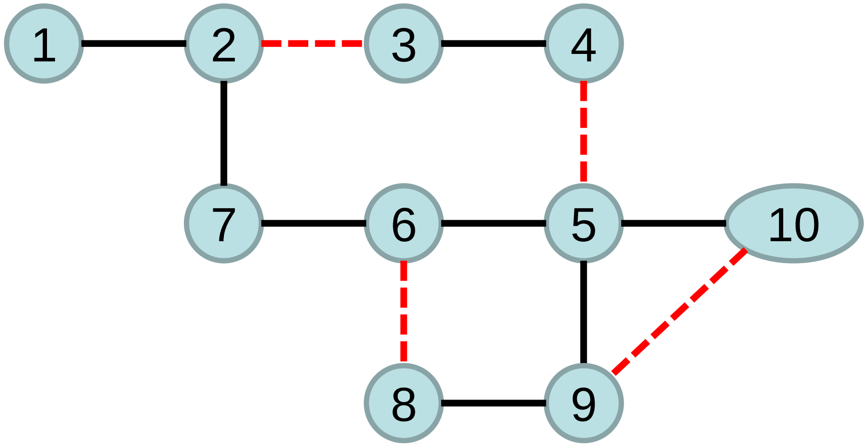


BFS from node 1 yields:

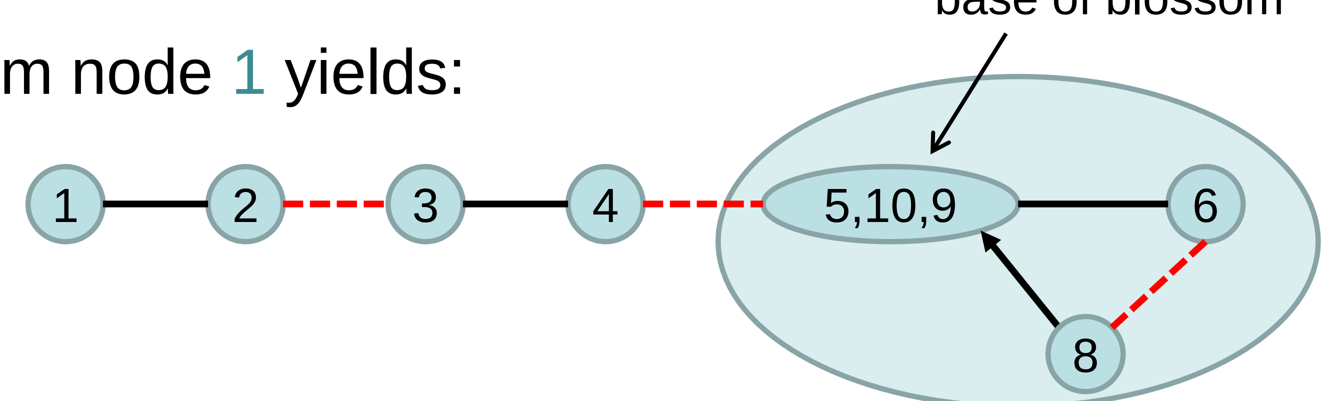


Matching in arbitrary Graphs

Example:

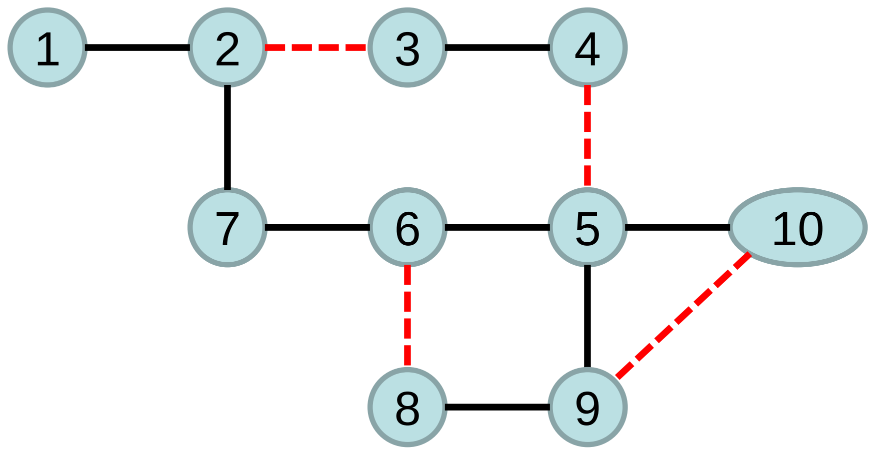


BFS from node 1 yields:

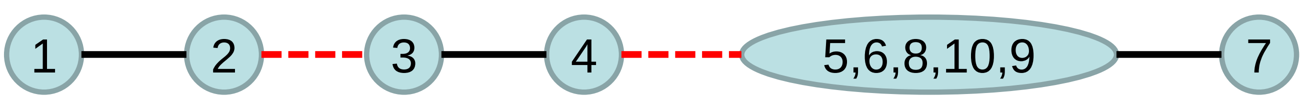


Matching in arbitrary Graphs

Example:

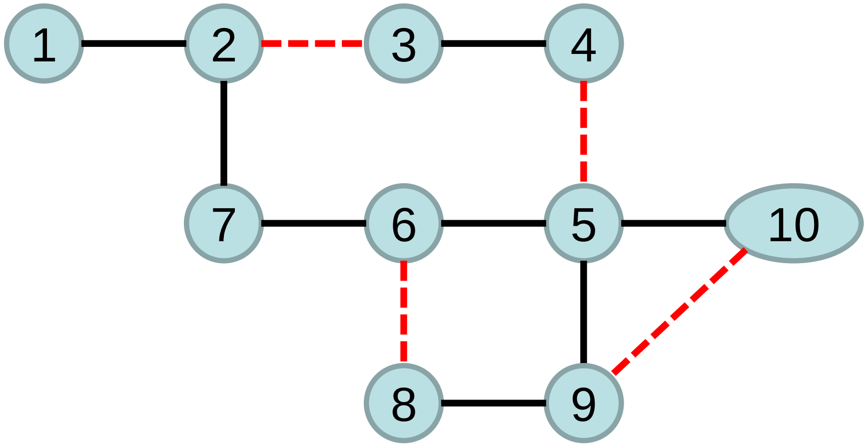


BFS from node 1 yields:

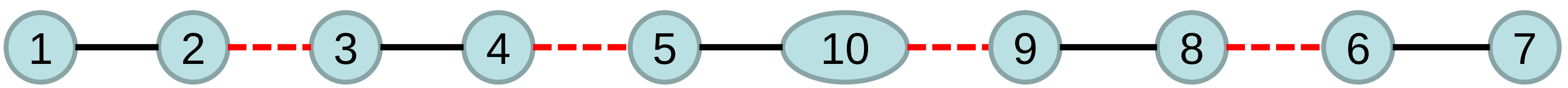


Matching in arbitrary Graphs

Example:

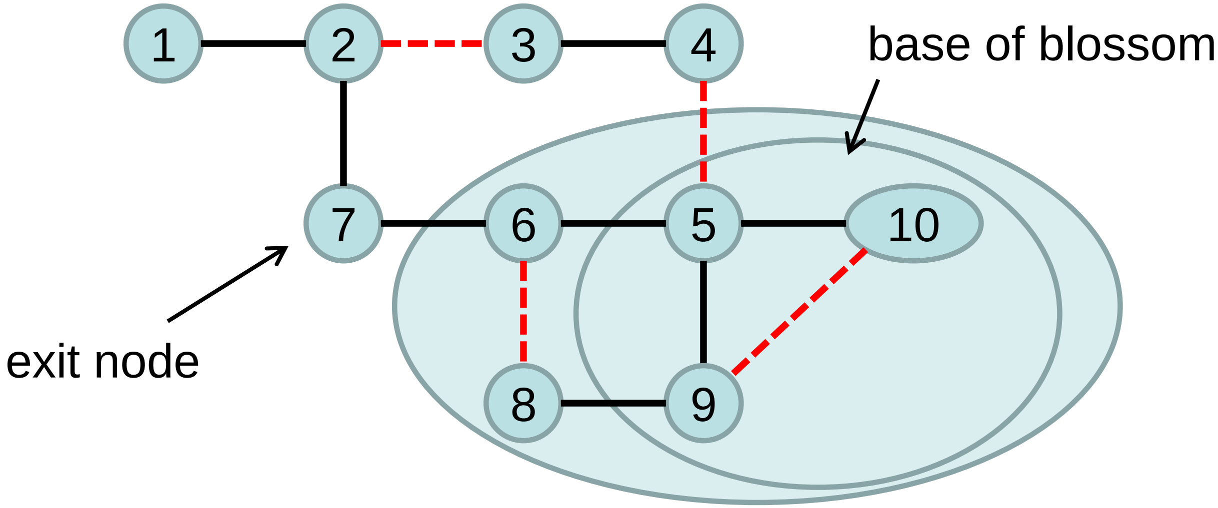


Unshrinking the nodes results in the following augm. path:



Matching in arbitrary Graphs

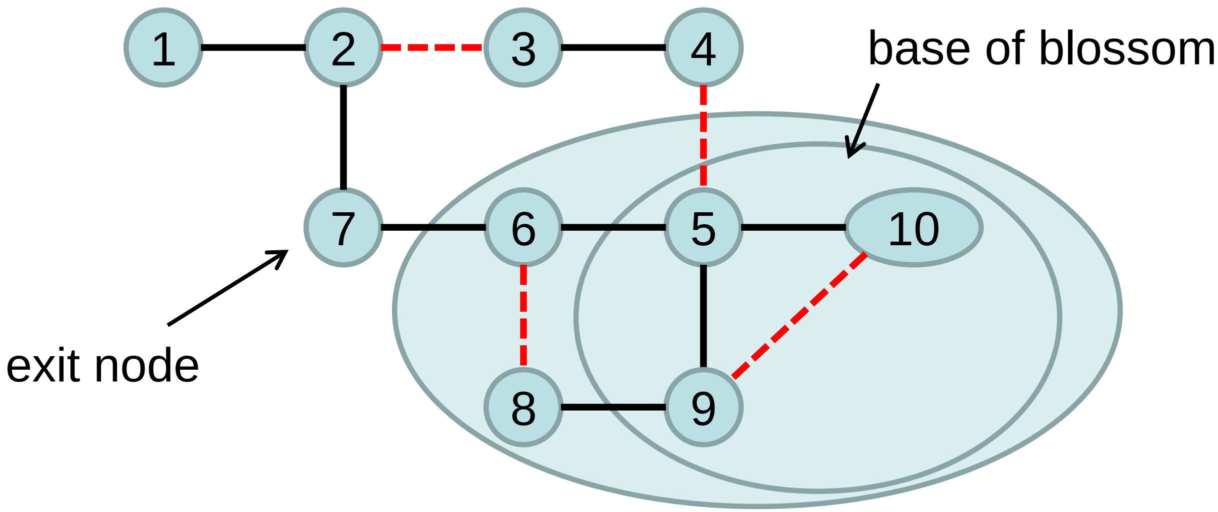
Unshrinking:



Problem: unshrink the blossoms to find augmenting path.

Matching in arbitrary Graphs

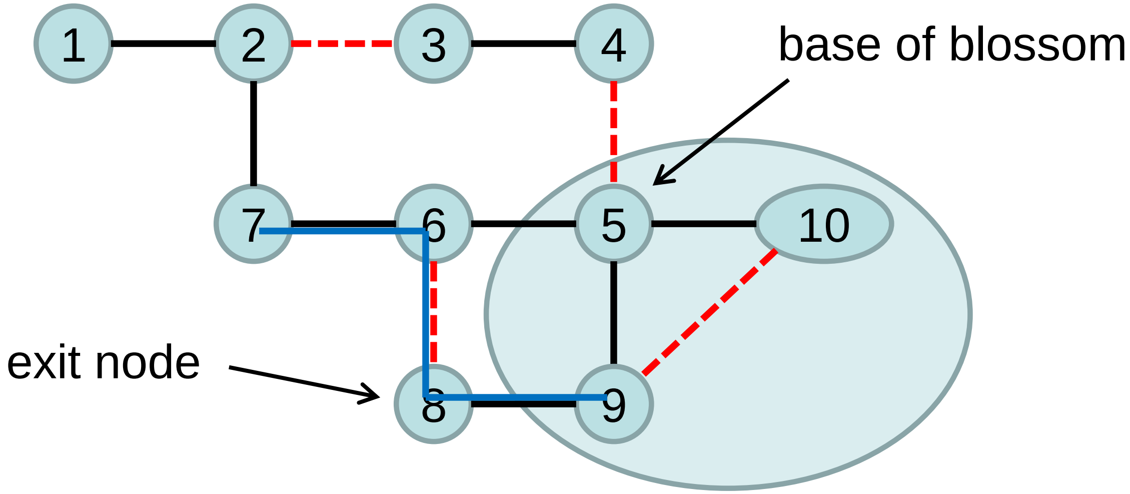
Unshrinking:



Solution: recursively find an augmenting path from base of blossom to the exit node.

Matching in arbitrary Graphs

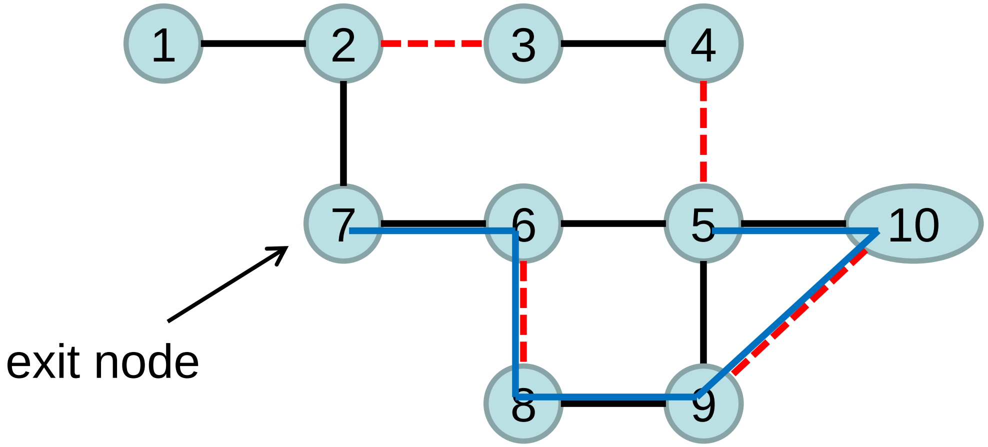
Unshrinking:



Solution: recursively find an augmenting path from base of blossom to the exit node.

Matching in arbitrary Graphs

Unshrinking:



Solution: recursively find an augmenting path from base of blossom to the exit node.

Easy because only blossom edges need to be considered!

Matching in arbitrary Graphs

Edmond´s algorithm:

$M := \emptyset$

repeat $\exists$augmenting P w.r.t. M do
    search for an augmenting path P w.r.t. M using Edmond`s blossom-based alternating BFS algorithm
    $M := M \ominus P$

output M

Runtime:

- The while-loop is executed at most n times.
- The blossom-based alternating BFS algorithm can be implemented in $O(n+m)$ time.

Therefore, a runtime of $O(n \cdot (n+m))$ is possible.

Matching in arbitrary Graphs

The Hopcroft-Karp approach can also be used for arbitrary graphs:

$M := \emptyset$

while $\exists$ augmenting path w.r.t. M do

- $l :=$ length of shortest augmenting path w.r.t. M
- determine w.r.t. „$\subseteq$“ maximal set of node-disjoint augmenting paths $Q_1, \ldots, Q_k$ w.r.t. M that have length l
- $M := M \ominus Q_1 \ominus \ldots \ominus Q_k$

- A runtime of $O(m)$ is possible per round, resulting in an overall runtime of $O(m \cdot \sqrt{n})$.
- Details can be found, for example, in:
  Paul Peterson and Michael Loui. The general maximum matching algorithm of Micali and Vazirani. Algorithmica 3:511-533, 1988.

Next Chapter

Network flow…