# Fundamental Algorithms

## Chapter 3:
## Advanced Search Structures

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(based on slides of Christian Scheideler)

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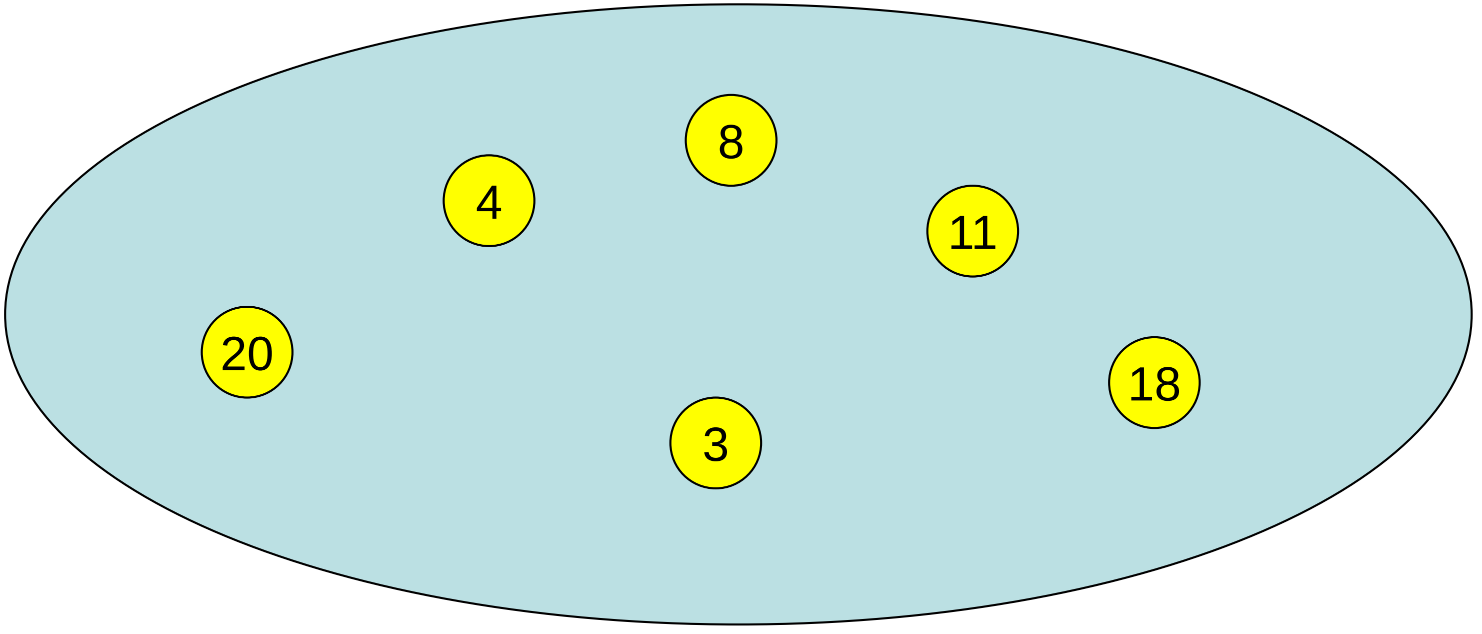
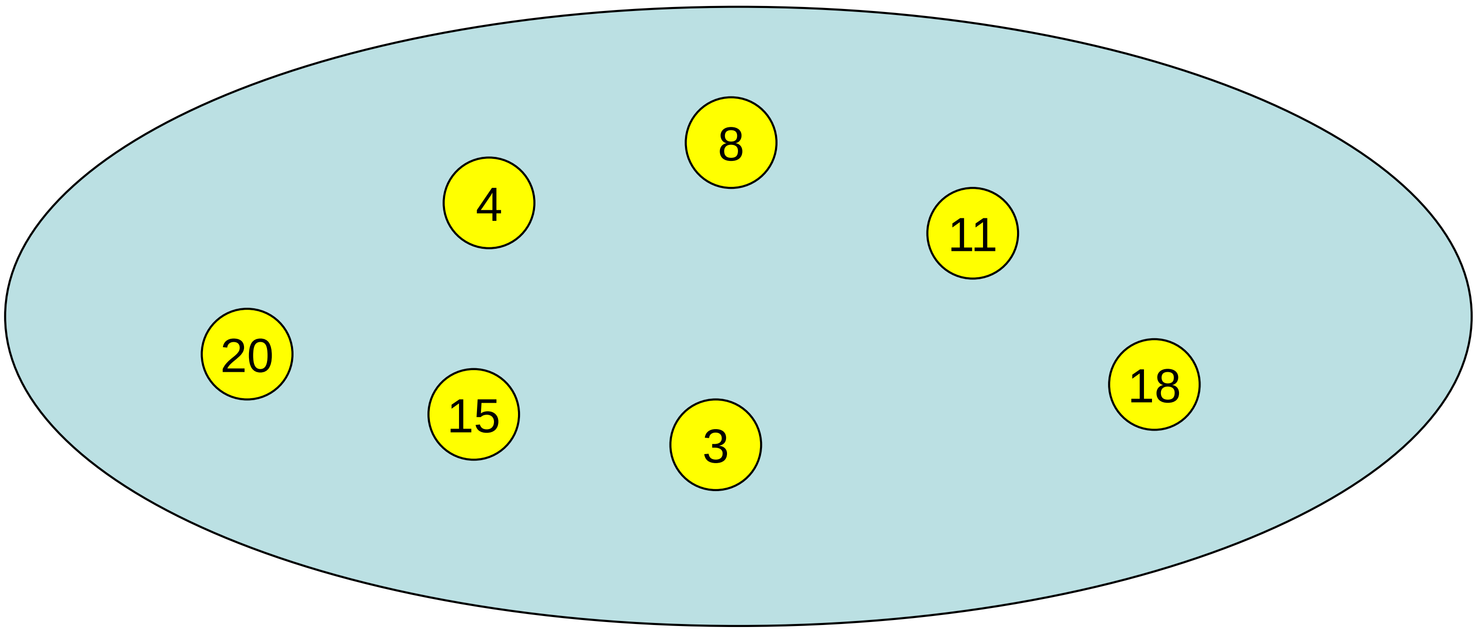
# Search Structure



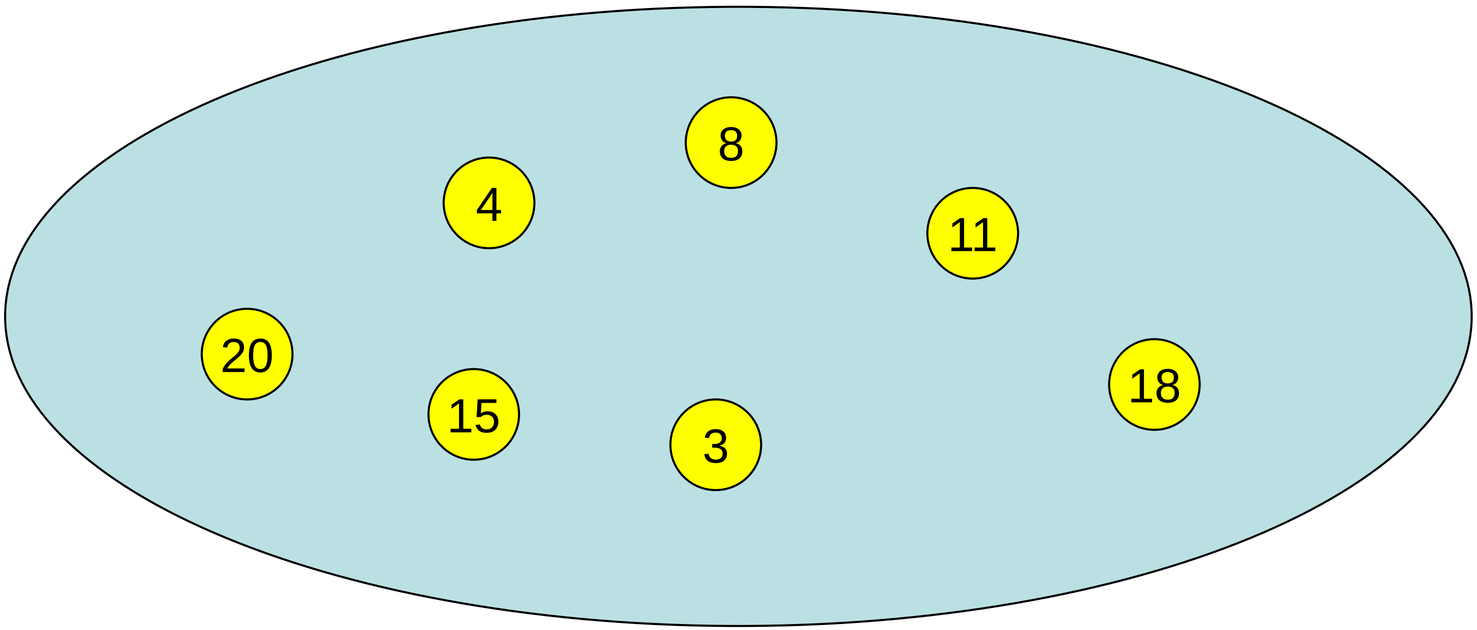
# Search Structure

insert(15)



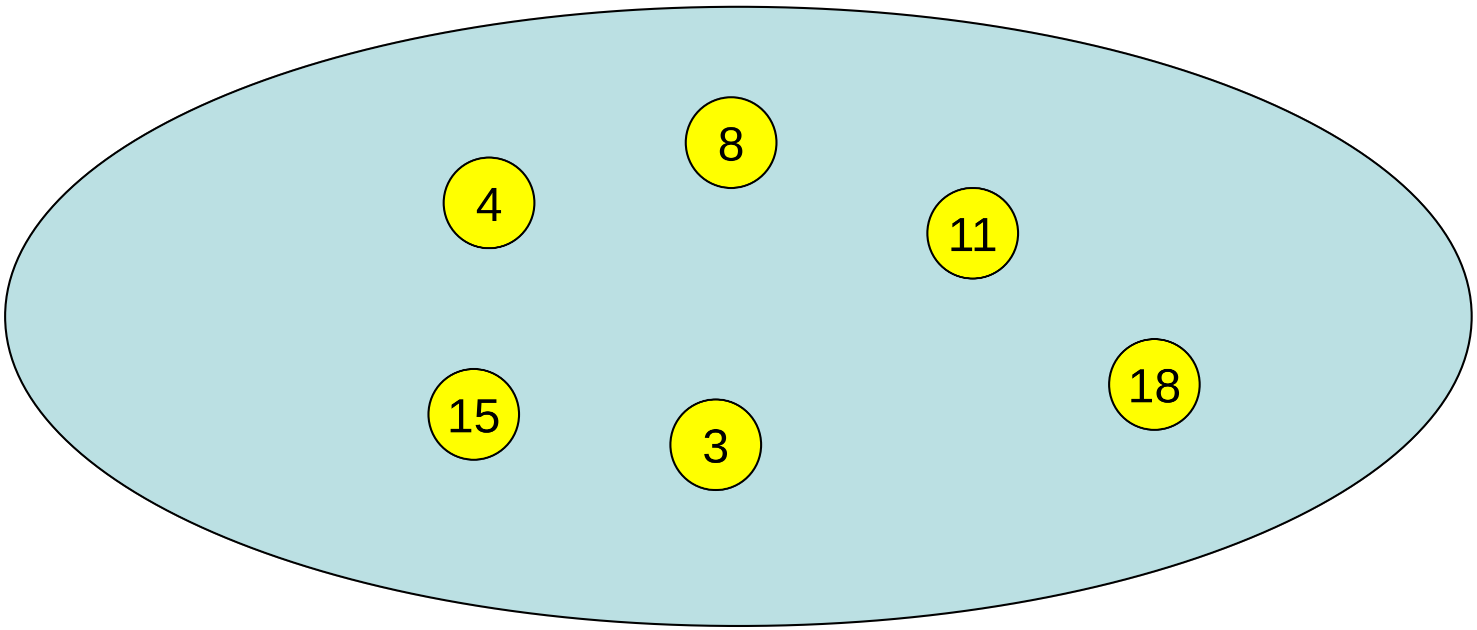
# Search Structure

delete(20)



# Search Structure

search(7) gives 8 (closest successor)



# Search Structure

S: set of elements

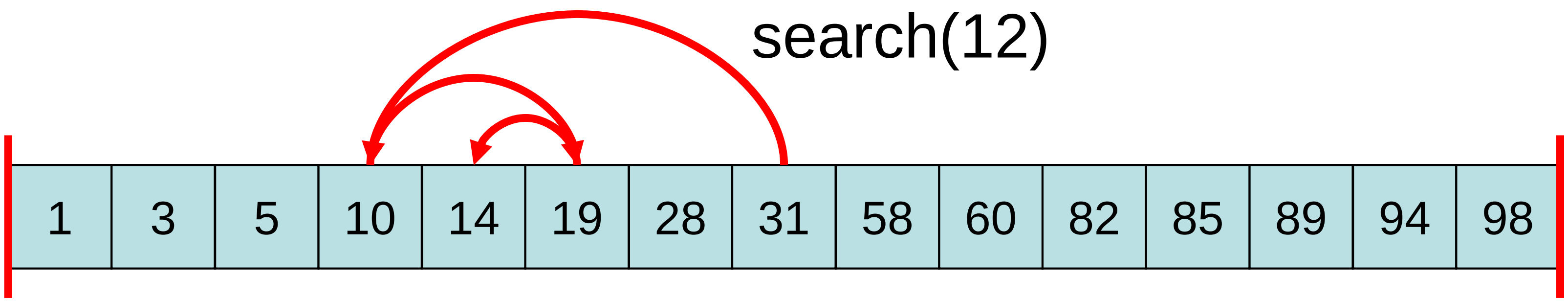
Every element e identified by key(e).

Operations:

- S.insert(e: Element): $S:=S\cup\{e\}$
- S.delete(k: Key): $S:=S\setminus\{e\}$, where e is the element with key(e)=k (note: now given *key*, not *pointer* to e!)
- S.search(k: Key): outputs $e\in S$ with minimal key(e) so that $key(e)\geq k$

# Static Search Structure

1. Store elements in sorted array.




search: via binary search (in O(log n) time)

# Binary Search

Input: number $x$ and sorted array $A[1],\ldots,A[n]$

Algorithm BinarySearch:

$l:=1$; $r:=n$

while $l < r$ do

    $m:=(r+l)$ div $2$

    if $A[m] = x$ then return $m$

    if $A[m] < x$ then $l:=m+1$

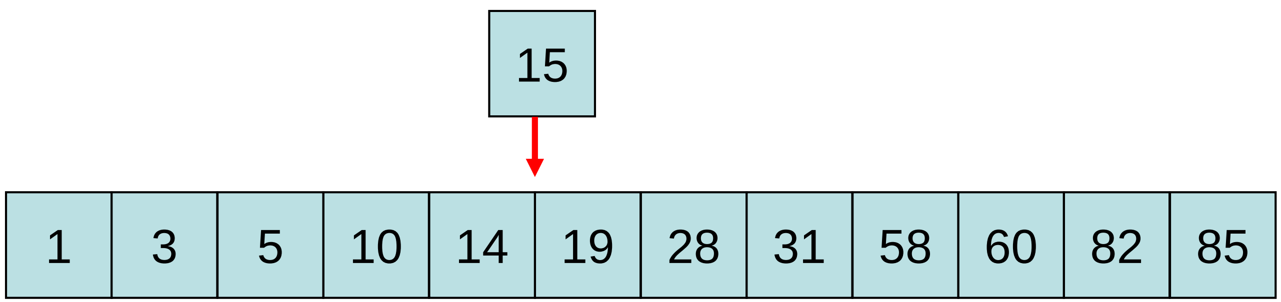
                  else $r:=m$

return $l$

# Dynamic Search Structure

insert und delete Operations:

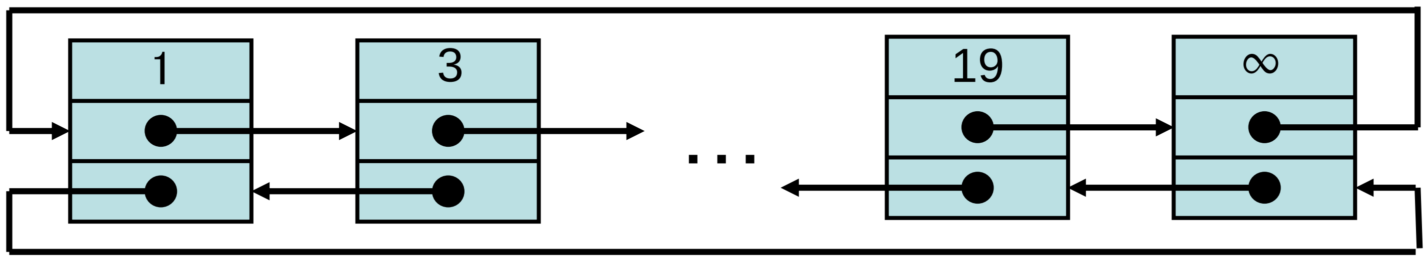
Sorted array difficult to update!




Worst case: $\Theta(n)$ time

# Search Structure

2. Sorted List (with an ∞-Element)

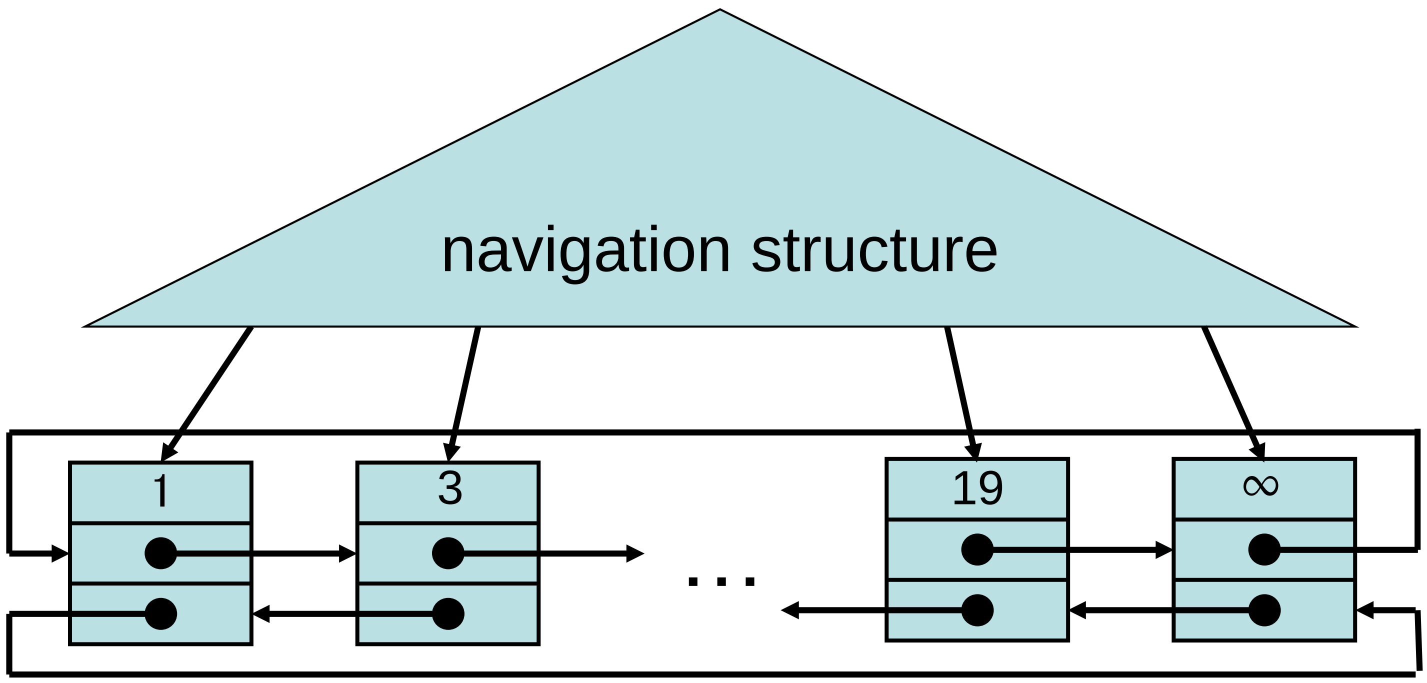



Problem: insert, delete and search take $\Theta(n)$ time in the worst case (why for insert/delete?)

Observation: If search could be implemented efficiently, then also all other operations

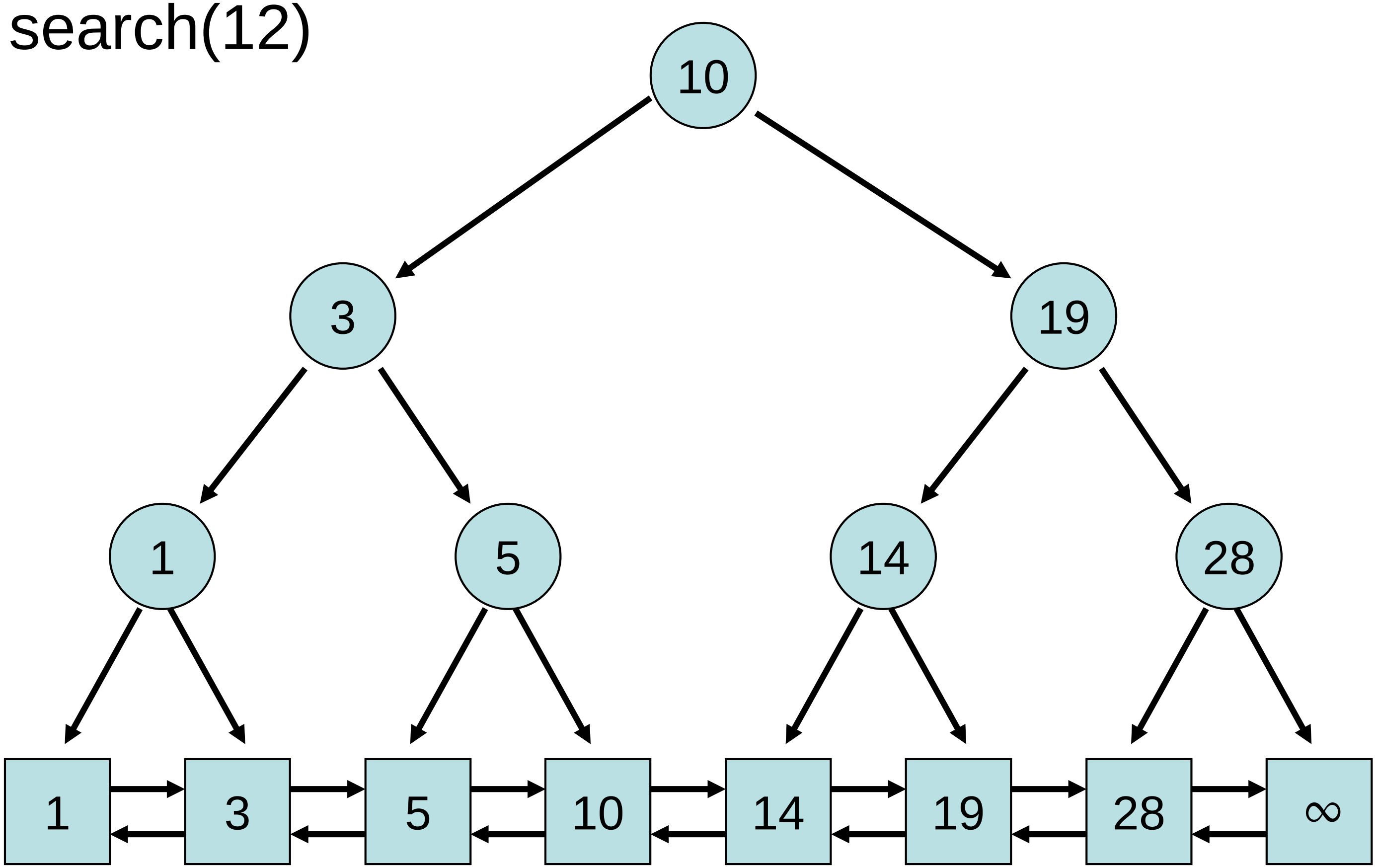
# Search Structure

Idea: add navigation structure that allows search to run efficiently



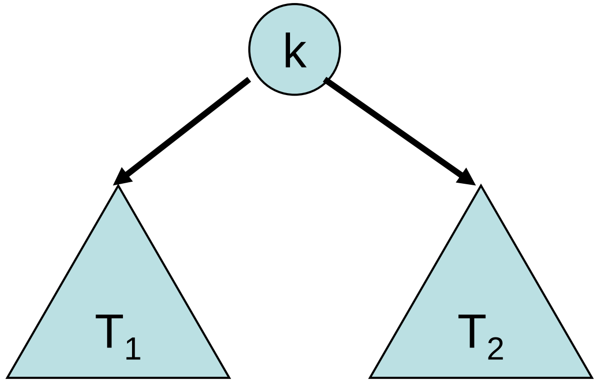
# Binary Search Tree (ideal)

search(12)



# Binary Search Tree

Search tree invariant:




For **all** keys $k´$ in $T_1$ and $k´´$ in $T_2$: $k´ \leq k < k´´$

# Binary Search Tree

Formally: for every tree node v let

- key(v) be the key stored at v
- d(v) the number of children (degree) of v

- Search tree invariant: (as above)

- Degree invariant:
  All tree nodes have exactly two children
  (as long as the number of elements in the list is >0, recall presence of ∞ node)

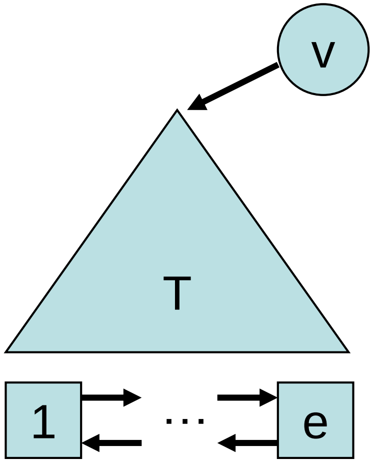
- Key invariant:
  For every element e in the list there is exactly one tree node v with key(v)=key(e).

# Binary Search Tree

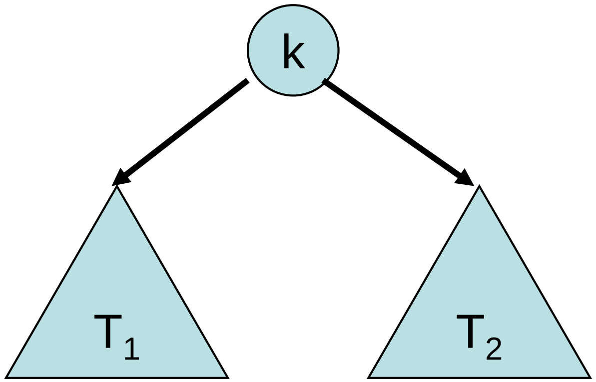
- Search tree invariant: (as before)
- Degree invariant:
  All tree nodes have exactly two children
  (as long as the number of elements is >0)
- Key invariant:
  For every element e in the list there is exactly one tree node v with key(v)=key(e).

From the search tree and key invariants it follows that for every left subtree T of a node v, the rightmost list element e under T satisfies key(v)=key(e).

(Why?)



# search(x) Operation

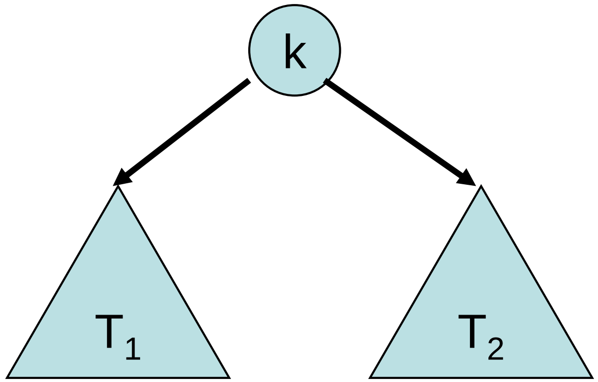



For all keys $k'$ in $T_1$ and $k''$ in $T_2$: $k' \leq k < k''$

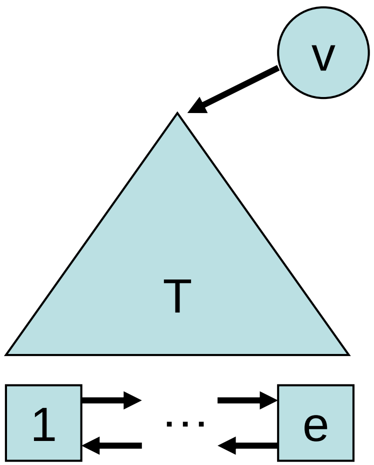
Search strategy:

- Start at the root, v, of the search tree
- while v is a tree node:
  - if $x \leq key(v)$ then let v be the left child of v,
    otherwise let v be the right child of v
- Output (list node) v

# search(x) Operation



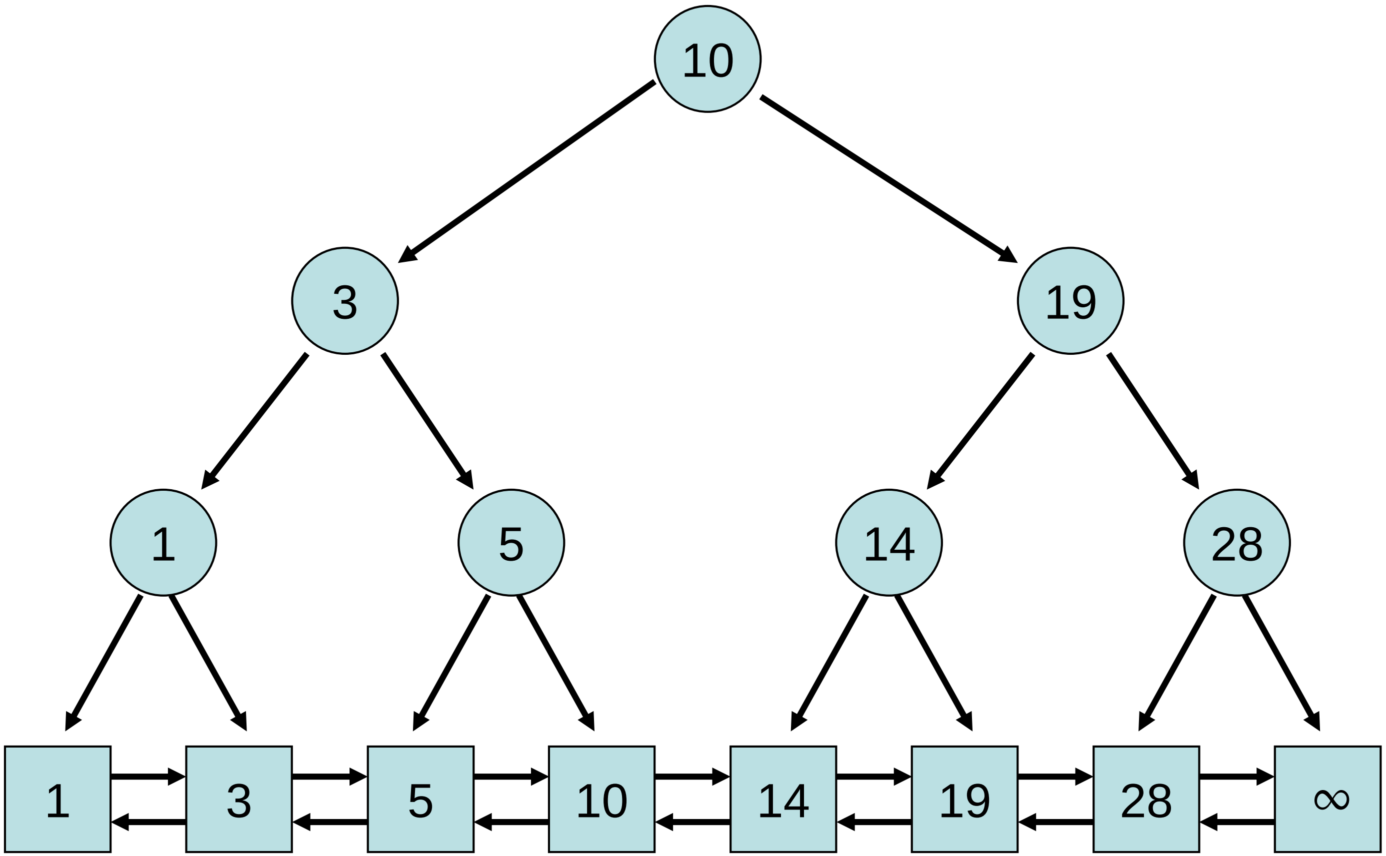

For all keys k´ in $T_1$ and k´´ in $T_2$: k´ $\leq$ k < k´´




Correctness of search strategy:

- For every left subtree T of a node v, the rightmost list element e under T satisfies key(v)=key(e).
- If search(x) enters T, since key(v)$\geq$x, there is an element e in the list below T with key(e)$\geq$x.

# Search(9)



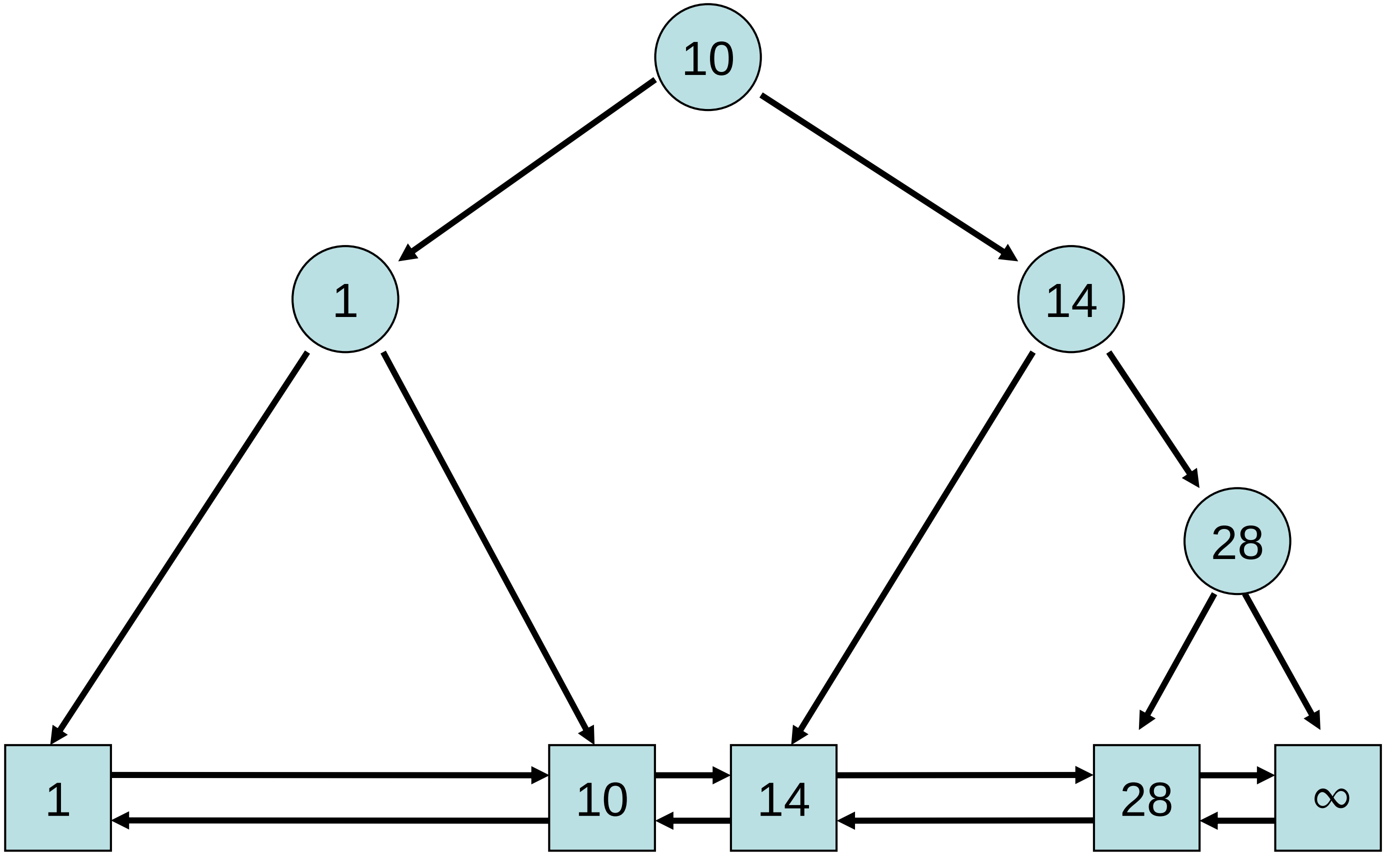
# Insert and Delete Operations

Strategy:

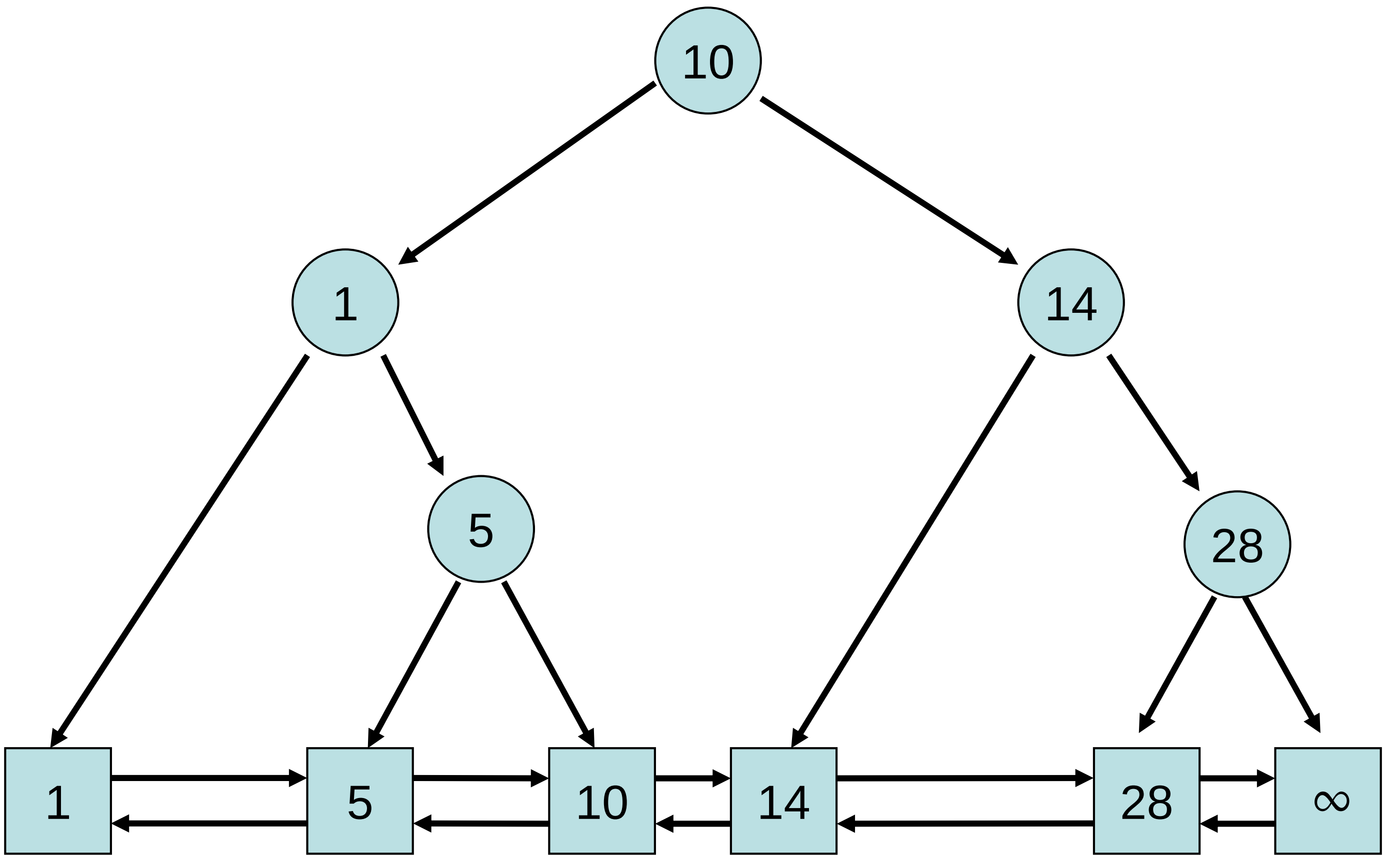
- insert(e):
  First, execute search(key(e)) to obtain a list element e´. If key(e)=key(e´), replace e´ by e, otherwise insert e between e´ and its predecessor in the list and add a new search tree leaf leading to e (left) and e´ (right) with key key(e).

- delete(k):
  First, execute search(k) to obtain a list element e. If key(e)=k, then delete e from the list and the parent v of e from the search tree, and relabel tree node w with key(w)=k as key(w):=key(v).

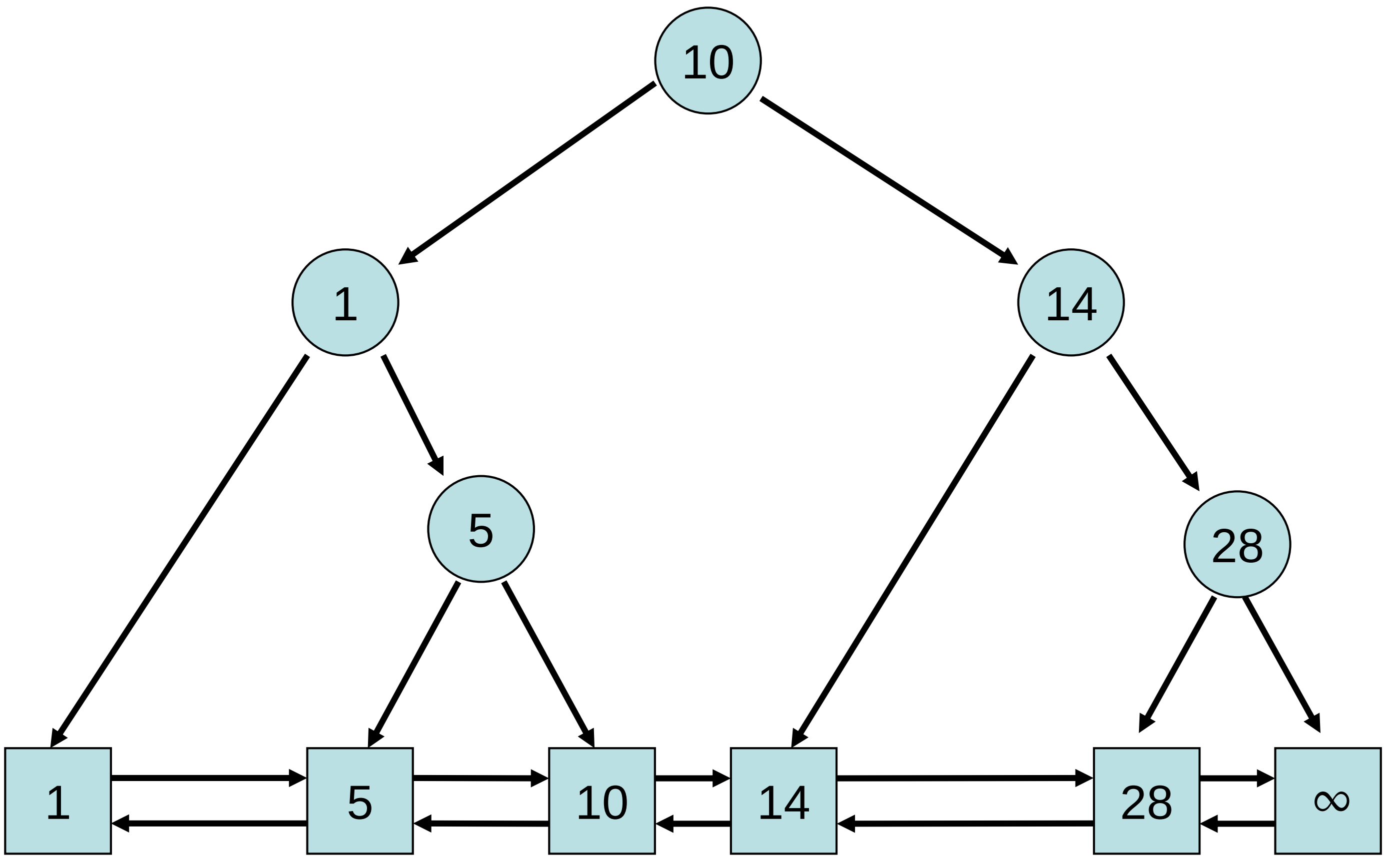
# Insert(5)



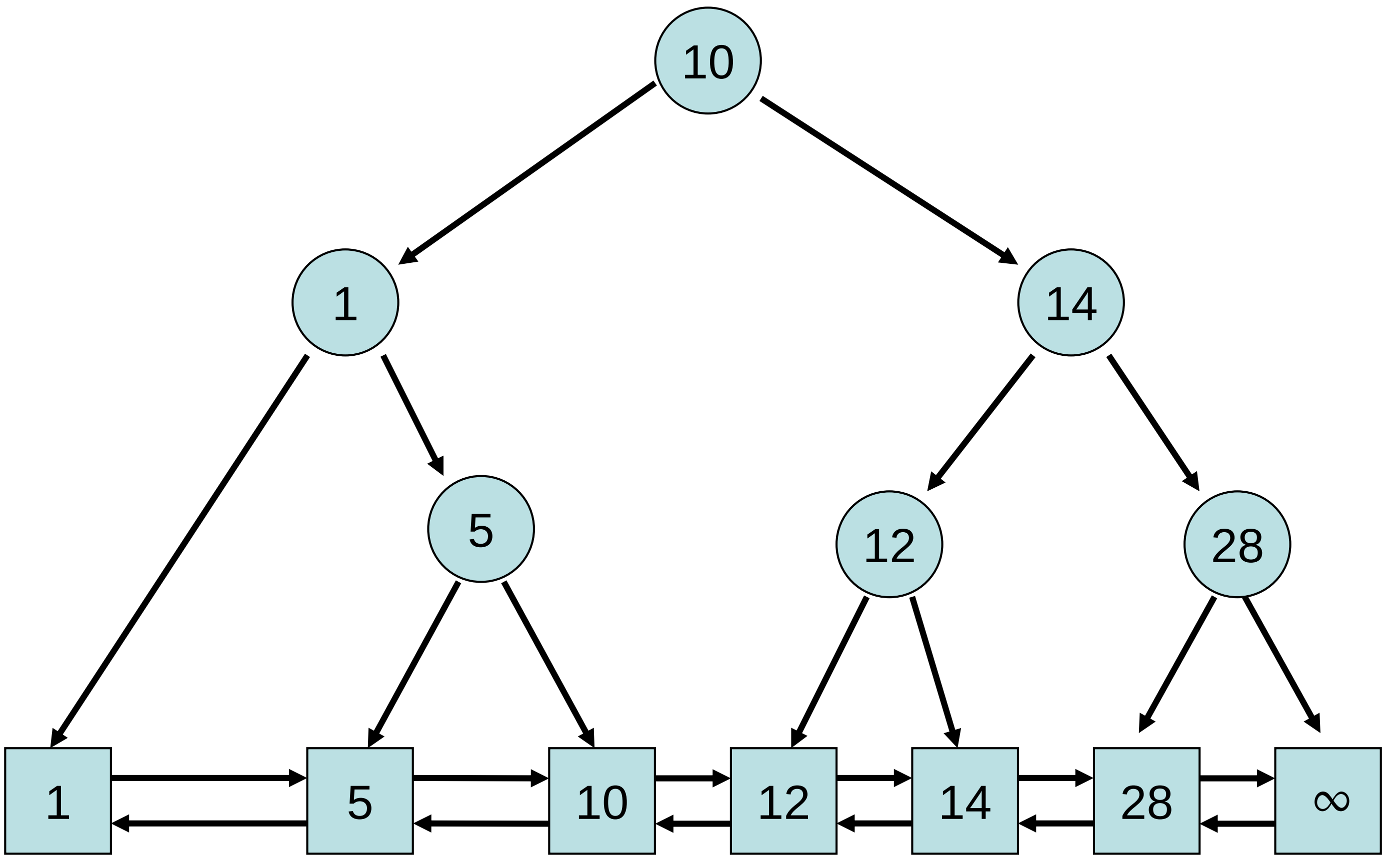
# Insert(5)



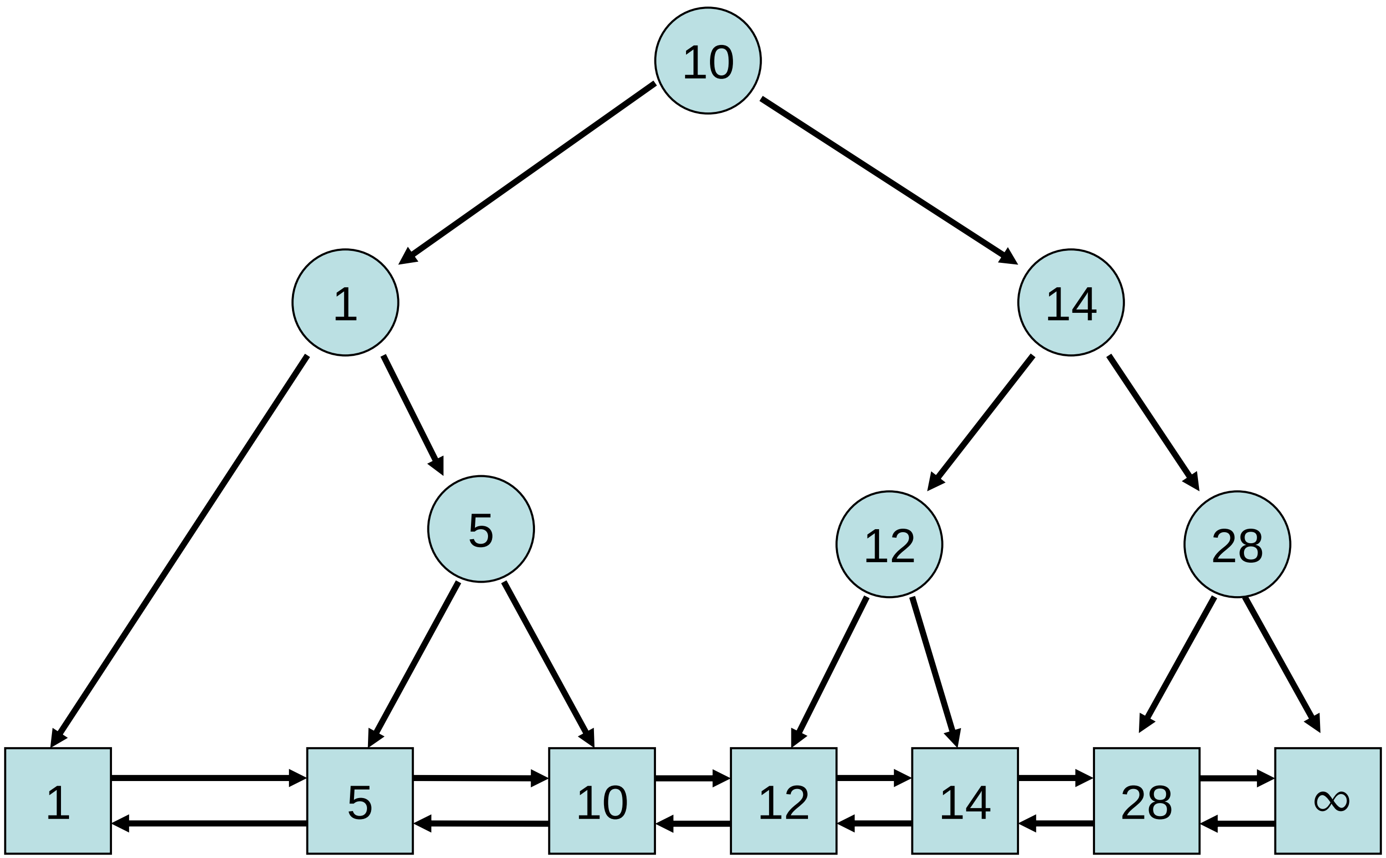
# Insert(12)



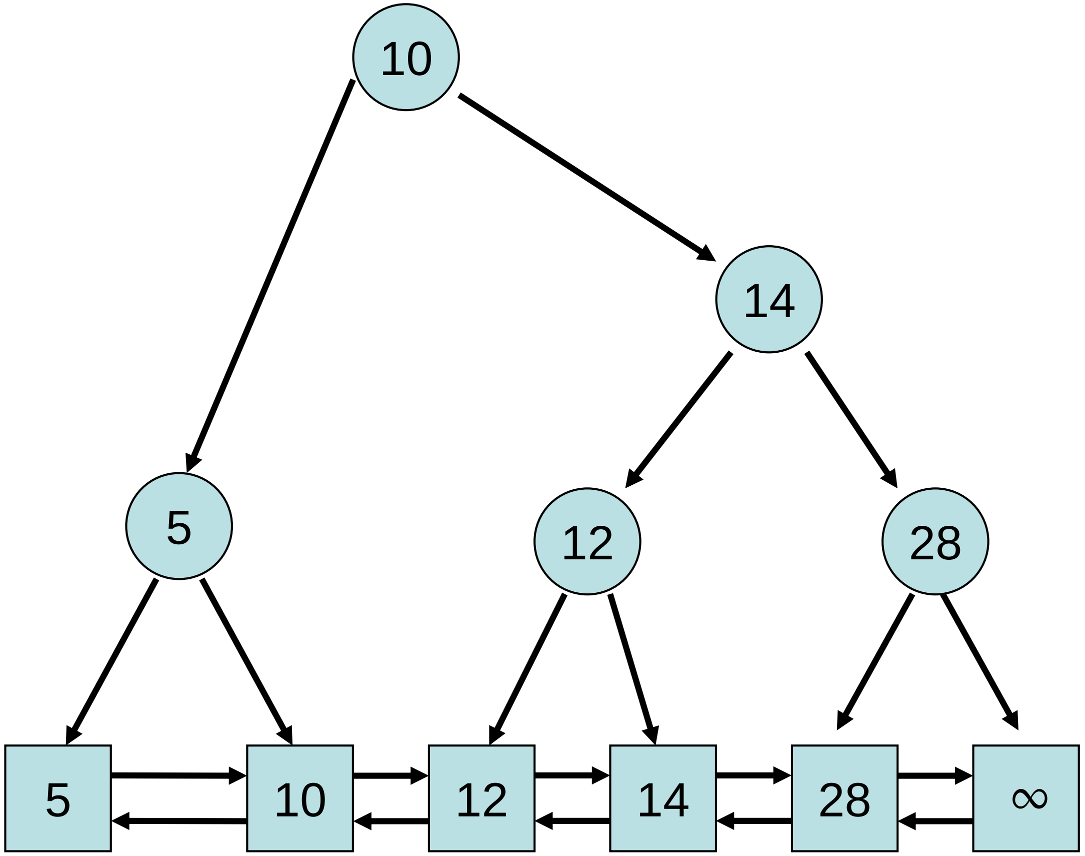
# Insert(12)



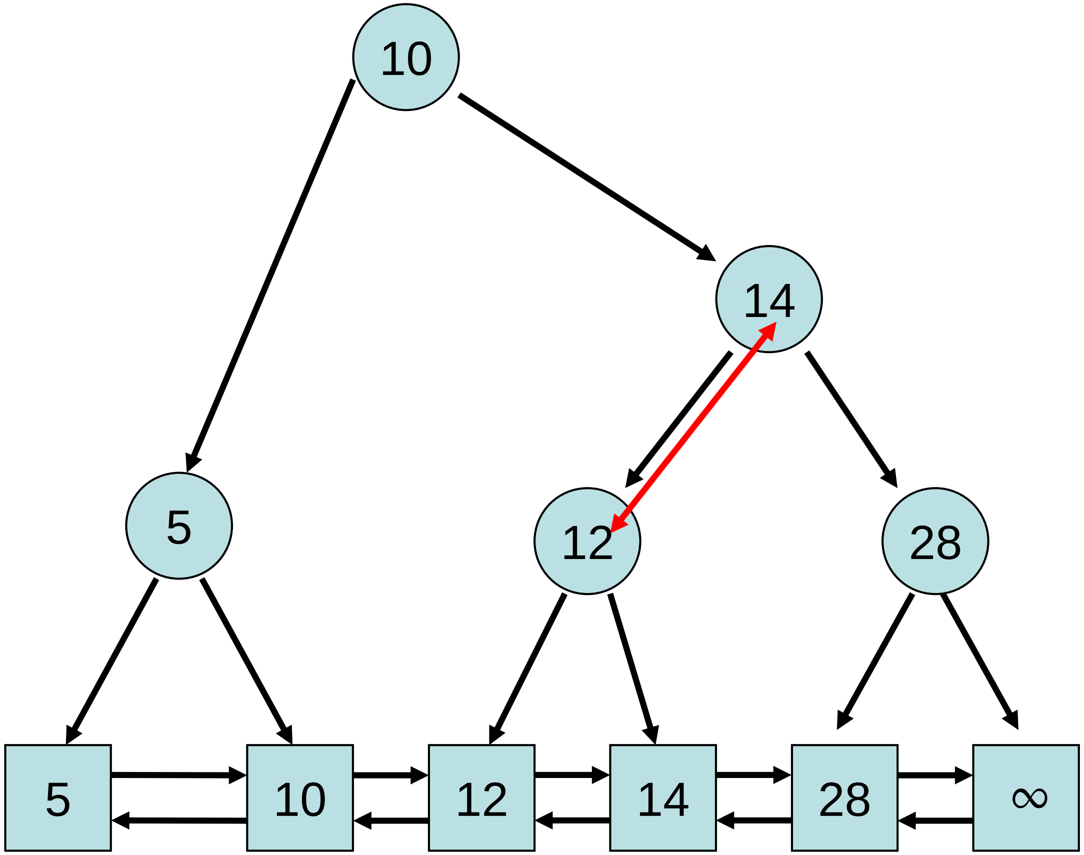
# Delete(1)



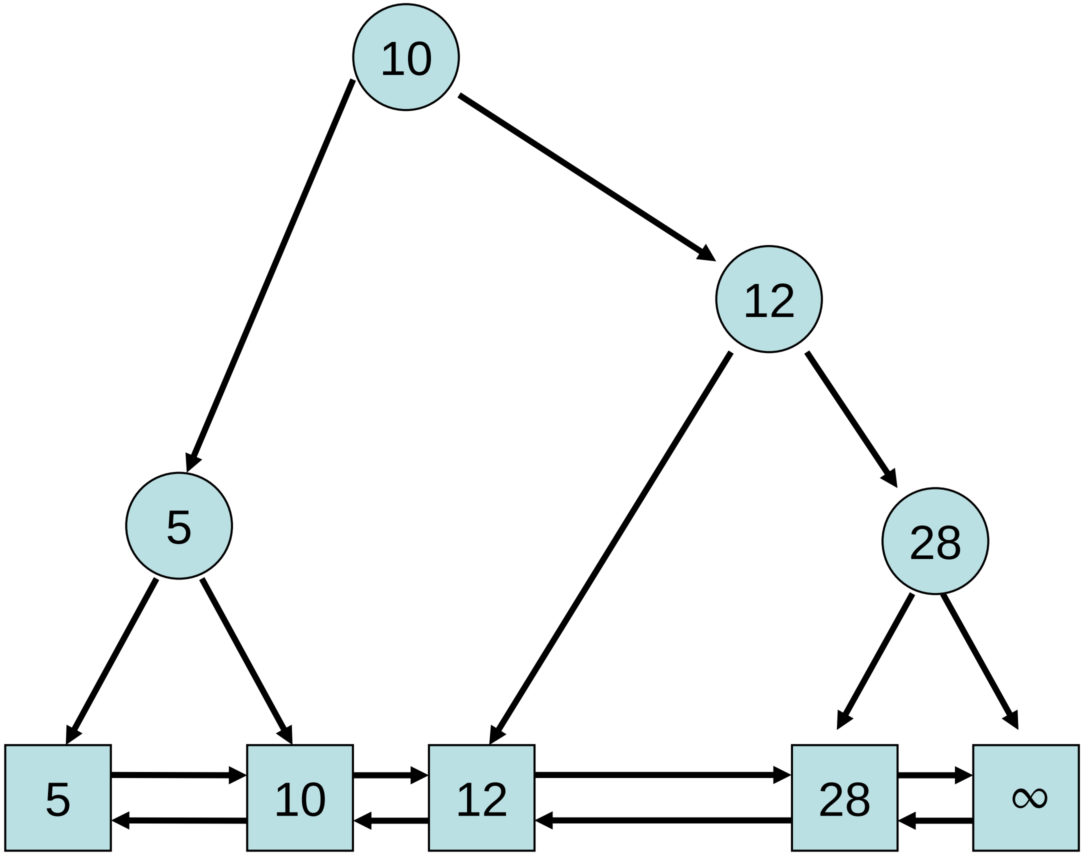
# Delete(1)



# Delete(14)



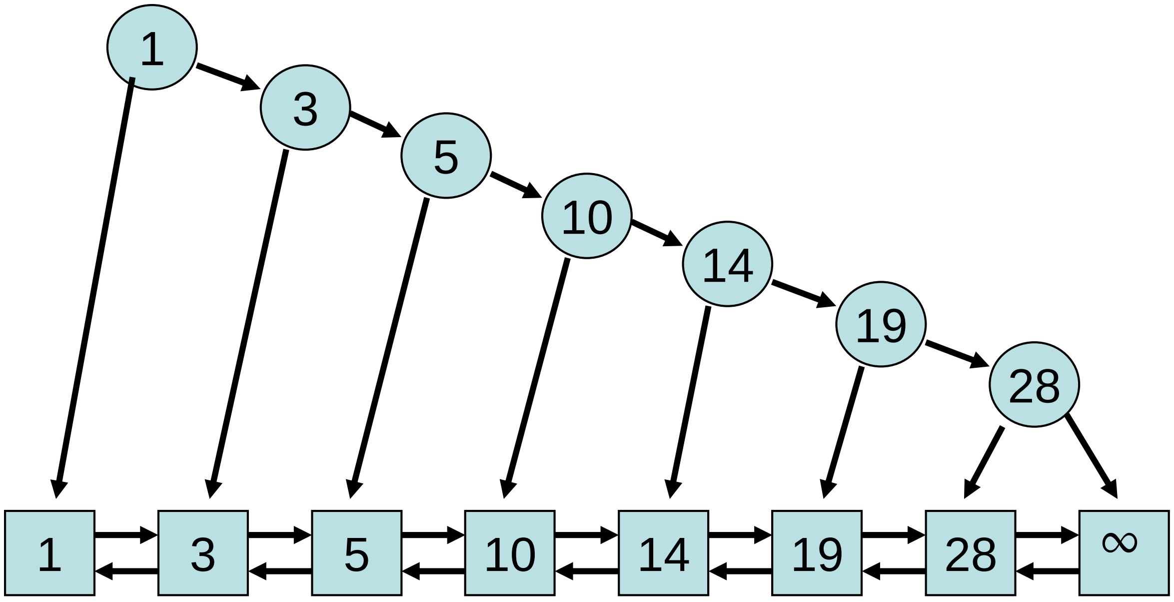
# Delete(14)



# Binary Search Tree

Problem: binary tree can degenerate!

Example: numbers are inserted in sorted order



# Pop quiz

Q1: What is the worst case runtime for binary search on a sorted array?

O(logn).

Q2: What is the worst case runtime for searching in a binary search tree?

O(n)! (see e.g. previous slide)

# Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- Patricia trie

# Splay Tree

Usually: Implementation as internal search tree (i.e., elements directly integrated into tree and not in an extra list)

Here: Implementation as external search tree (like for the binary search tree above)

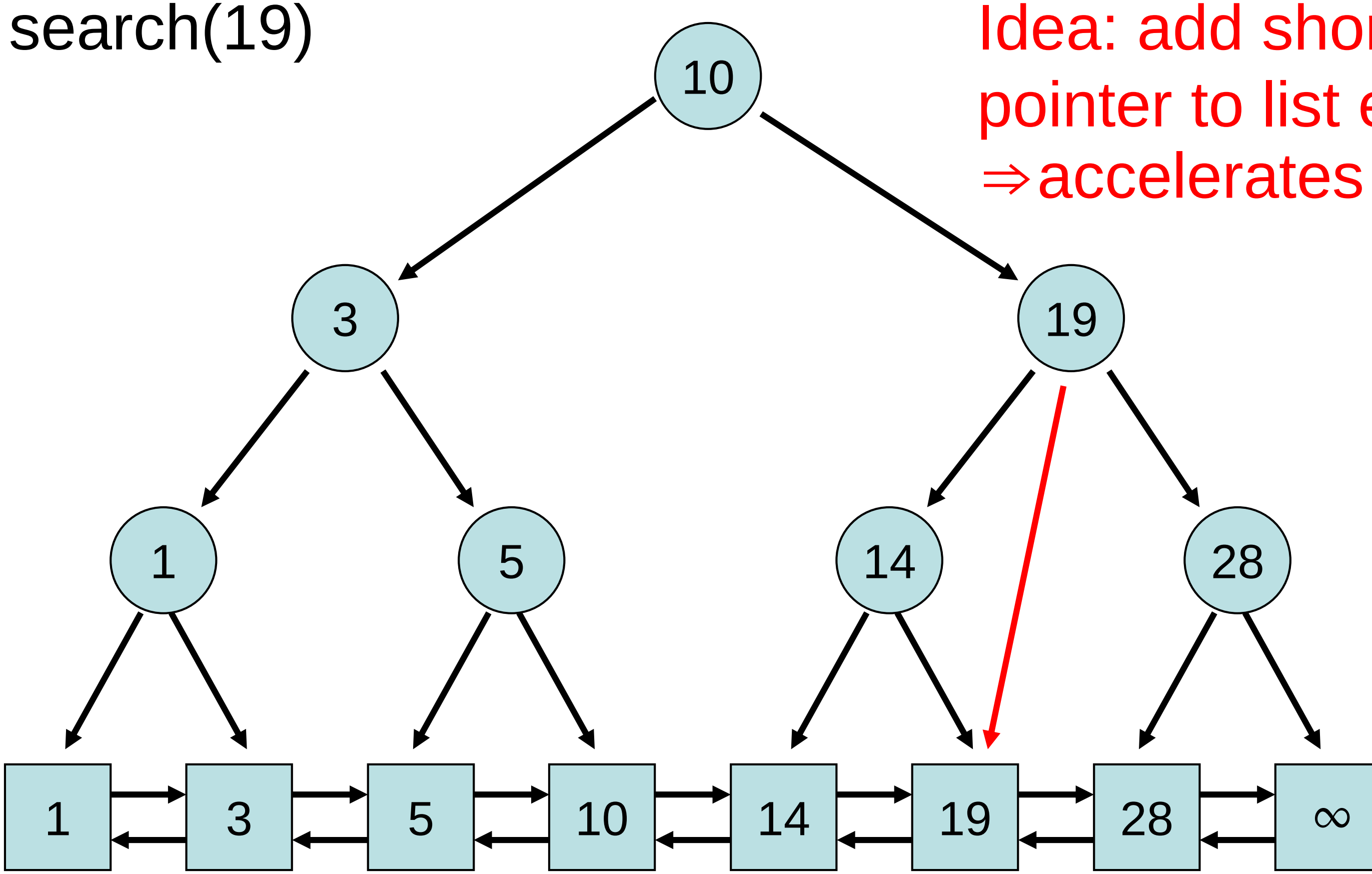
# Why Splay Trees?

- Self-adjusting binary search tree
- Invented by Sleator and Tarjan (1985)
- Pros:
  - Recently accessed elements quick to access again. (Great for caches, garbage collection!)
  - Low amortized costs
- Cons:
  - Can still have highly unbalanced trees, hence worst-case linear time search.

# Splay Tree

search(19)

Idea: add shortcut pointer to list element
⇒accelerates search



# Splay Tree

Ideas:

1. Add shortcut pointers in tree to list elements
2. For every search(k) operation, move pred(k) (the closest predecessor of k in T) to the root (why?)

Movement for (2): via Splay operation

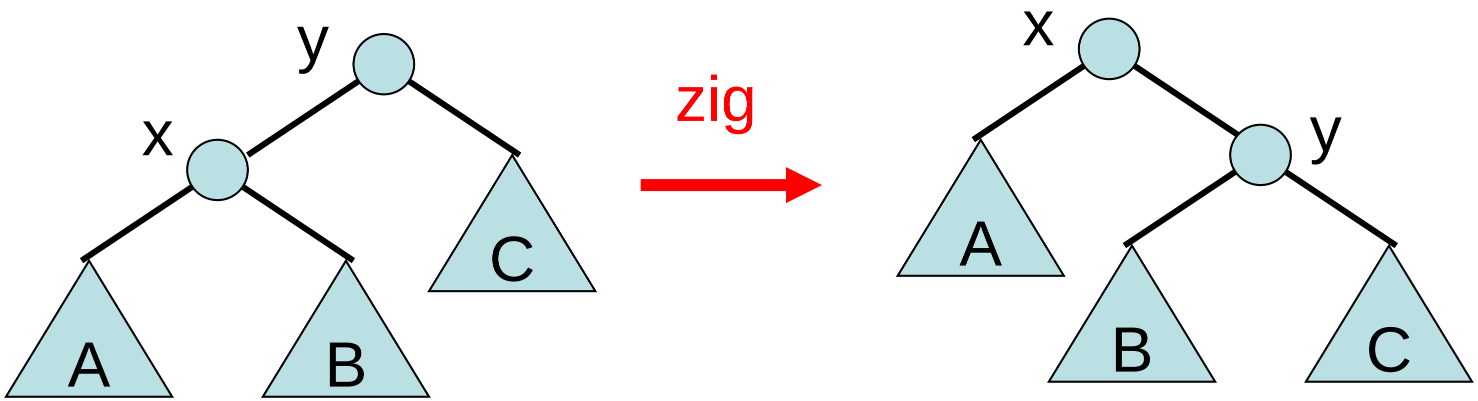
For simplicity: we focus on search(k) for keys k *already in the search tree*.

# Splay Operation

Movement of key x to the root: 3 cases.

Case 1:

1a. x is a *left* child of the root:

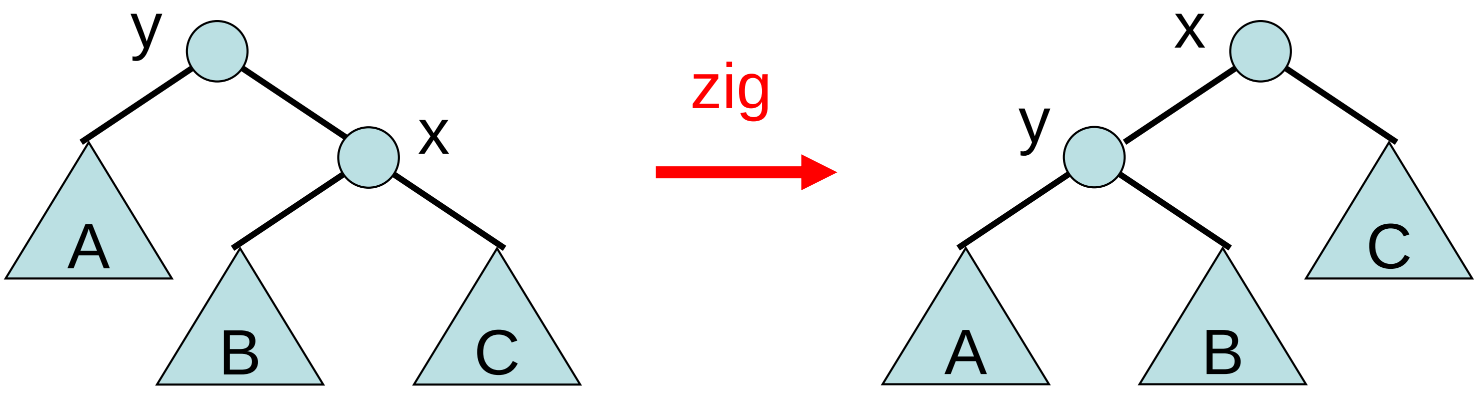


# Splay Operation

Movement of key x to the root: 3 cases

Case 1:

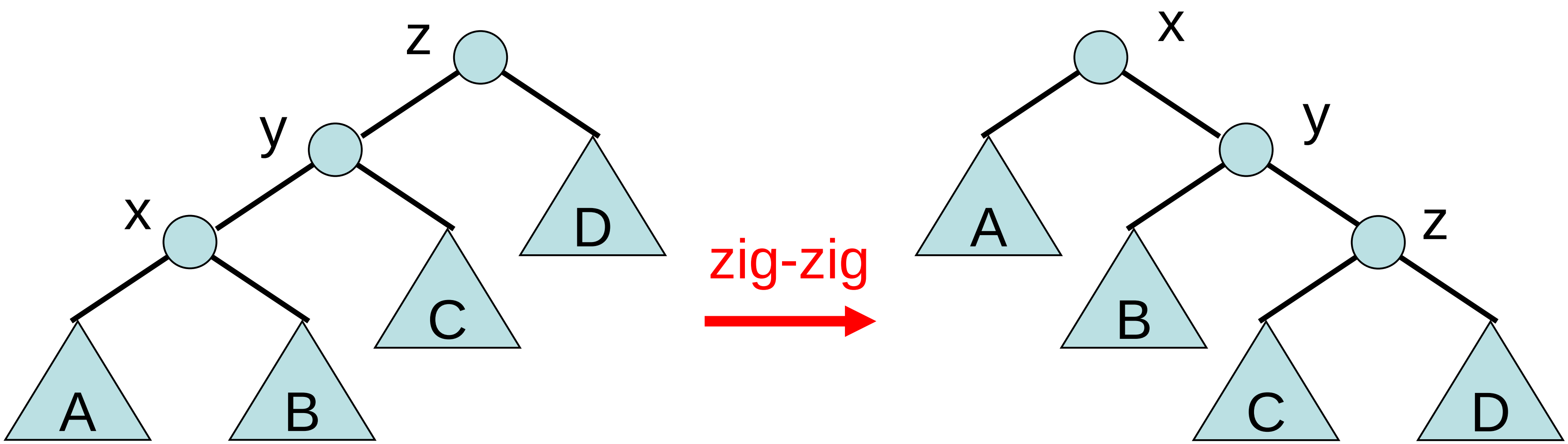
1b. x is a *right* child of the root:



# Splay Operation

Case 2:

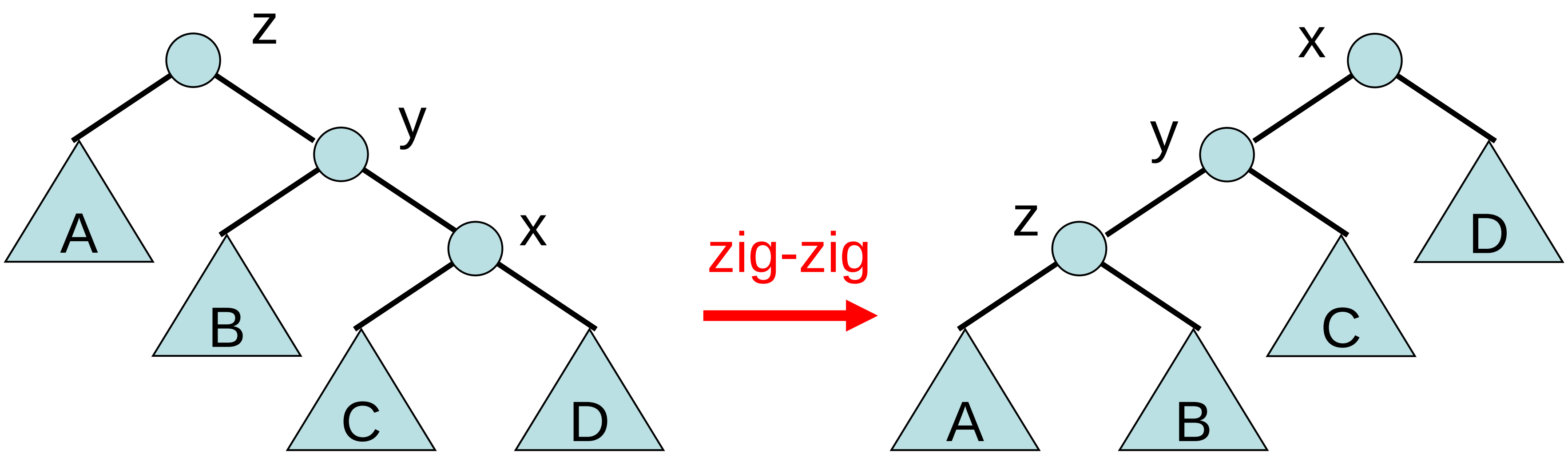
2a. x has father and grand father to the *right*



# Splay Operation

Case 2:

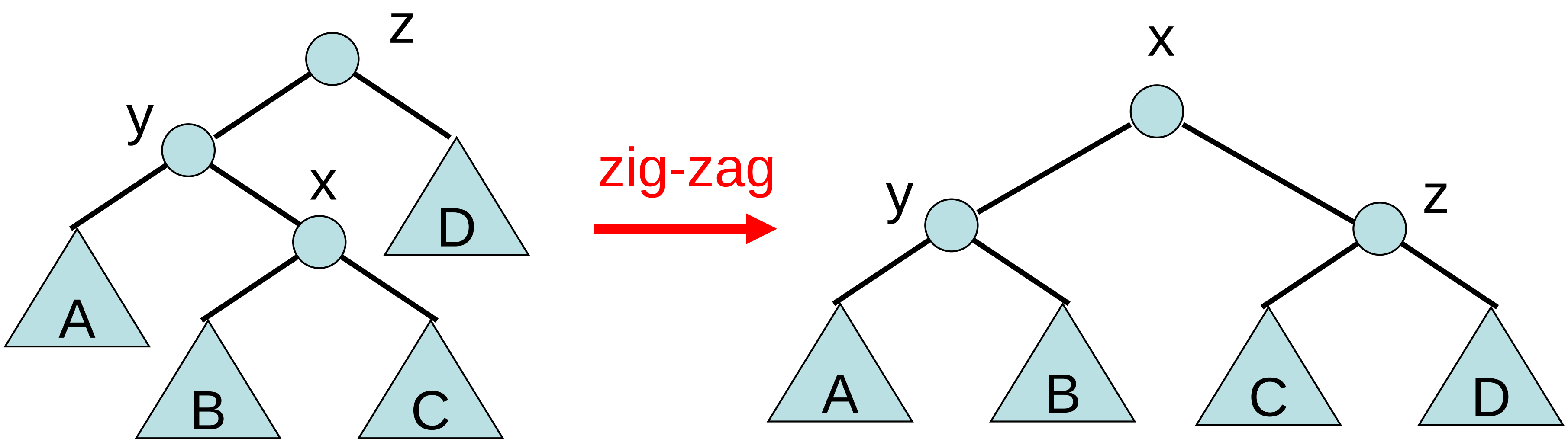
2b. x has father and grand father to the *left*



# Splay Operation

Case 3:

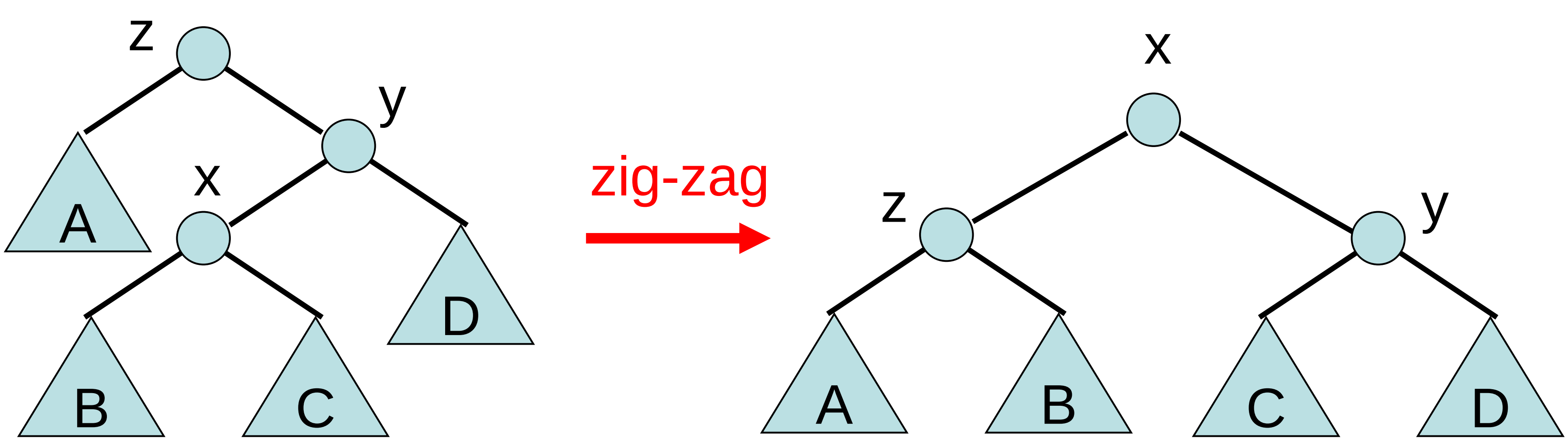
3a. x: father *left*, grand father *right*



# Splay Operation

Case 3:

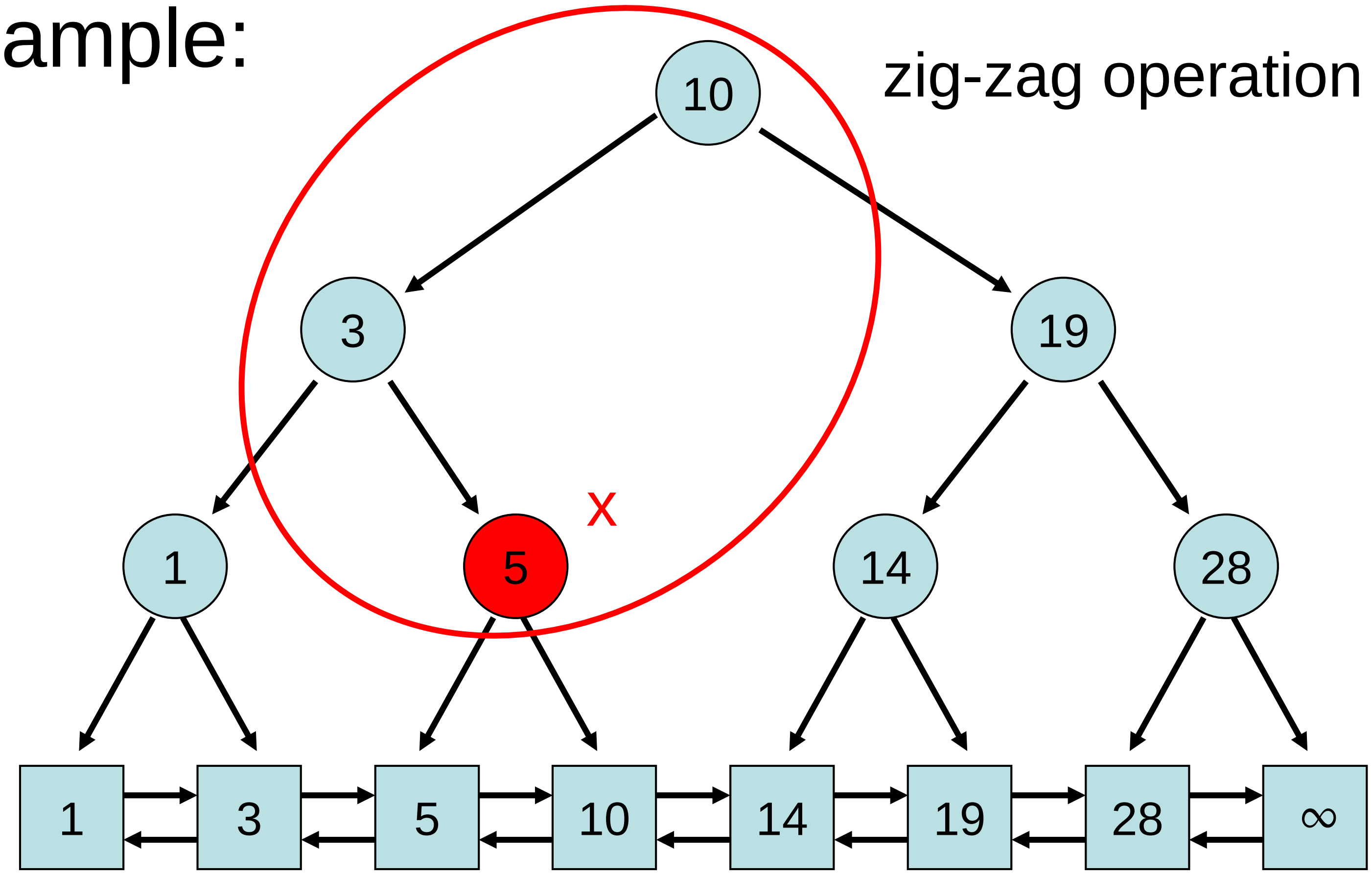
3b. x: father *right*, grand father *left*



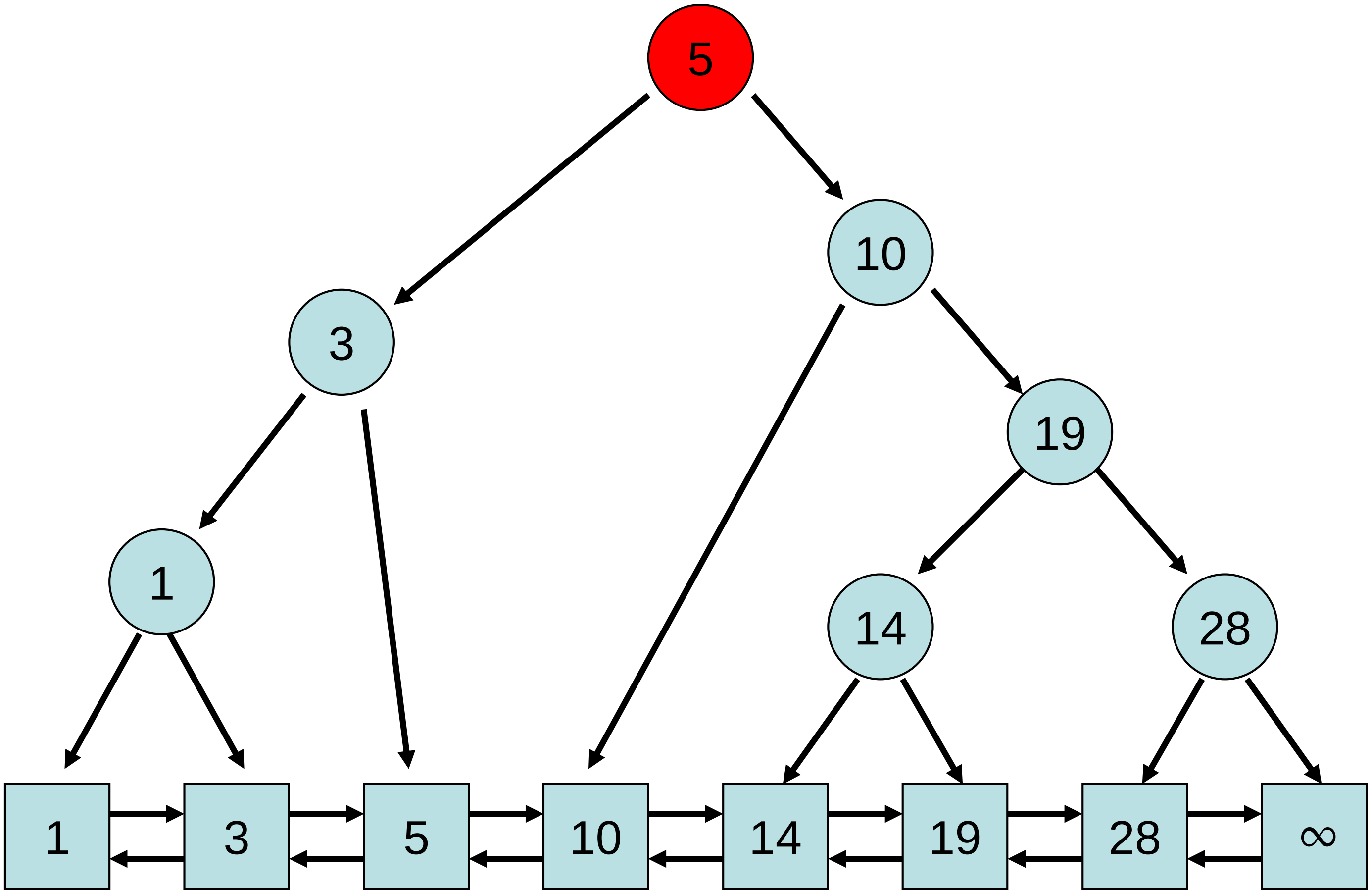
# Splay Operation

Example:

zig-zag operation (3a)

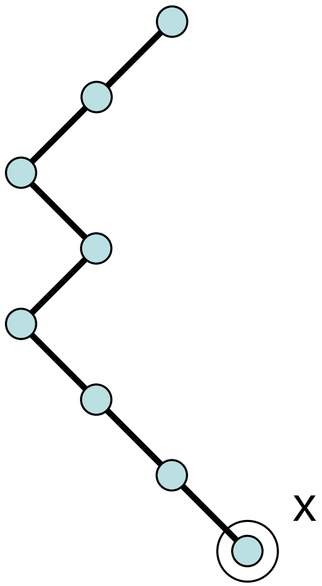


# Splay Operation

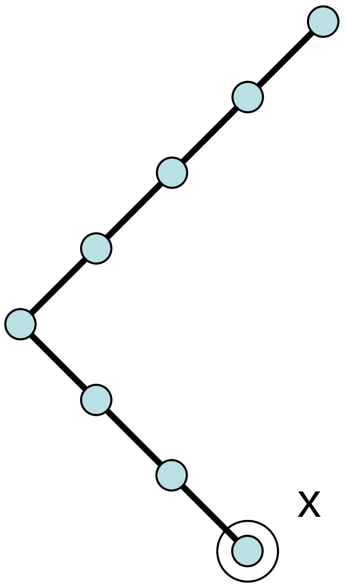


# Splay Operation

Examples:



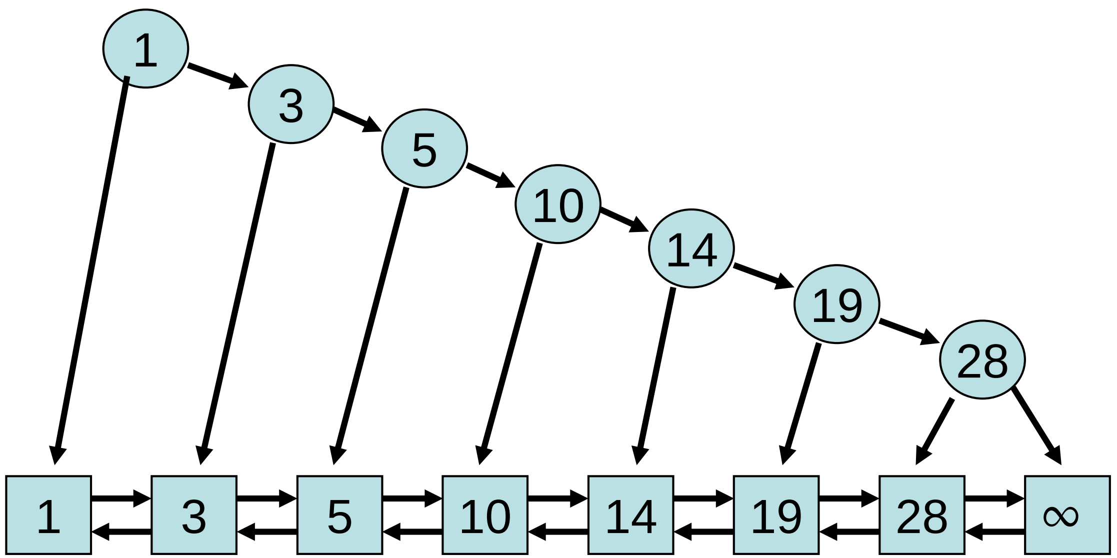

zig-zig, zig-zag, zig-zag, zig




zig-zig, zig-zag, zig-zig, zig

# Splay Operation

Observation: Tree can still be highly imbalanced! But amortized costs are low.



# Splay Operation

search(k)-operation:

- Move downwards from the root (as in standard binary tree) till pred(k) found in search tree (which can be checked via shortcut to the list) or the list is reached
- call splay(pred(k)), output next successor, succ(k) (recall we assume k exists in tree for simplicity: pred(k)=succ(k)=k)

Amortized Analysis:

- Note: runtime of search(k) is O(runtime of splay(pred(k))).
- Our goal: bound runtime of m Splay operations on arbitrary binary search tree with n elements (m>n)

# Splay Operation

- Weight of node $x$: $w(x)>0$
- Tree weight of tree $T$ with root $x$: $tw(x)= \sum_{y\in T} w(y)$
- Rank of node $x$: $r(x) = \log(tw(x))$
- Potential of tree $T$: $\phi(T) = \sum_{x\in T} r(x)$

Lemma 3.1: Let $T$ be a Splay tree with root $x$ and $u$ be a node in $T$. The amortized cost for splay($u$,$T$) is at most $1+3(r(x)-r(u))$.

# Splay Operation

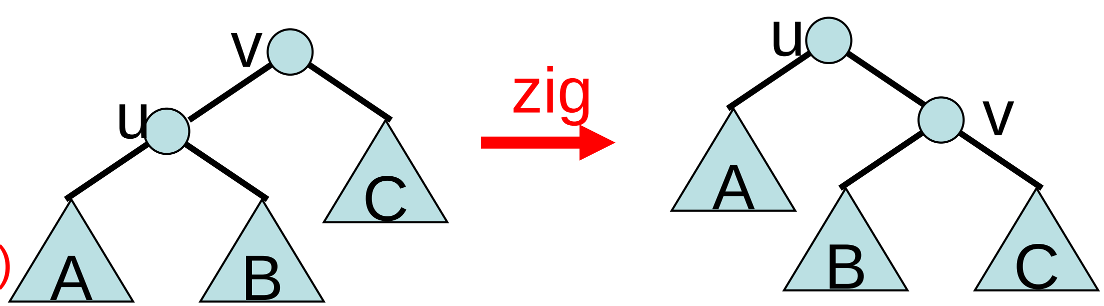
(Recall: Amortized cost $A_X(s) := T_X(s) + (\phi(s´) - \phi(s))$)

Proof of Lemma 3.1:

Induction over the sequence of rotations.

- r and tw : rank and weight before the rotation
- r’ and tw’: rank and weight after the rotation

Case 1:




Runtime
(# rotations)

Amortized cost:

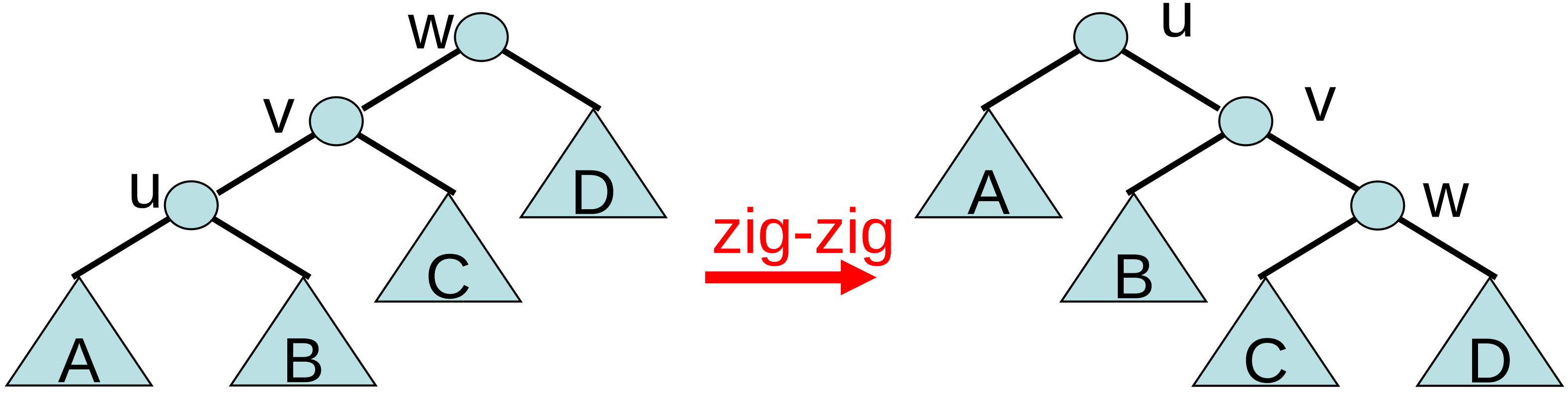
$\leq 1+r'(u)+r'(v)-r(u)-r(v) \leq 1+r'(u)-r(u)$ since $r'(v) \leq r(v)$

$\leq 1+3(r'(u)-r(u))$ since $r'(u) \geq r(u)$

Change in $\phi$

# Splay Operation

Case 2:




Amortized cost:

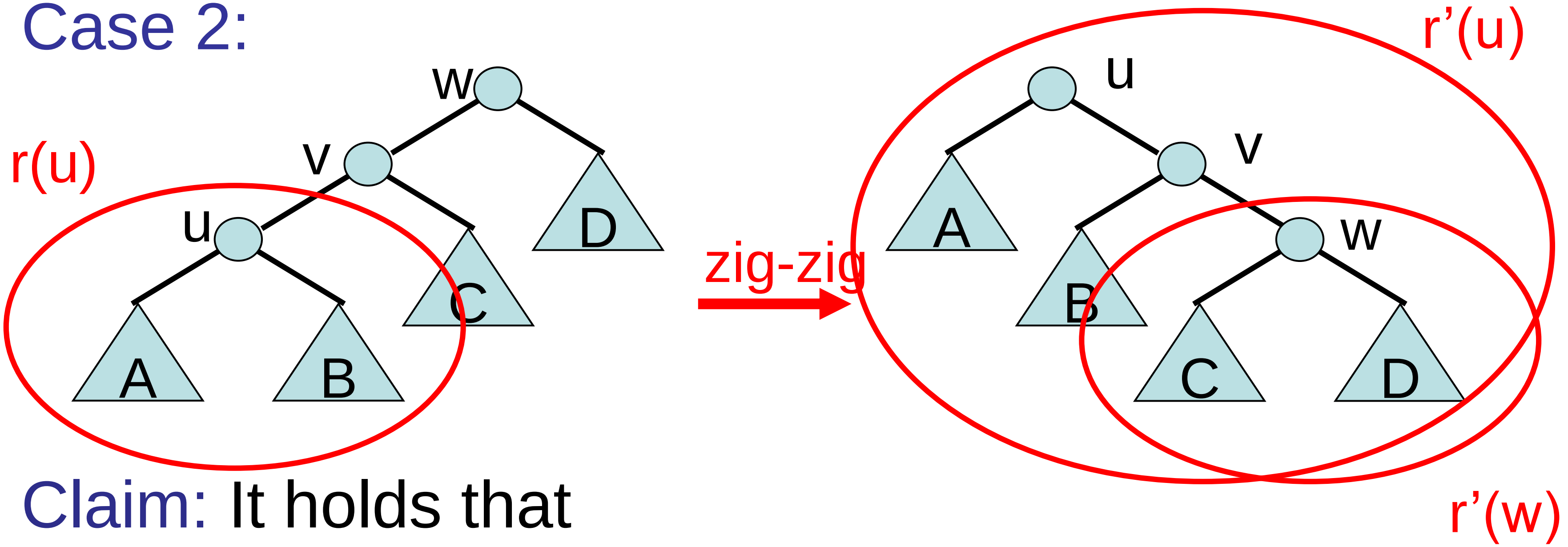
$\leq$ 2+r'(u)+r'(v)+r'(w)-r(u)-r(v)-r(w)

= 2+r'(v)+r'(w)-r(u)-r(v) since r'(u)=r(w)

$\leq$ 2+r'(u)+r'(w)-2r(u) since r'(u)$\geq$r'(v) and r(v)$\geq$r(u)

# Splay Operation

Case 2:




Claim: It holds that

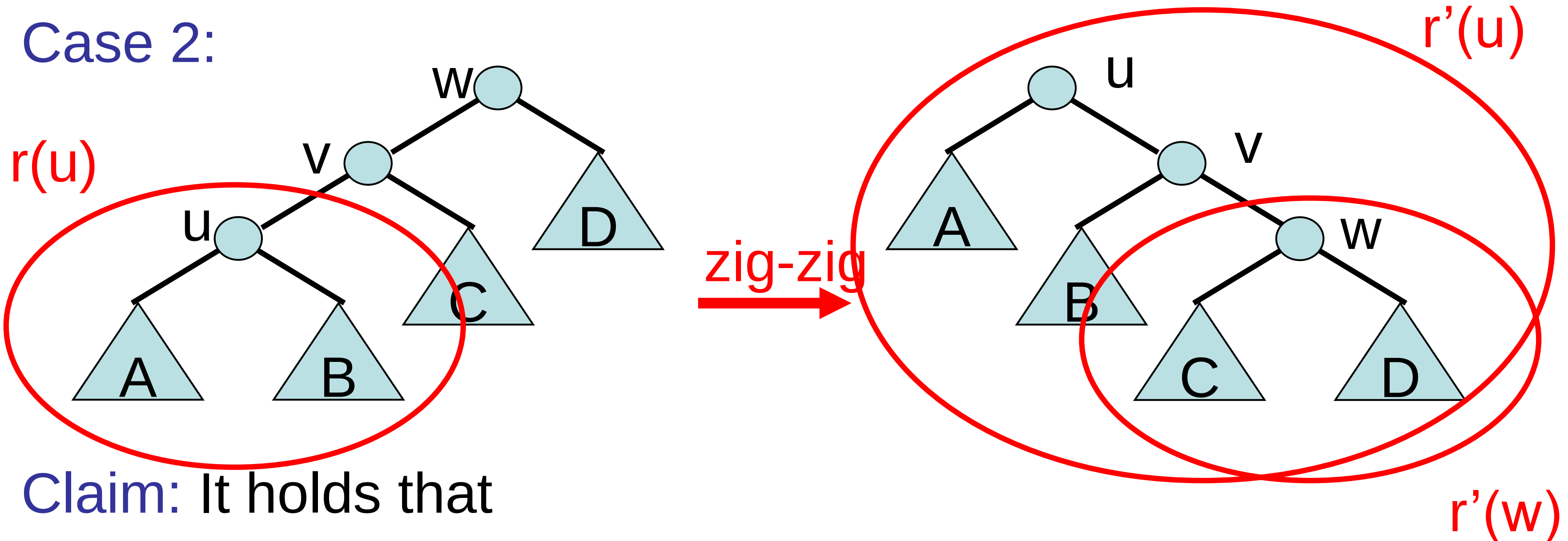
$$2+r'(u)+r'(w)-2r(u) \leq 3(r'(u)-r(u))$$

i.e.

$$r(u)+r'(w) \leq 2(r'(u)-1)$$

# Splay Operation

Case 2:



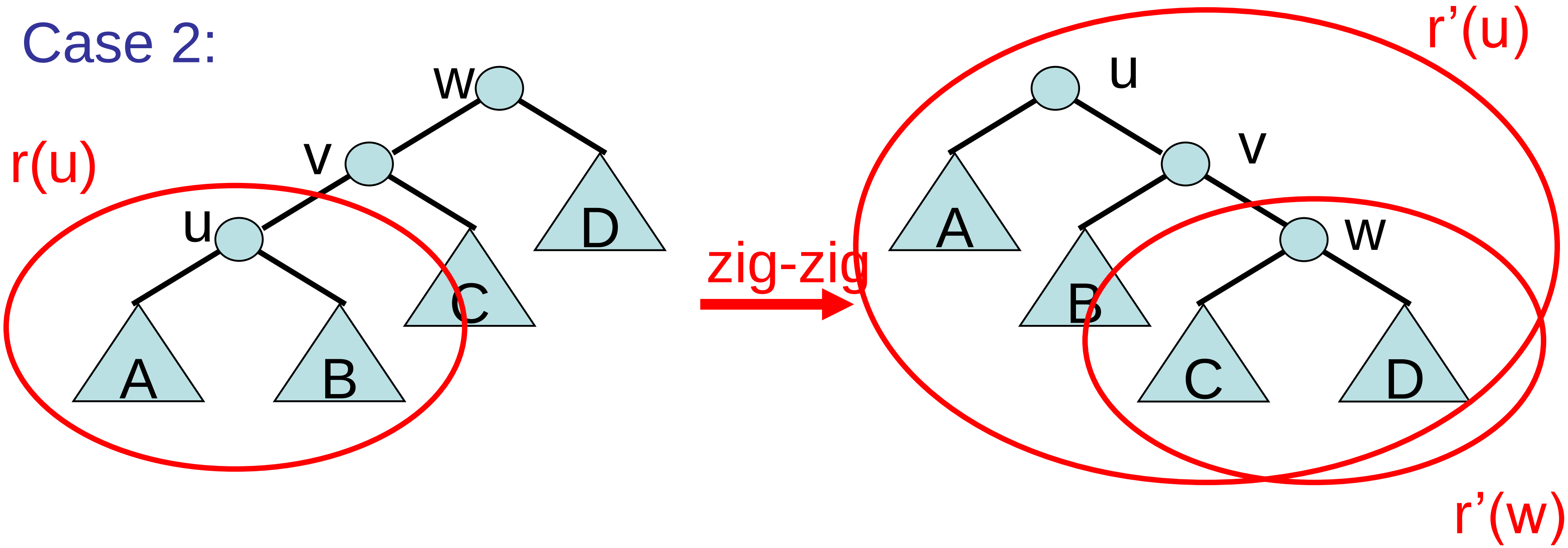

Claim: It holds that

$$r(u)+r'(w) \leq 2(r'(u)-1)$$

- Observe: There exist $0<x,y<1$ and scaling factor $c>0$ with $r(u)=\log(c\cdot x)$, $r'(w)=\log(c\cdot y)$, and $r'(u)\geq\log(c(x+y))$.
- Hence, the claim holds if $\log(c\cdot x)+\log(c\cdot y) \leq 2(\log(c(x+y))-1)$ for all $0<x,y<1$ and $c>0$.

# Splay Operation

Case 2:



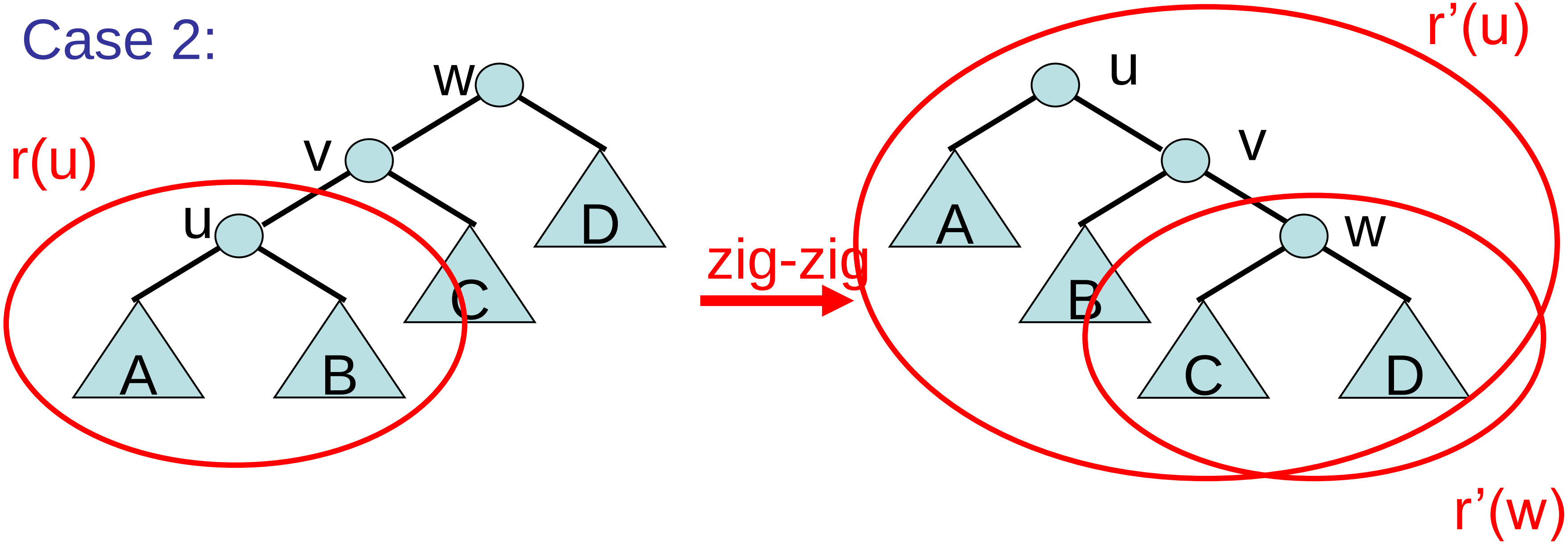

- For all 0<x,y<1 and c>0 holds:

  $\log(c\cdot x)+\log(c\cdot y) \le 2(\log(c(x+y))-1)$

  $\Leftrightarrow \log(x)+\log(y) \le 2(\log(x+y)-1)$

- WLOG set c so that c(x+y)=1. Let x'=c·x and y'=c·y.

# Splay Operation




- To show: for all $0<x',y'\leq 1$,with $x'+y'=1$:

  $\log(x')+\log(y') \leq 2(\log(1)-1) = -2$

- Or more generally: show for $f(x,y)=\log(x)+\log(y)$ that

  $f(x,y)\leq -2$ for all $x,y>0$ with $x+y\leq 1$

# Splay Operation

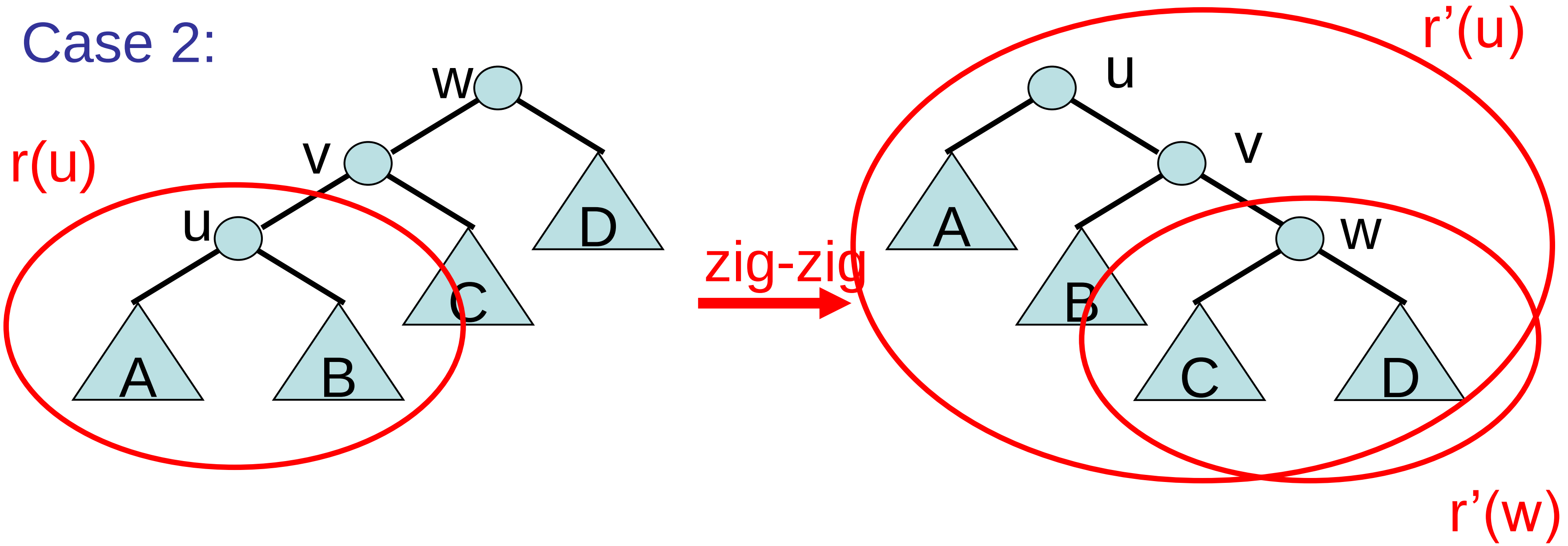
Lemma 3.2: In the area $x,y>0$ with $x+y\leq 1$, the function $f(x,y)=\log x + \log y$ has its maximum at $(½,½)$.

Proof:

- Reduce to univariate problem:
  - $\log x$ is monotonically increasing. Hence, WLOG maximum satisfies $x+y=1$, $x,y>0$.
  - Consider determining the maximum for $g(x) = \log x + \log (1-x)$
- High school calculus: (note base of log WLOG is e)
  - The only root of $g'(x) = 1/x - 1/(1-x)$ is at $x=1/2$.
  - For $g''(x)= -(1/x^2 + 1/(1-x)^2))$ it holds that $g''(1/2)<0$.
- Hence, $f$ has its maximum at $(½,½)$.

# Splay Operation

Case 2:

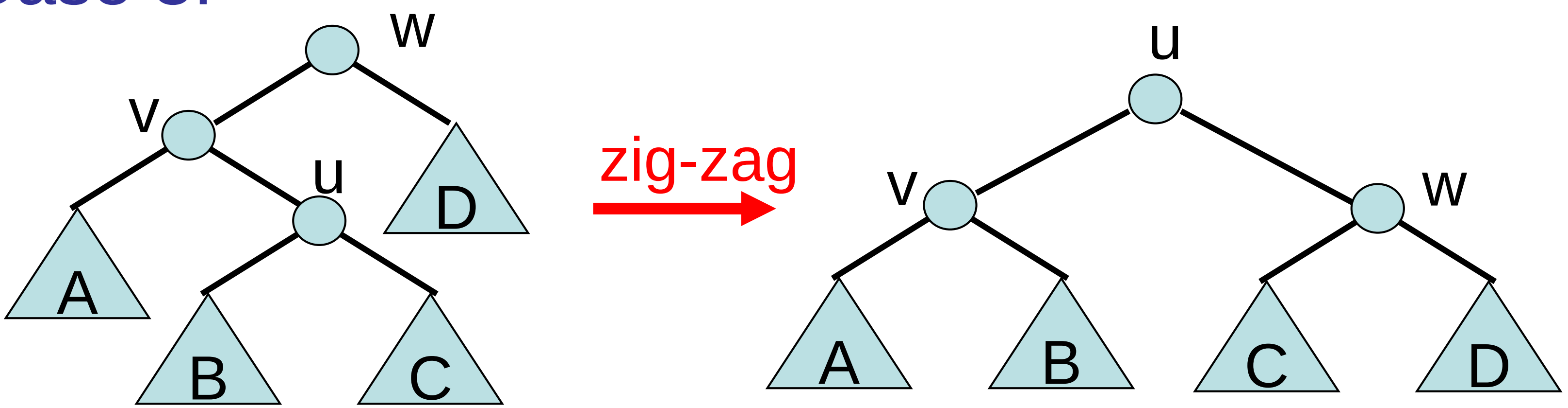



Hence, it holds that $f(x,y) \leq -2$ for all $x,y>0$ with $x+y\leq 1$, which implies the claim that $r(u)+r'(w) \leq 2(r'(u)-1)$, which was equivalent to obtaining upper bound

$$3(r'(u)-r(u)).$$

# Splay Operation

Case 3:




Amortized cost:

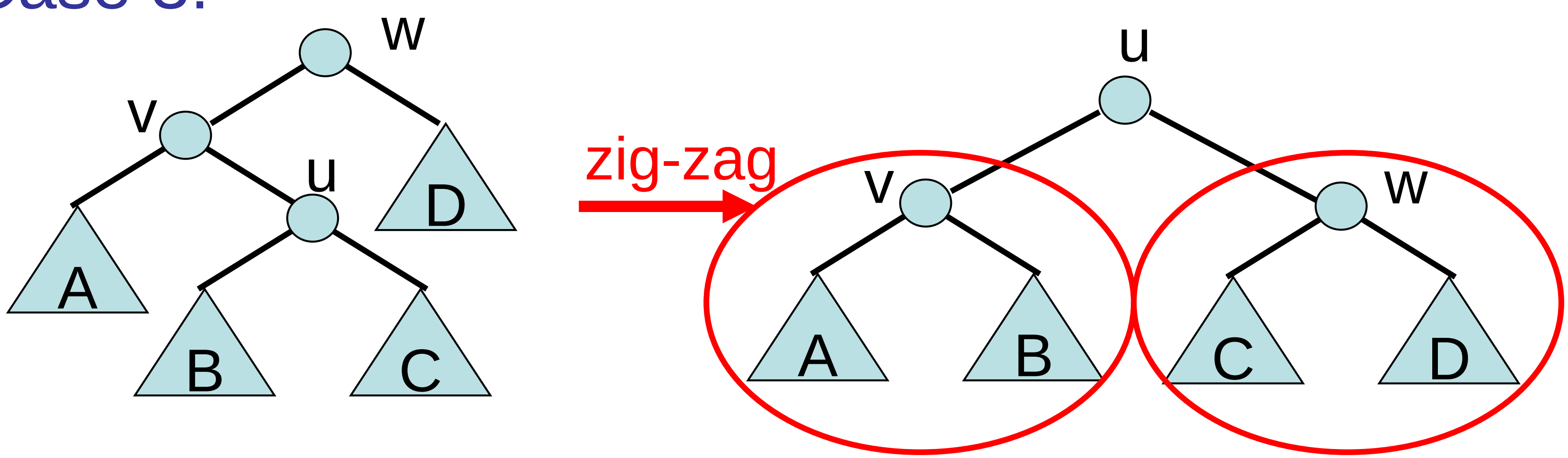
$\leq$ 2+r'(u)+r'(v)+r'(w)-r(u)-r(v)-r(w)

$\leq$ 2+r'(v)+r'(w)-2r(u) since r'(u)=r(w) and r(u)$\leq$r(v)

$\leq$ 2(r'(u)-r(u)) because…

# Splay Operation

Case 3:




…it holds that:

$$2+r'(v)+r'(w)-2r(u) \leq 2(r'(u)-r(u))$$

$$\Leftrightarrow \quad 2r'(u)-r'(v)-r'(w) \geq 2$$

$$\Leftrightarrow \quad r'(v)+r'(w) \leq 2(r'(u)-1)$$, which can be shown to hold

# Splay Operation

**Proof of Lemma 3.1:** (Follow-up)

Induction over the sequence of rotations.

- r and tw : rank and weight before the rotation
- r' und tw': rank and weight after the rotation
- For every rotation (i.e. zig, zig-zig, or zig-zag), the amortized cost is <= $1+3(r'(u)-r(u))$ (case 1) resp. $3(r'(u)-r(u))$ (cases 2 and 3)
- Summation of the costs gives at most (x: root)
  $1 + \sum_{Rotations} 3(r'(u)-r(u)) = 1+3(r(x)-r(u))$
  - 1. Why do we only add 1 before the summation?
  - 2. Why do we get a telescoping series above?

# Splay Operation

- Tree weight of tree $T$ with root $x$:
  $tw(x) = \sum_{y \in T} w(y)$
- Rank of node $x$: $r(x) = \log(tw(x))$
- Potential of tree $T$: $\phi(T) = \sum_{x \in T} r(x)$

Lemma 3.1: Let $T$ be a Splay tree with root $x$ and $u$ be a node in $T$. The amortized cost for $splay(u,T)$ is at most $1+3(r(x)-r(u)) = 1+3 \cdot \log(tw(x)/tw(u))$.

Corollary 3.3: Let $W = \sum_x w(x)$ and $w_i$ be the weight of key $k_i$ in the $i$-th search call (recall we assume $k_i$ is in tree). For $m$ search operations, the amortized cost is $O(m + \sum_{i=1}^m \log (W/w_i))$.

# Splay Tree

Theorem 3.4: The runtime for $m$ successful search operations in a Splay tree $T$ with $n$ elements is at most

$$O(m+(m+n)\log n).$$

Proof:

- Let $w(x) = 1$ for all nodes $x$ in $T$.
- Then $W=n$ and $r(x) \leq \log W = \log n$ for all $x$ in $T$.
- For sequence $F$ of operations, total runtime satisfies $T(F) \leq A(F) + \phi(s_0)$ for any amortized cost function $A$ and any initial state $s_0$ (Recall: $A_X(s) := T_X(s) + (\phi(s') - \phi(s))$)
- $\phi(s_0) = \sum_{x\in T} r_0(x) \leq n \log n$
- Hence, Corollary 3.3 implies Theorem 3.4.

# Splay Tree

Suppose we have a probability distribution for the search requests, where each key in tree is searched for at least once.

- $p(x)$ : probability of searching for key $x$
- $H(p) = \sum_x p(x) \cdot \log(1/p(x))$ : entropy of $p$

Theorem 3.5: The expected runtime for $m$ successful search operations in a Splay tree $T$ with $n$ elements is at most

$$O(m \cdot (1+H(p))).$$

Proof: Follows from proof of Theorem 3.4 with $w(x) = p(x)$ for all $x$, and assuming each item $x$ is searched for $m \cdot p(x)$ times.

Note: This proof requires us to relax our requirement that the potential function $\phi$ is non-negative. Why?

# Splay Tree

Something amazing:

For a *fixed* optimal Binary Search Tree where each key x in tree is searched for with probability p(x), one can show expected cost of a successful search is

$\Omega(H(p))$ (*entropy bound*).

Our Theorem 3.5 says Splay Trees are almost optimal, in that the cost per search scales as $O(1+H(p))$!

Note: $0 \le H(p) \le \log n$

Question: How does this O(1+H(p)) support the idea that Splay trees would be good for applications like caching?

# Splay Tree

So far, we assumed all searches were successful, i.e. the key we were searching for was in the tree.

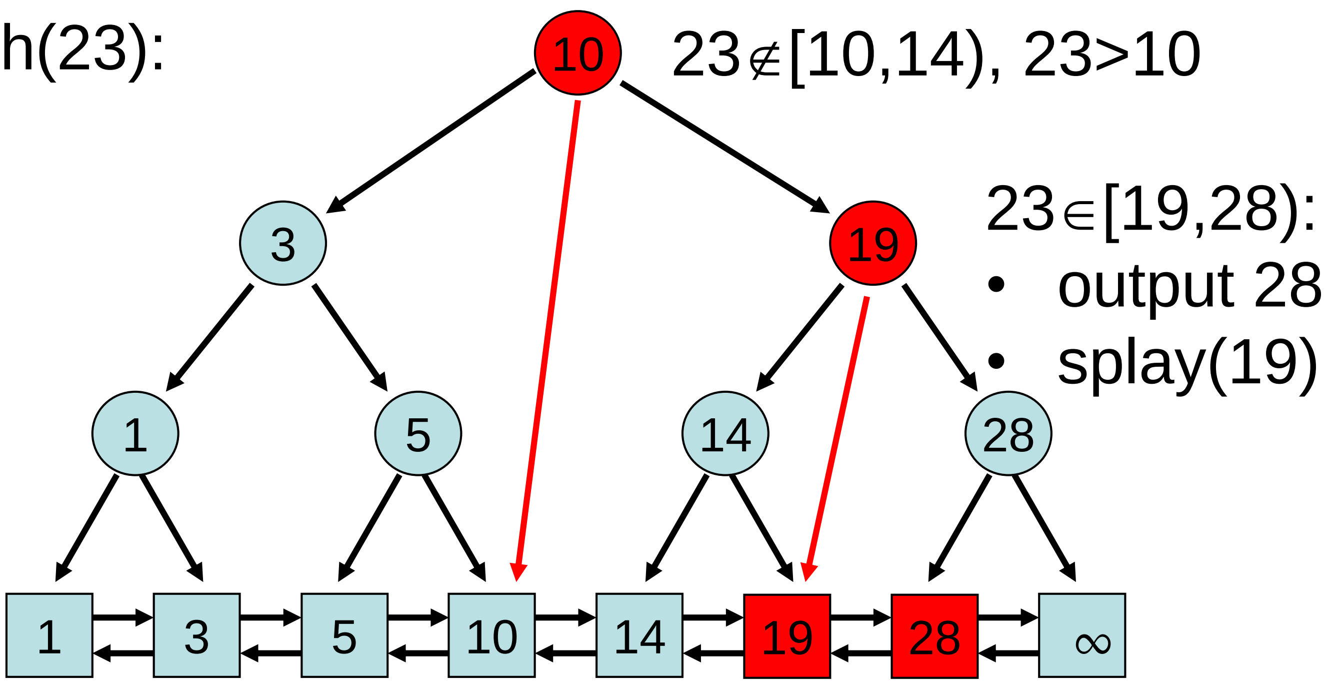
Q1: Where in our analysis did this assumption play a role?

Q2: What if we consider the more general case of allowing unsuccessful searches?

# Splay Tree – Unsuccessful Searches

- Instead of just successful searches, the Splay tree T should also support the search for the closest successor.

search(23):



# Splay Tree – Unsuccessful Searches

- To obtain a low amortized time bound, we associate with a key $x$ in $T$ the search range $[x,x_+)$ (including $x$ but excluding $x_+$), where $x_+$ is closest successor of $x$ in $T$.
- Each search range $[x,x_+)$ is associated with a weight $w([x,x_+))$. Using that, we can revise Corollary 3.3 to:

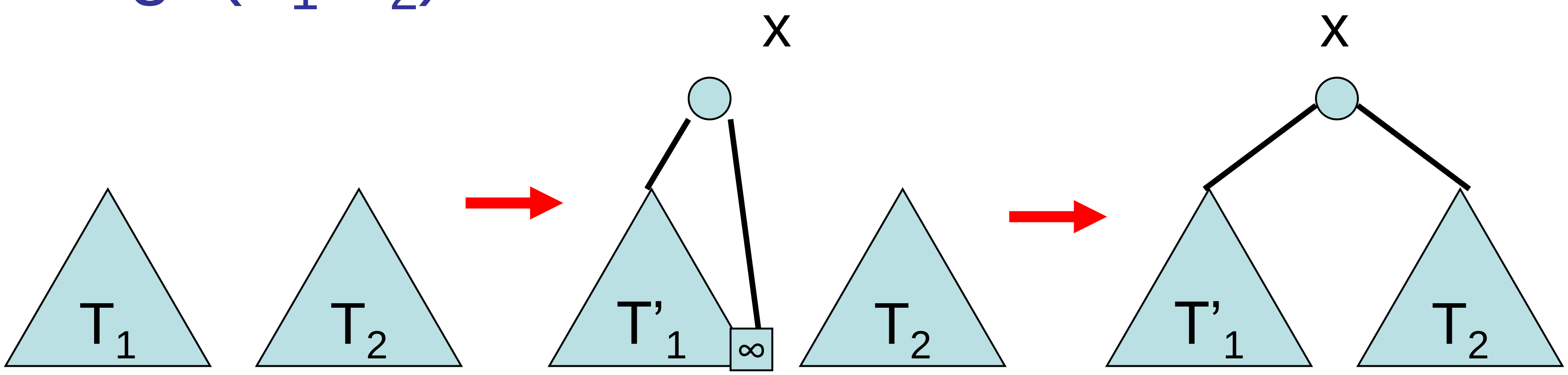
Corollary 3.3': Let $W=\sum_x w(x)$ and $w_i$ be the weight of the range $[x,x_+)$ containing the $i$-th search key. For $m$ search operations, the amortized cost is

$$O(m + \sum_{i=1}^{m} \log (W/w_i)).$$

# Splay Tree Operations

Let $T_1$ and $T_2$ be two Splay trees with key(x)<key(y) for all $x \in T_1$ und $y \in T_2$.

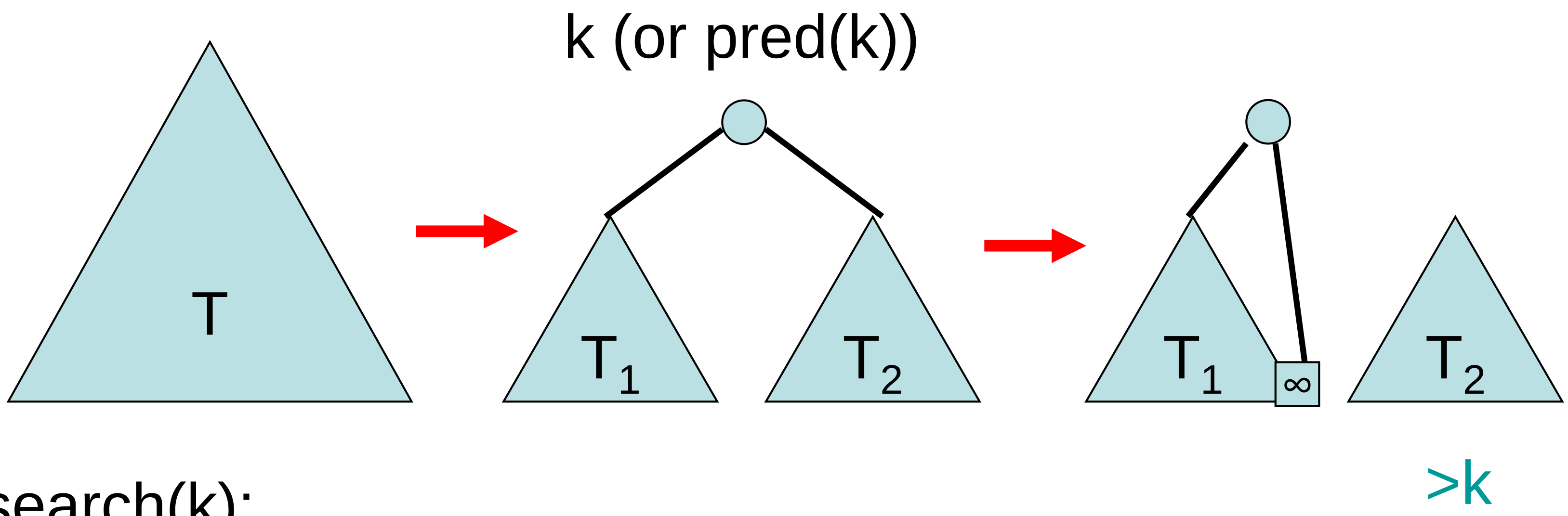
merge($T_1$,$T_2$):




Take max. element $x<\infty$ in $T_1$ and splay it up to root

# Splay Tree Operations

split(k,T): returns two trees as follows




search(k):
causes splay(k)
or splay(pred(k))

# Splay Tree Operations

insert(e):

- insert like in binary search tree
- Splay operation to move key(e) to the root

delete(k):

- execute search(k) (splays k to the root)
- remove root and execute merge($T_1$,$T_2$) of the two resulting subtrees

# Splay Operations

- $k_-$: closest predecessor $\leq k$ in $T$
- $k_+$: closest successor $>k$ in $T$

Theorem 3.6: The amortized cost of the following operations in the Splay tree are:

- search(k): $O(1+\log(W/w([k_-,k_+))))$
- insert(e): $O(1+\log(W/w([key(e),key(e)_+))))$
- delete(k): $O(1+\log(W/w([k,k_+))) + \log((W-w([k,k_+)))/w([k_-,k))) )$

# Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- Patricia trie

# (a,b)-Trees

Problem: how to maintain balanced search tree

Idea:

- All nodes $v$ (except for the root) have degree $d(v)$ with $a \le d(v) \le b$, where $a \ge 2$ and $b \ge 2a-1$ (otherwise this cannot be enforced)
- All leaves have the same depth

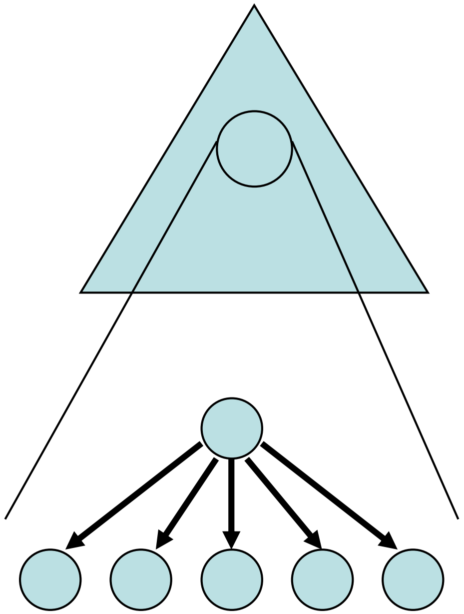
# (a,b)-Trees

Formally: for a tree node $v$ let

- $d(v)$ be the number of children of $v$
- $t(v)$ be the depth of $v$ (root has depth 0)

- Form Invariant:
  For all leaves $v,w$: $t(v)=t(w)$

- Degree Invariant:
  For all inner nodes $v$
  except for root: $d(v)\in[a,b]$,
  for root $r$: $d(r)\in[2,b]$
  (as long as #elements $>1$)



# (a,b)-Trees

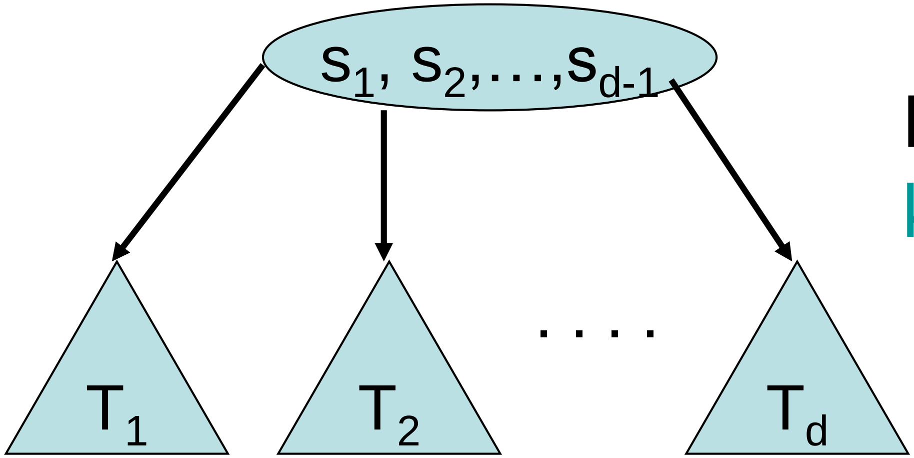
Lemma 3.10: An (a,b)-tree with $n$ elements has depth at most $1+\lfloor \log_a (n/2) \rfloor$

Proof:

- The root has degree $\geq 2$ and every other inner node has degree $\geq a$.
- At depth $t$ there are at least $2a^{t-1}$ nodes
- $n \geq 2a^{t-1} \Leftrightarrow t \leq 1+\lfloor \log_a(n/2) \rfloor$

# (a,b)-Trees

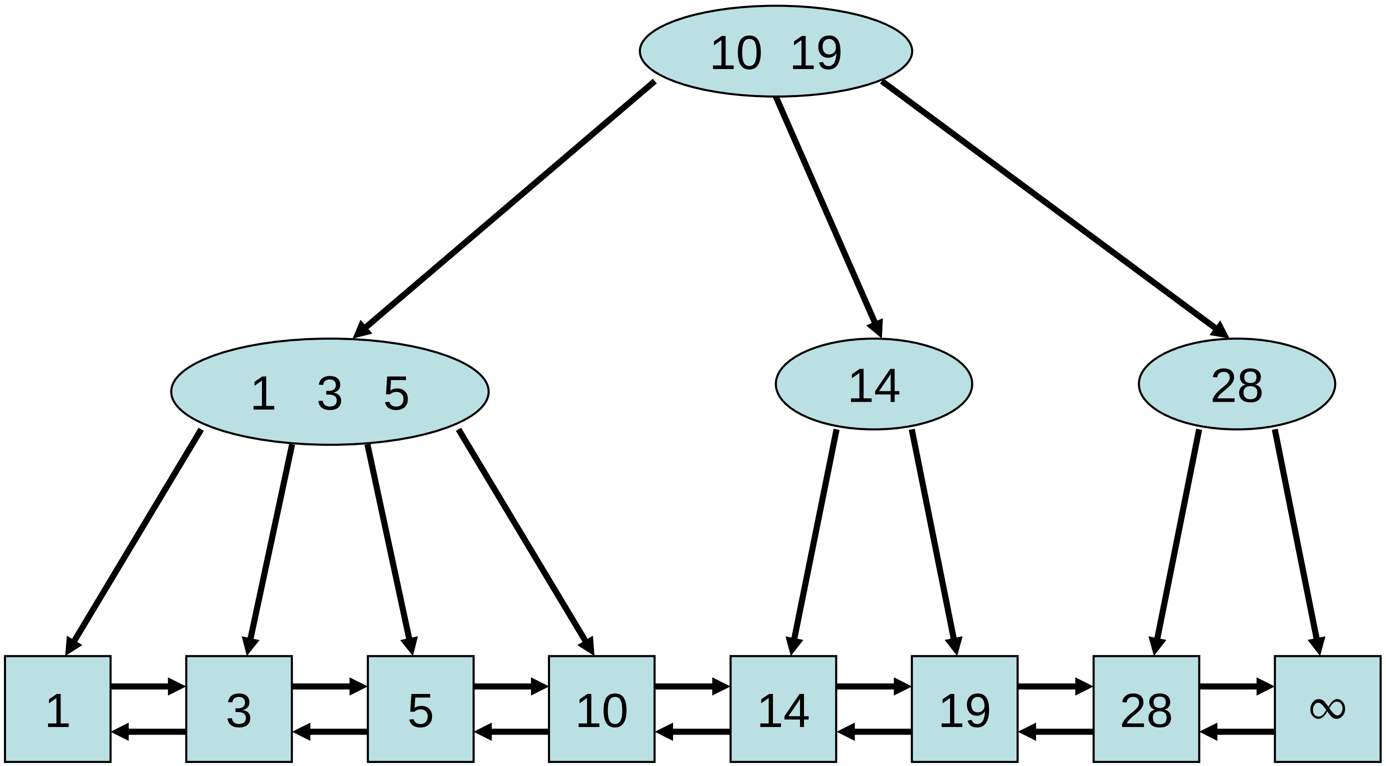
(a,b)-Tree-Rule:




For all keys $k$ in $T_i$ and $k'$ in $T_{i+1}$: $k \leq s_i < k'$

Then search operation easy to implement.

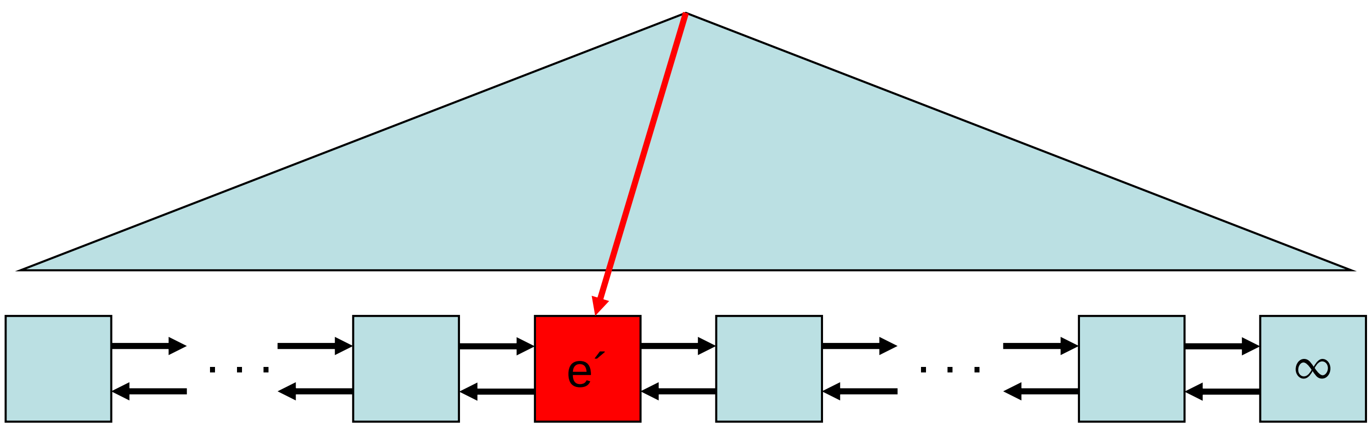
# Search(9)



# Insert(e) Operation

Strategy:

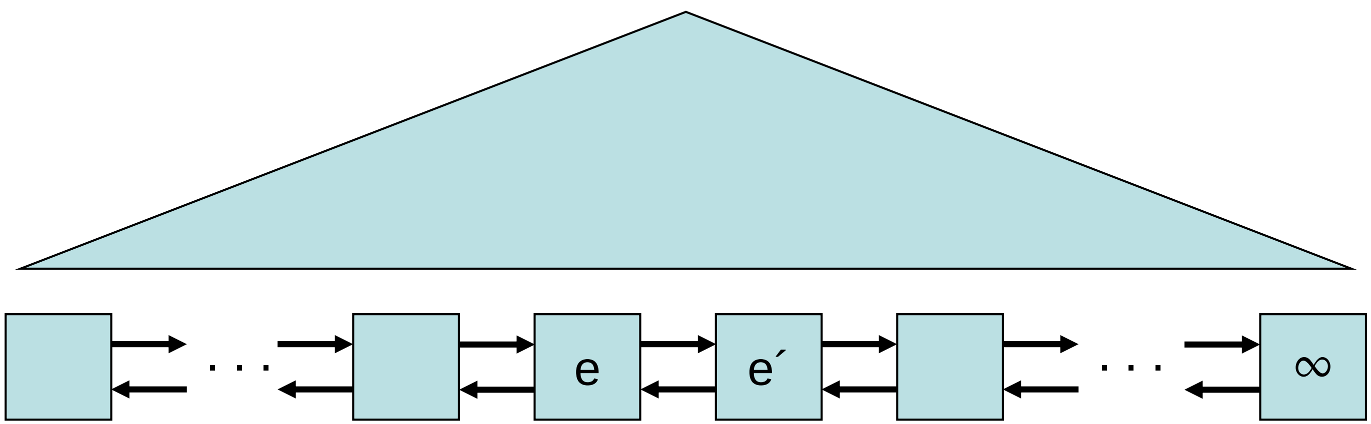
- First search(key(e)) until some e´ found in the list. If key(e´)>key(e), insert e in front of e´, otherwise replace e´ by e.



# Insert(e) Operation

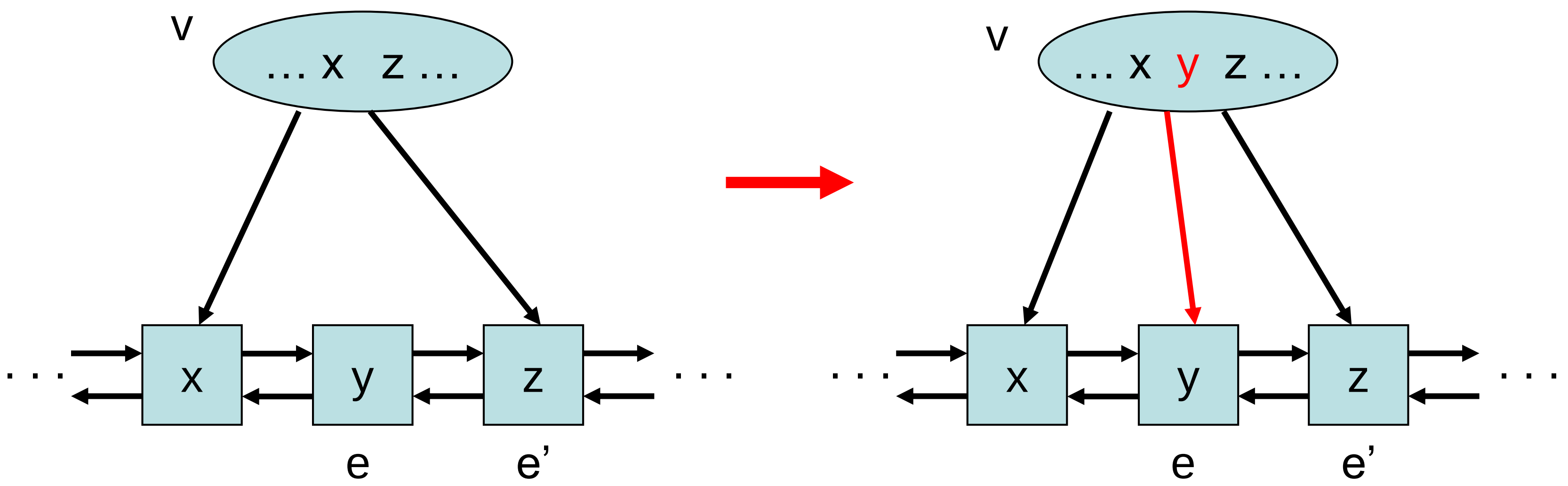
Strategy:

- First search(key(e)) until some e´ found in the list. If key(e´)>key(e), insert e in front of e´, otherwise replace e´ by e.



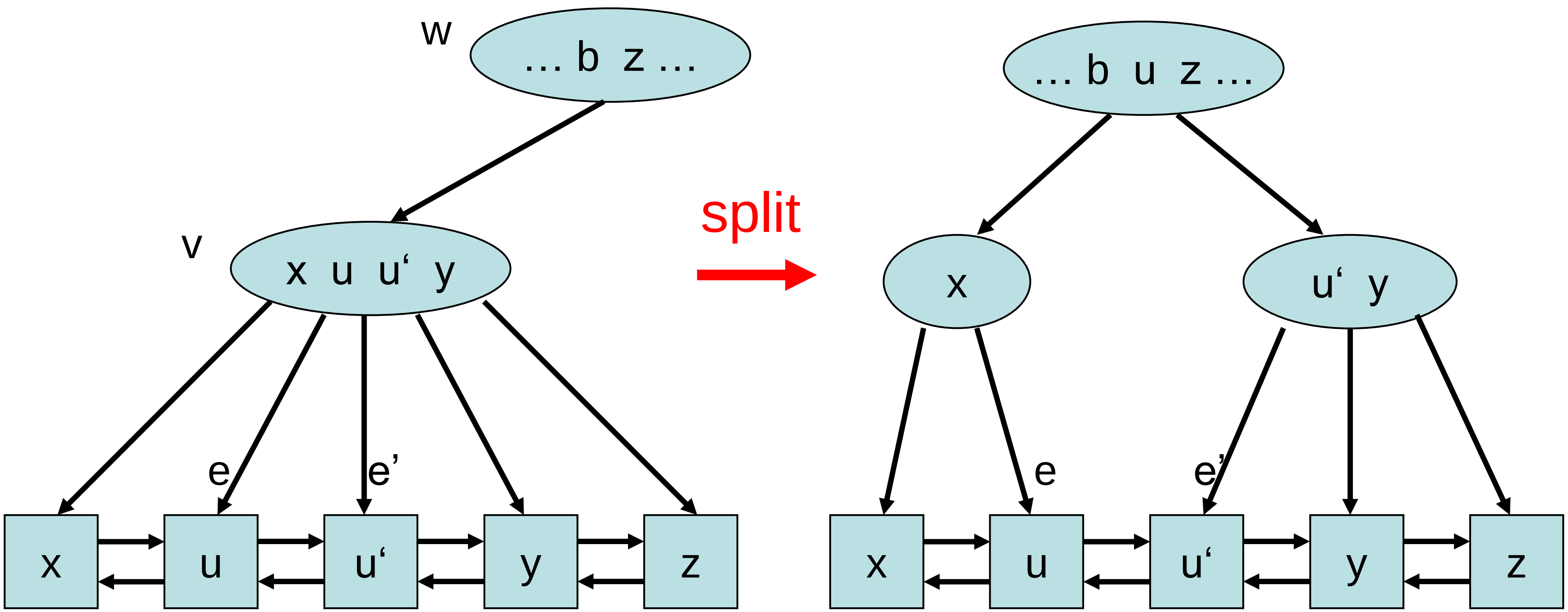
# Insert(e) Operation

- Add key(e) and pointer to e in tree node v which is parent of e´. If we still have $d(v) \in [a,b]$ after-wards, then we are done.



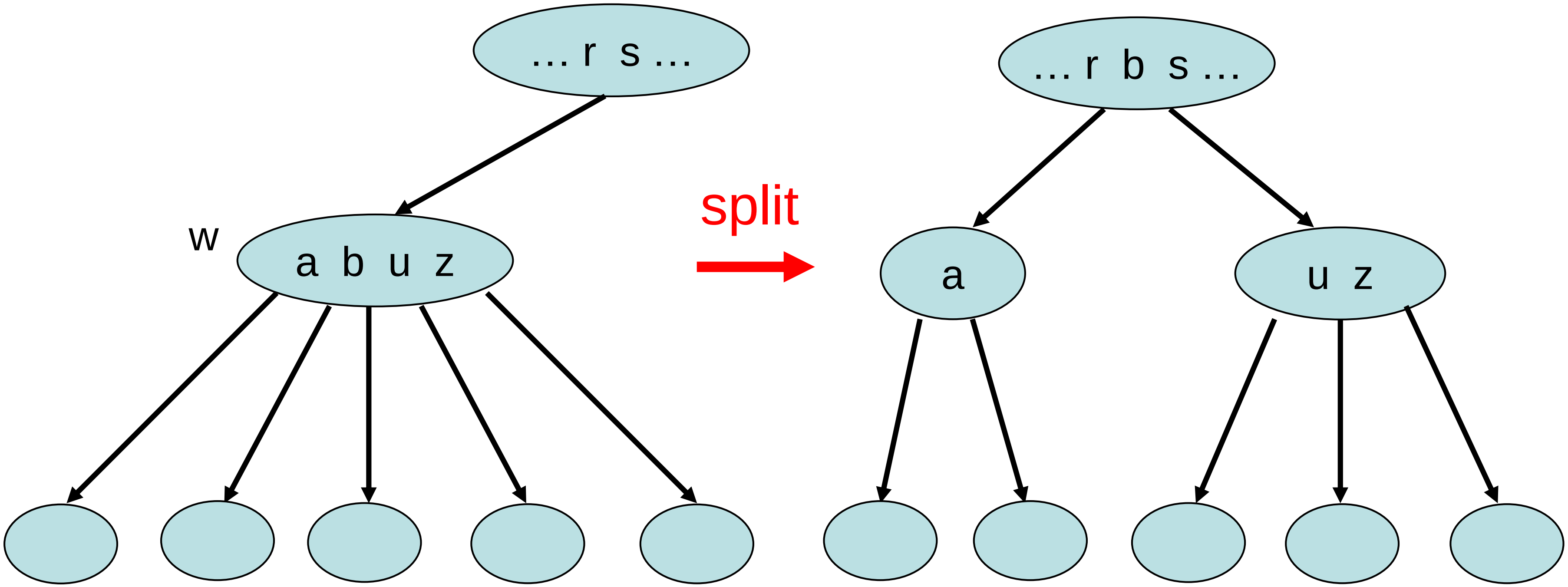
# Insert(e) Operation

- If d(v)>b, then cut v into two nodes. (Example: a=2, b=4)



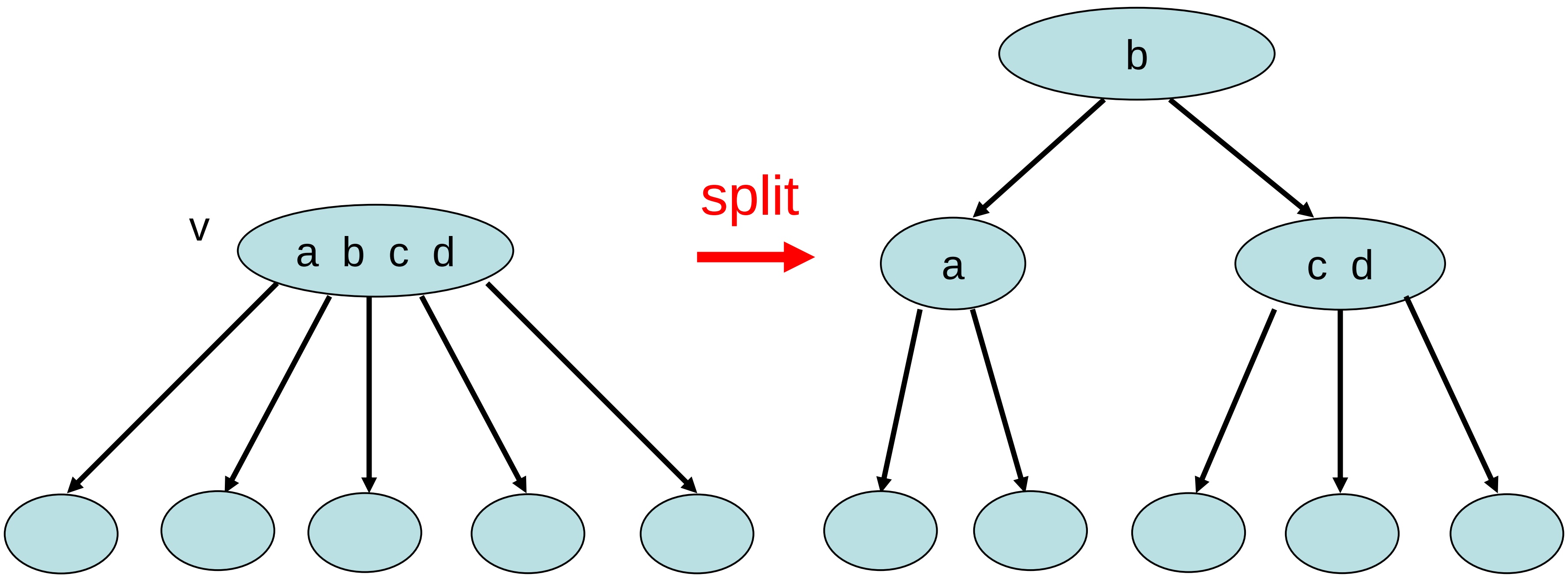
# Insert(e) Operation

- If after splitting v, d(w)>b, then cut w into two nodes (and so on, until all nodes have degree ≤b or we reached the root)



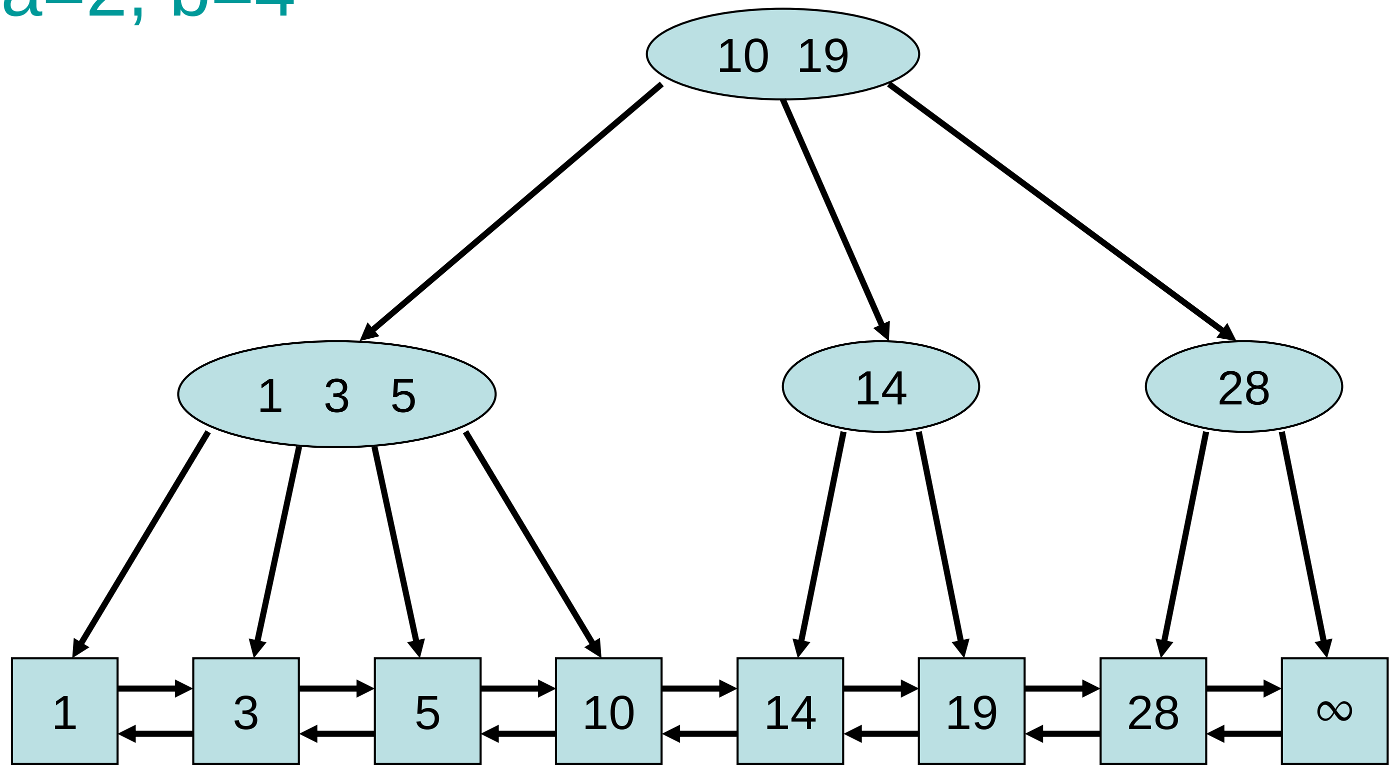
# Insert(e) Operation

- If for the root v of T, d(v)>b, then cut v into two nodes and create a new root node.



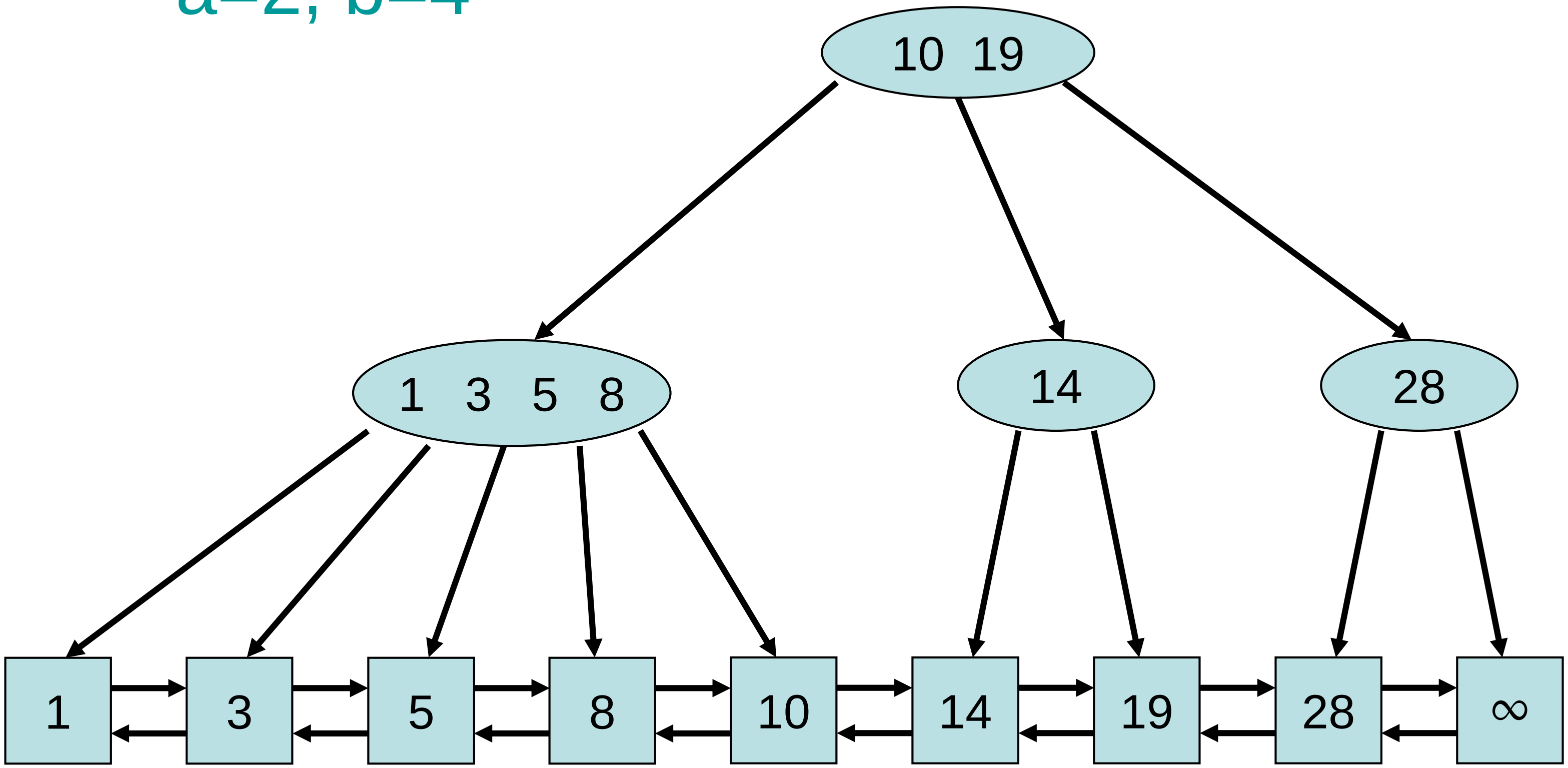
# Insert(8)

a=2, b=4



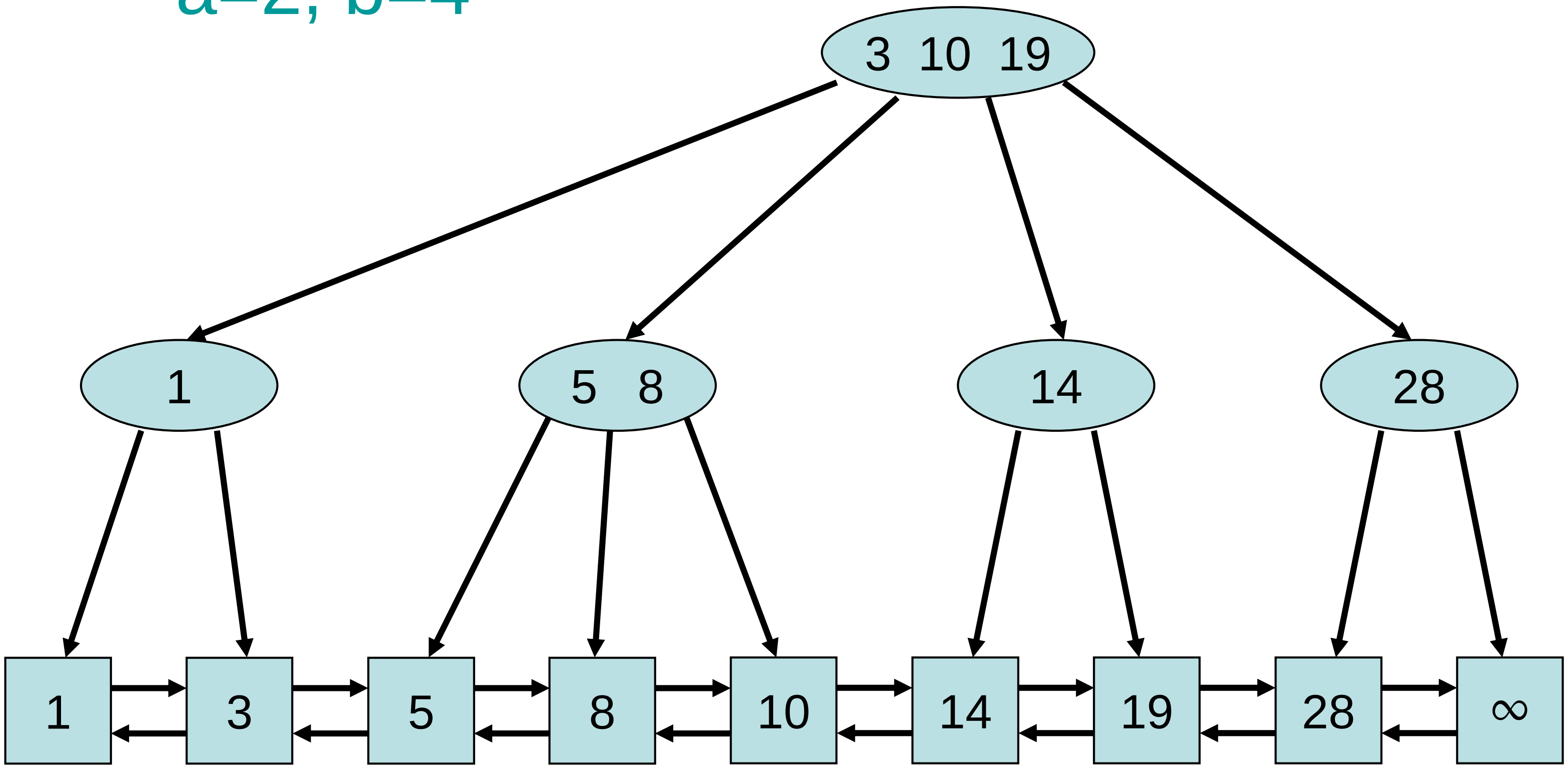
# Insert(8)

a=2, b=4



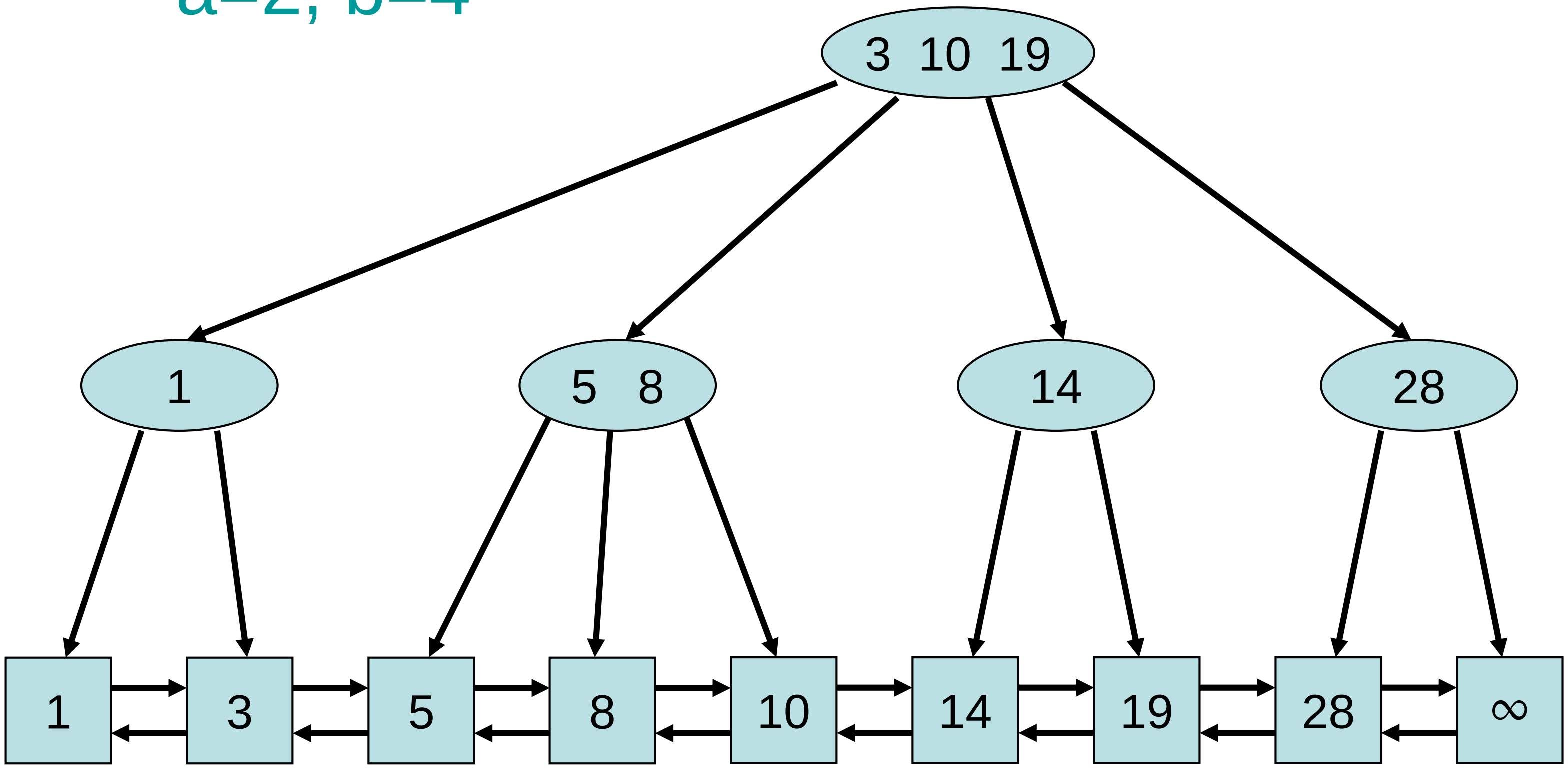
# Insert(8)

a=2, b=4



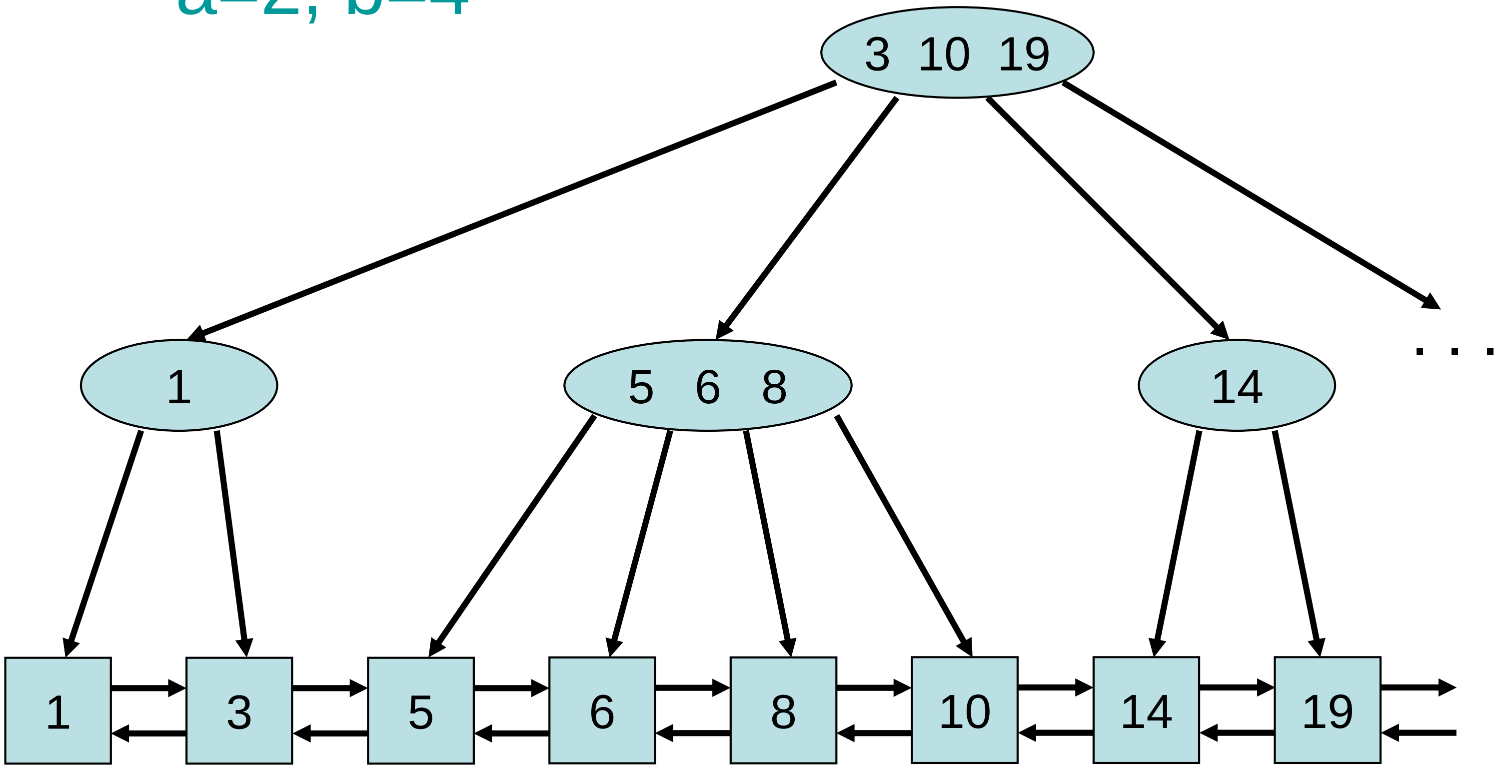
# Insert(6)

a=2, b=4



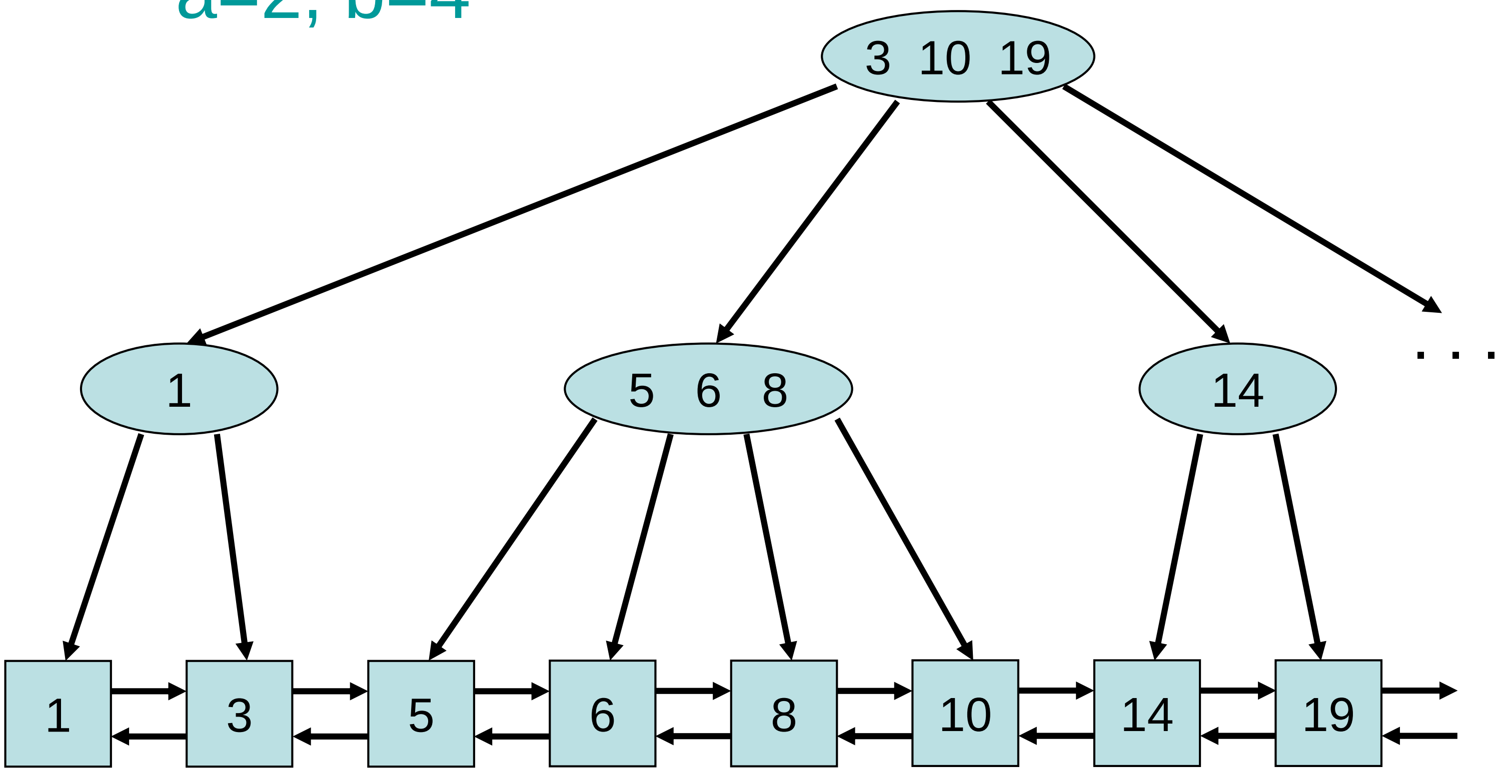
# Insert(6)

a=2, b=4



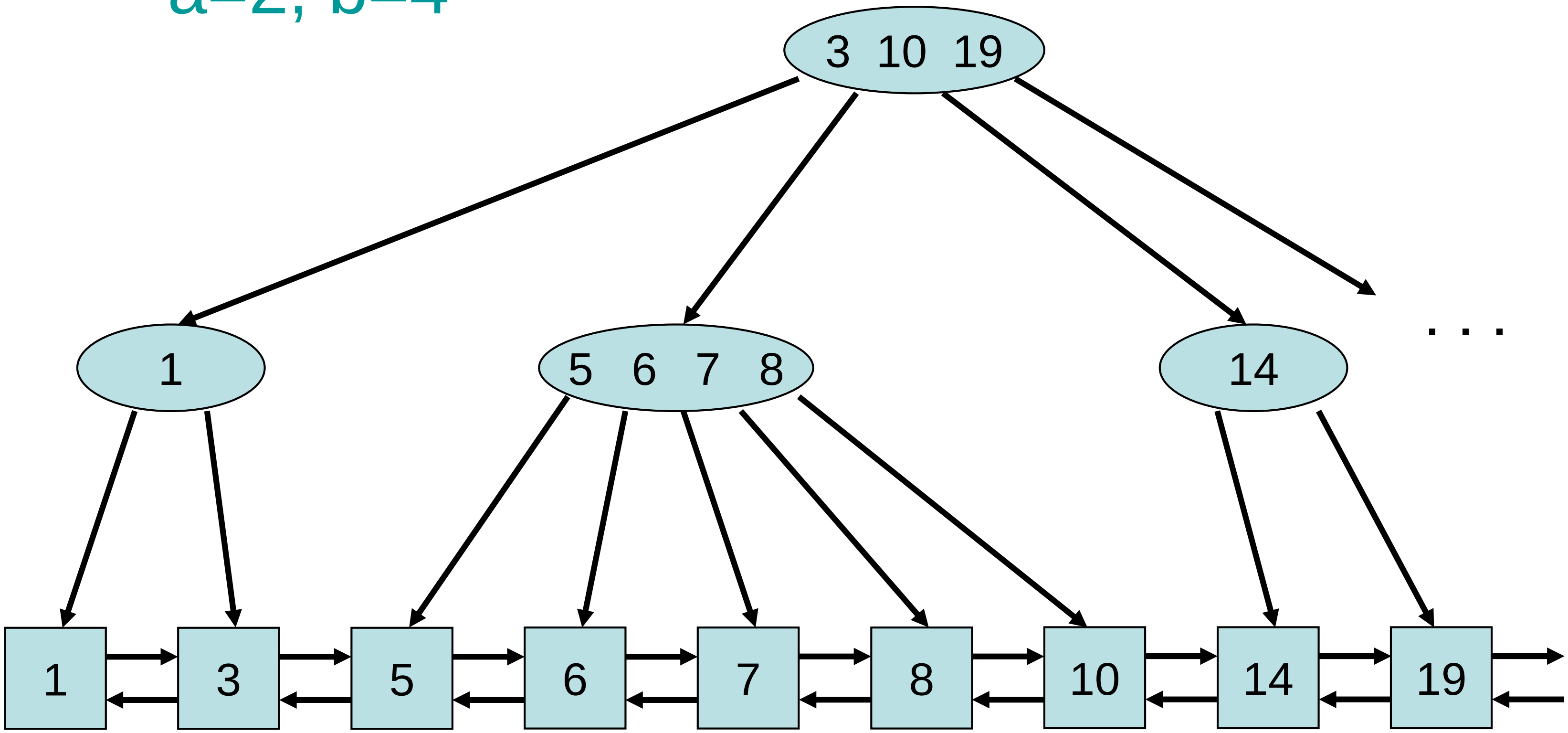
# Insert(7)

a=2, b=4



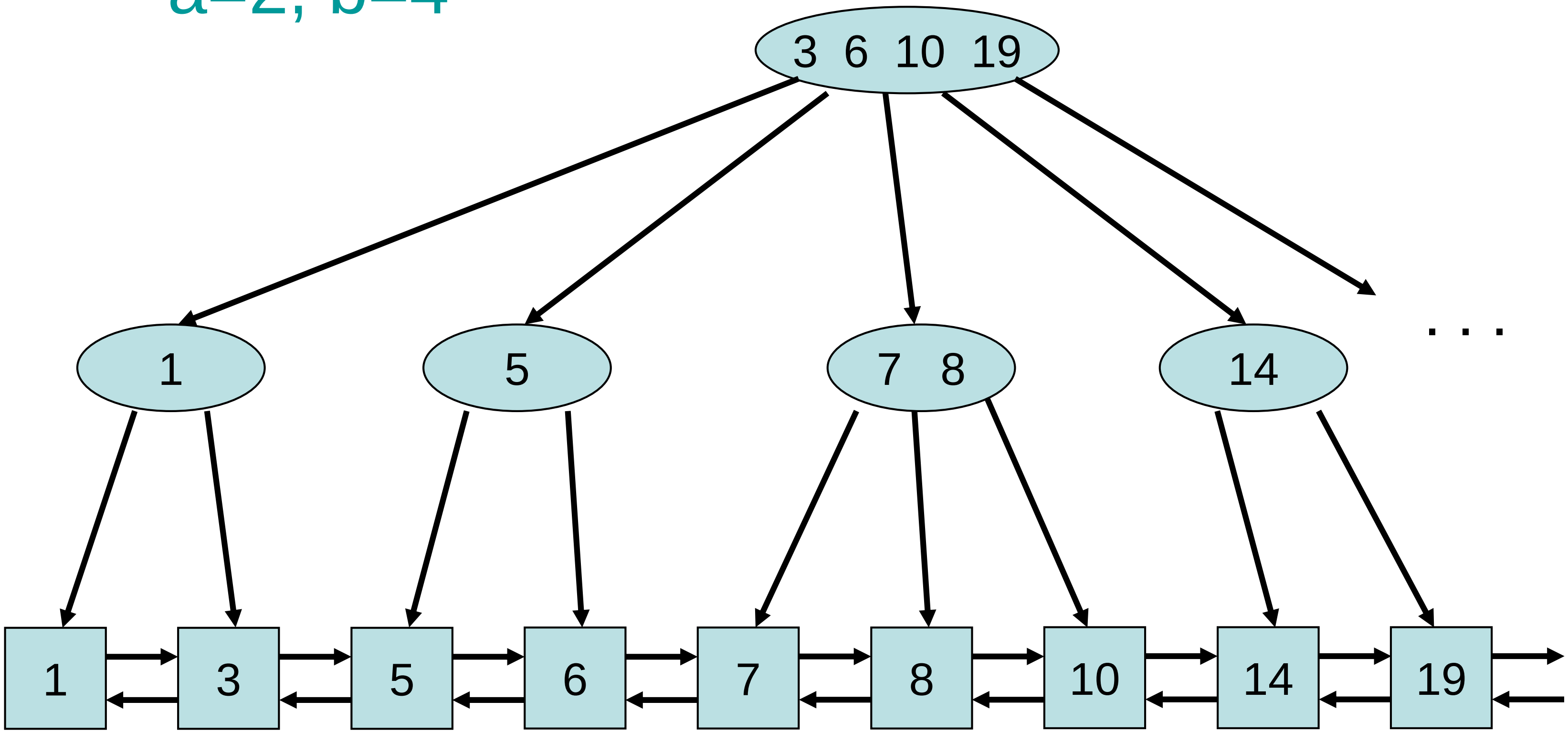
# Insert(7)

a=2, b=4

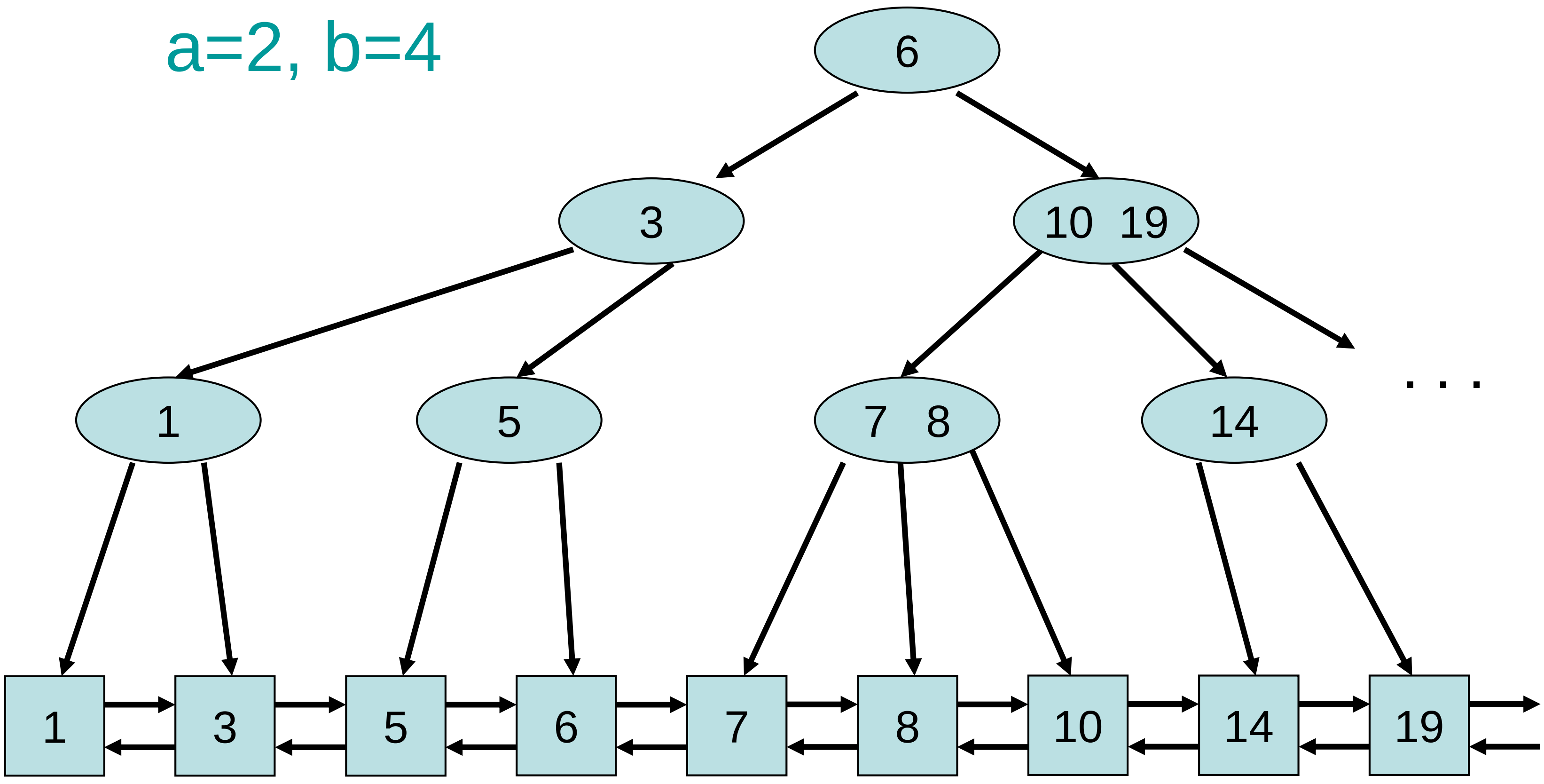


# Insert(7)

a=2, b=4



# Insert(7)



# Insert Operation

- Form Invariant:
  For all leaves $v,w$: $t(v)=t(w)$
  Satisfied by Insert!

- Degree Invariant:
  For all inner nodes $v$ except for the root:
  $d(v)\in[a,b]$, for root $r$: $d(r)\in[2,b]$

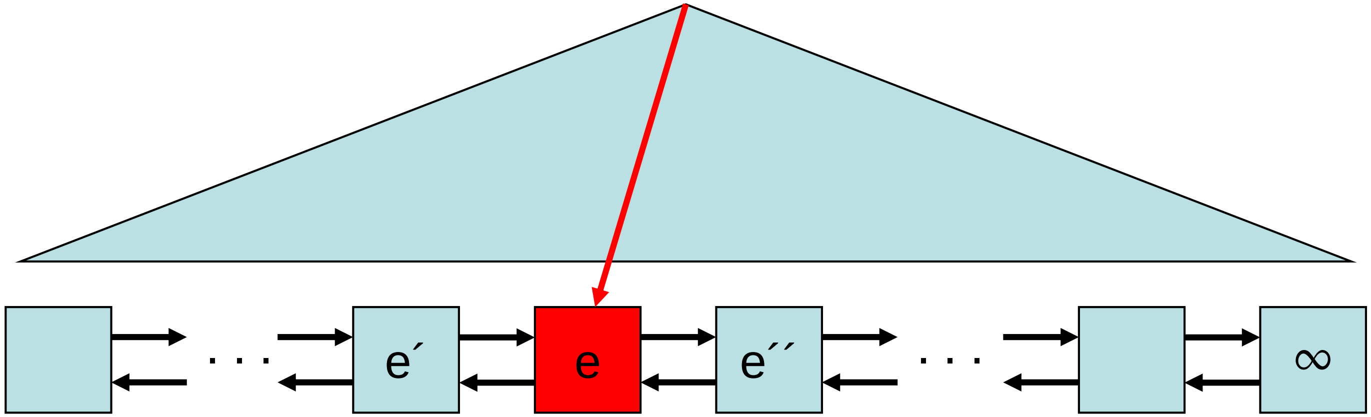
1) Insert splits nodes of degree $b+1$ into nodes of degree $\lfloor(b+1)/2\rfloor$ and $\lceil(b+1)/2\rceil$. If $b\geq 2a-1$, then both values are at least $a$.

2) If root has reached degree $b+1$, then a new root of degree $2$ is created.

# Delete(k) Operation

Strategy:

- First search(k) until some element e is reached in the list. If key(e)=k, remove e from the list, otherwise we are done.



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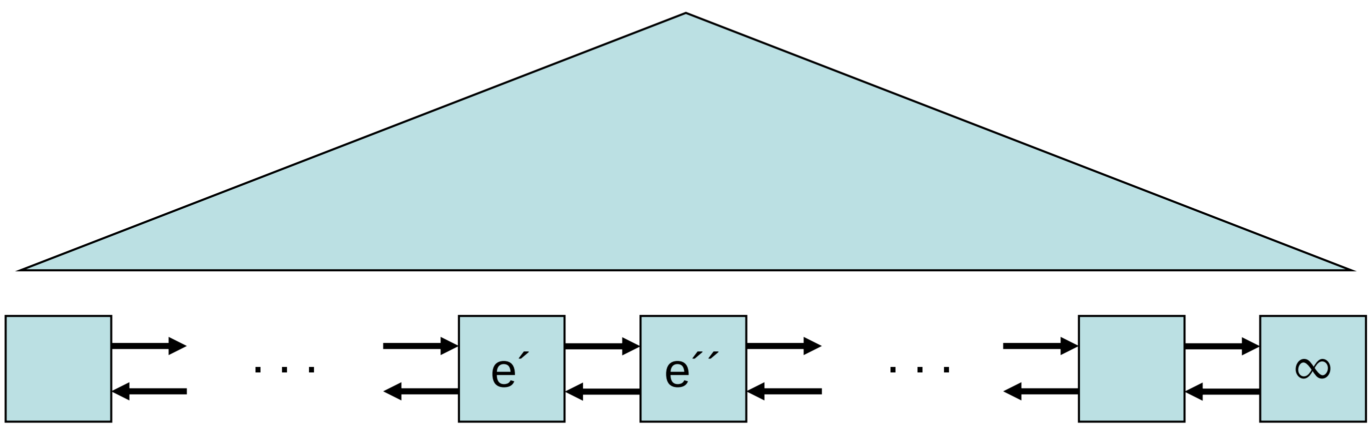
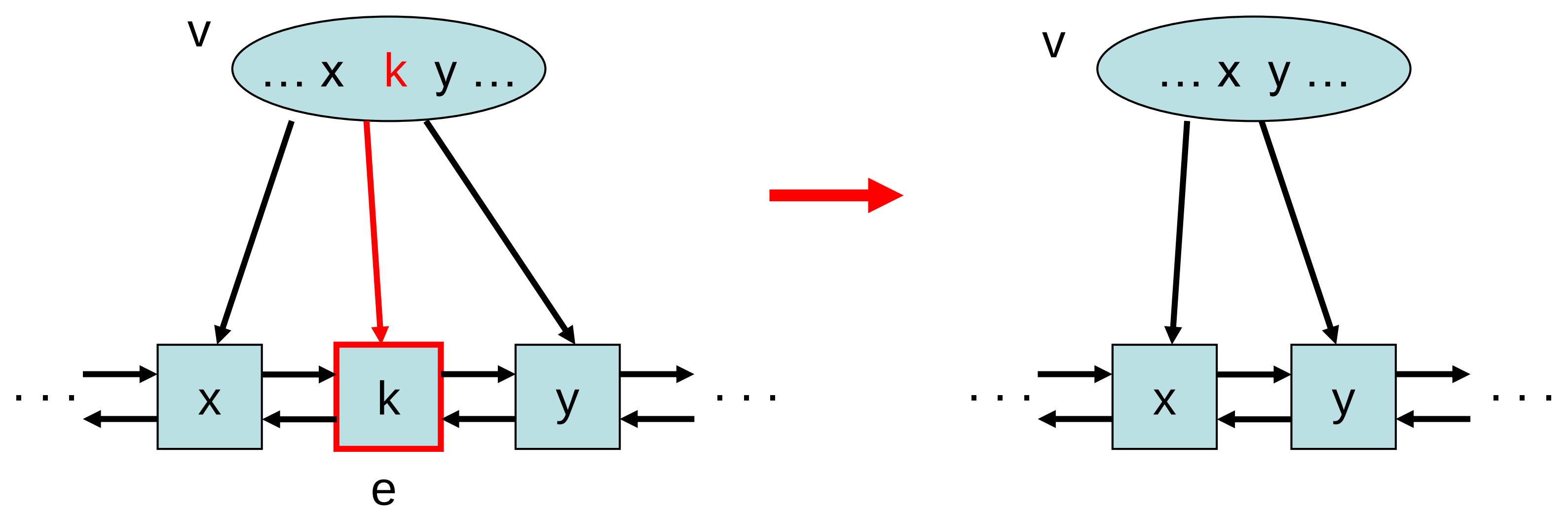


# Delete(k) Operation

- Remove pointer to e and key k from the leaf node v above e. (e rightmost child: perform key exchange like in binary tree!) If afterwards we still have $d(v) \geq a$, we are done.



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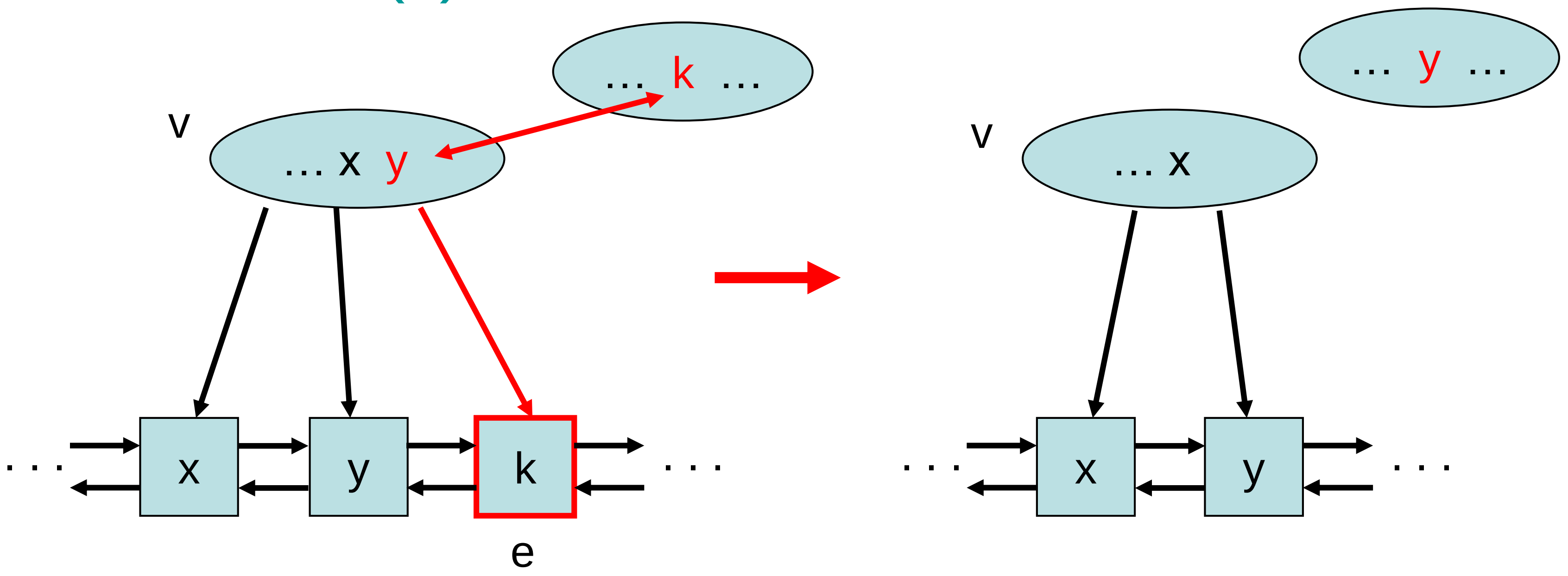
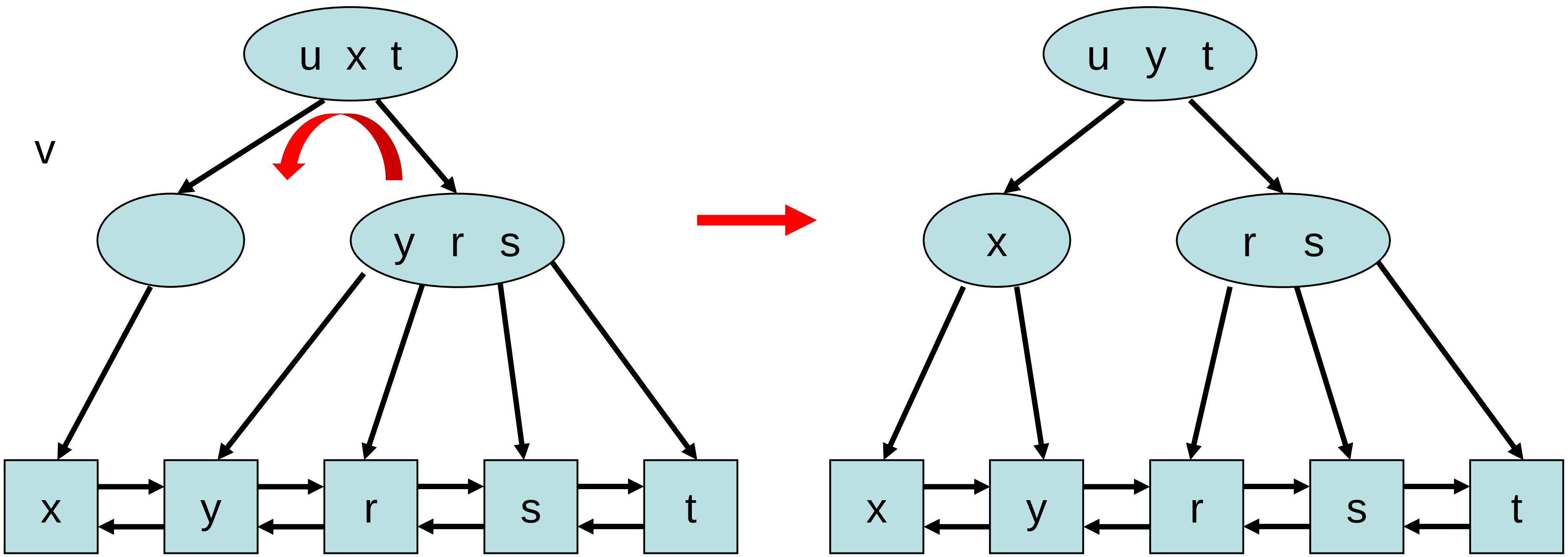


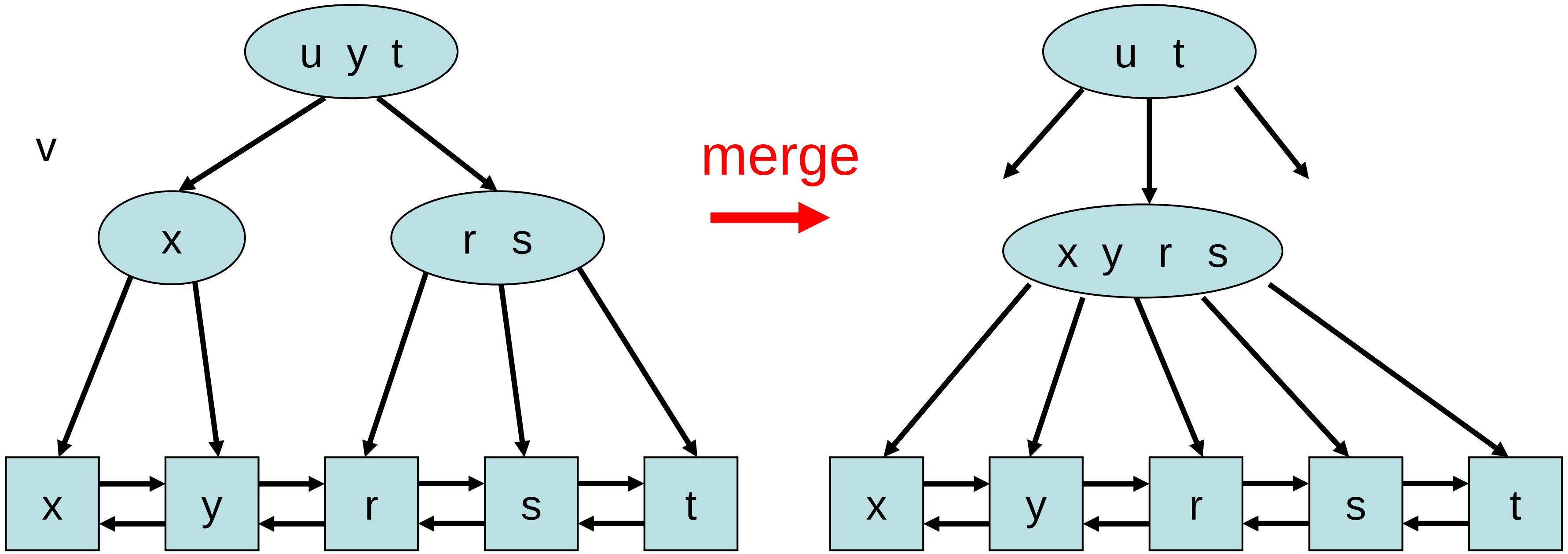
# Delete(k) Operation

- If d(v)<a and the preceding or succeeding sibling of v has degree >a, steal an edge from that sibling. (Example: a=2, b=4)



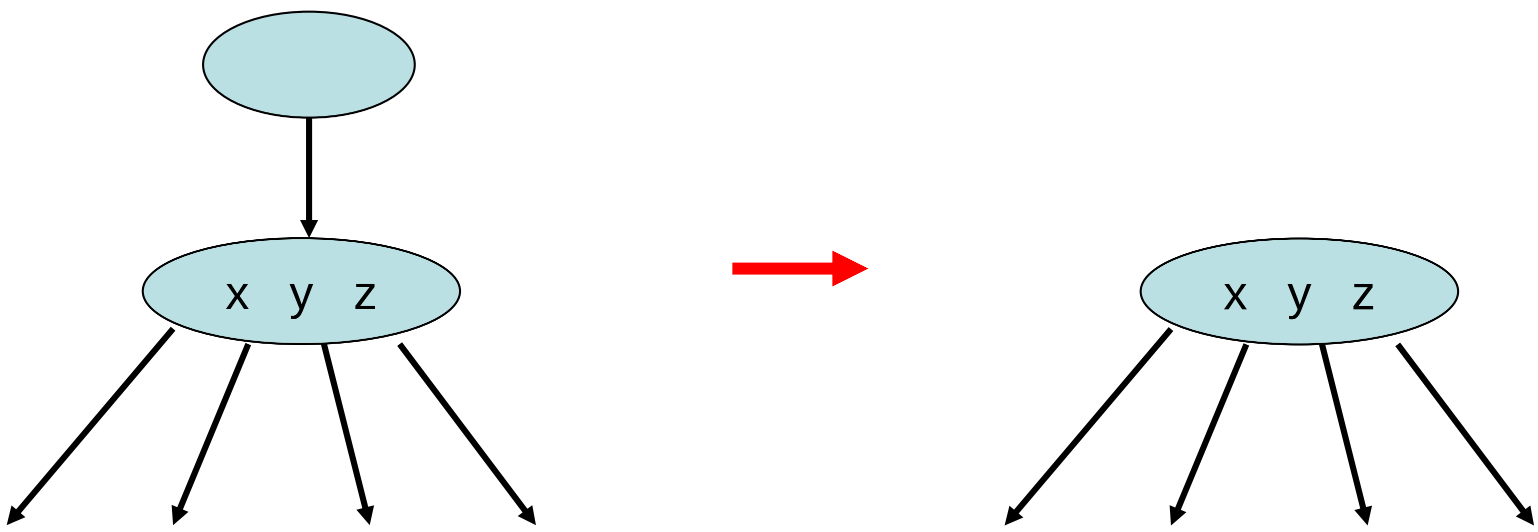
# Delete(k) Operation

- If $d(v)<a$ and the preceding and succeeding siblings of $v$ have degree $a$, merge $v$ with one of these. (Example: $a=3$, $b=5$)



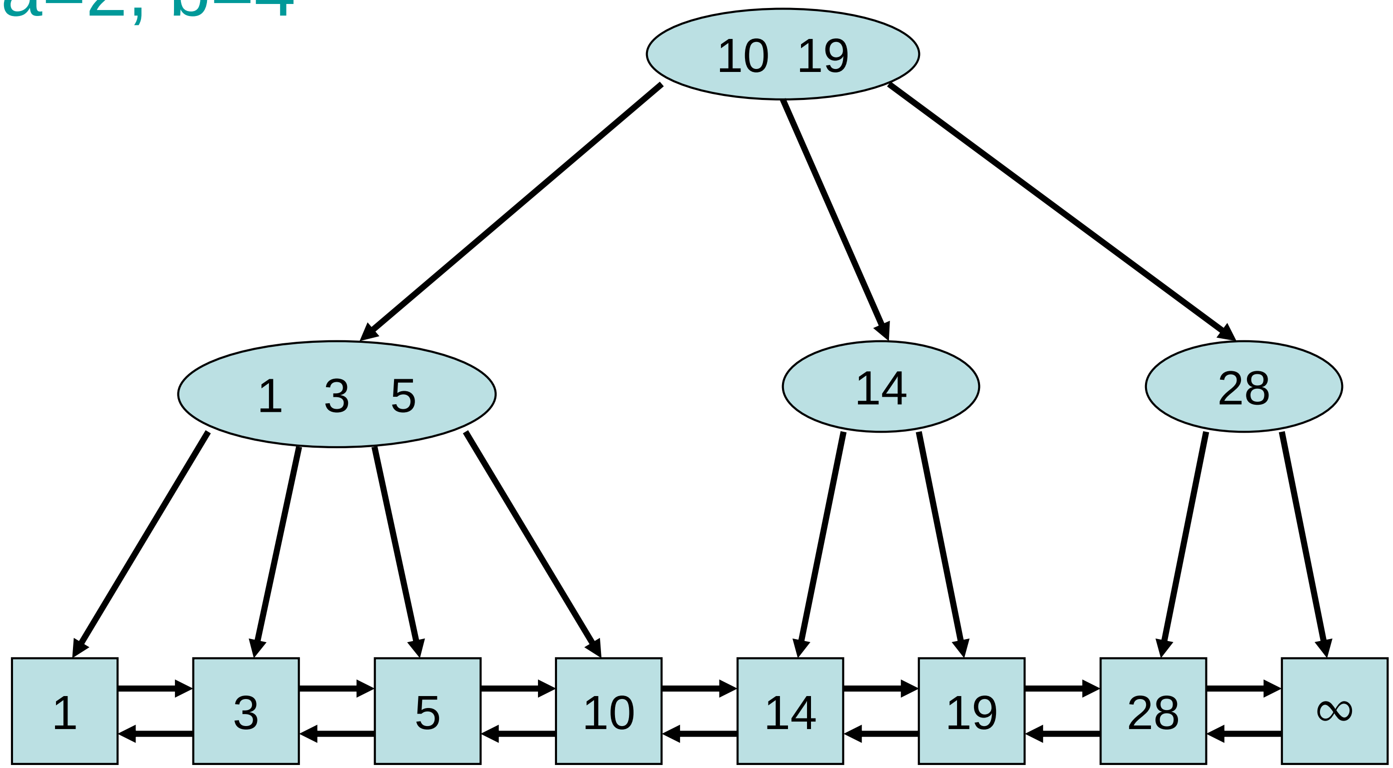
# Delete(k) Operation

- Perform changes upwards until all inner nodes (except for the root) have degree $\geq a$. If root has degree $<2$: remove root.



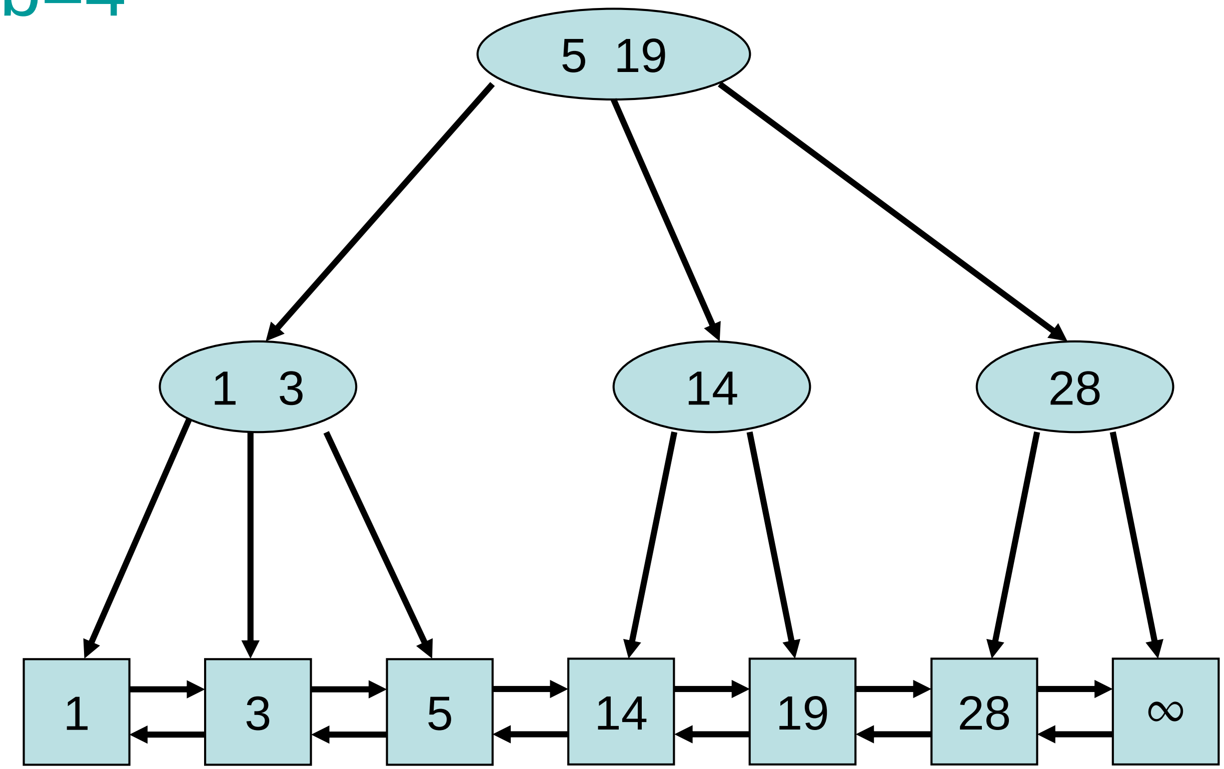
# Delete(10)

a=2, b=4



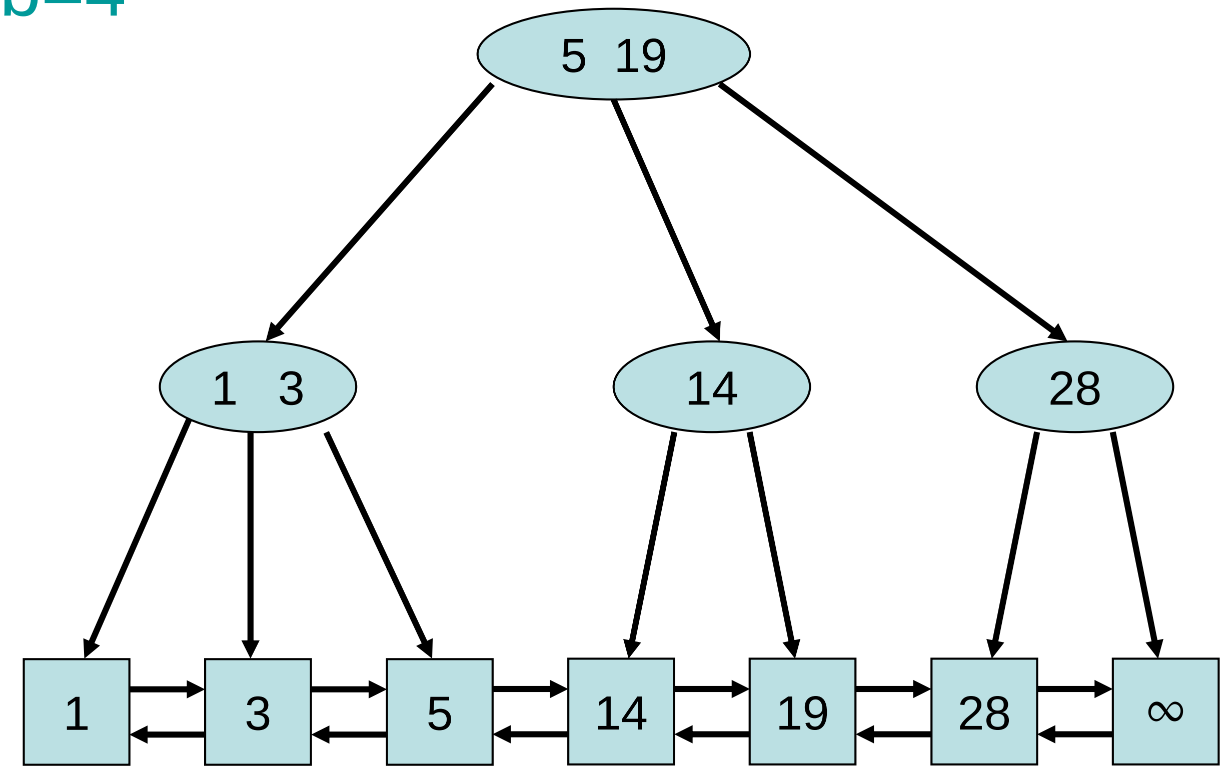
# Delete(10)

a=2, b=4



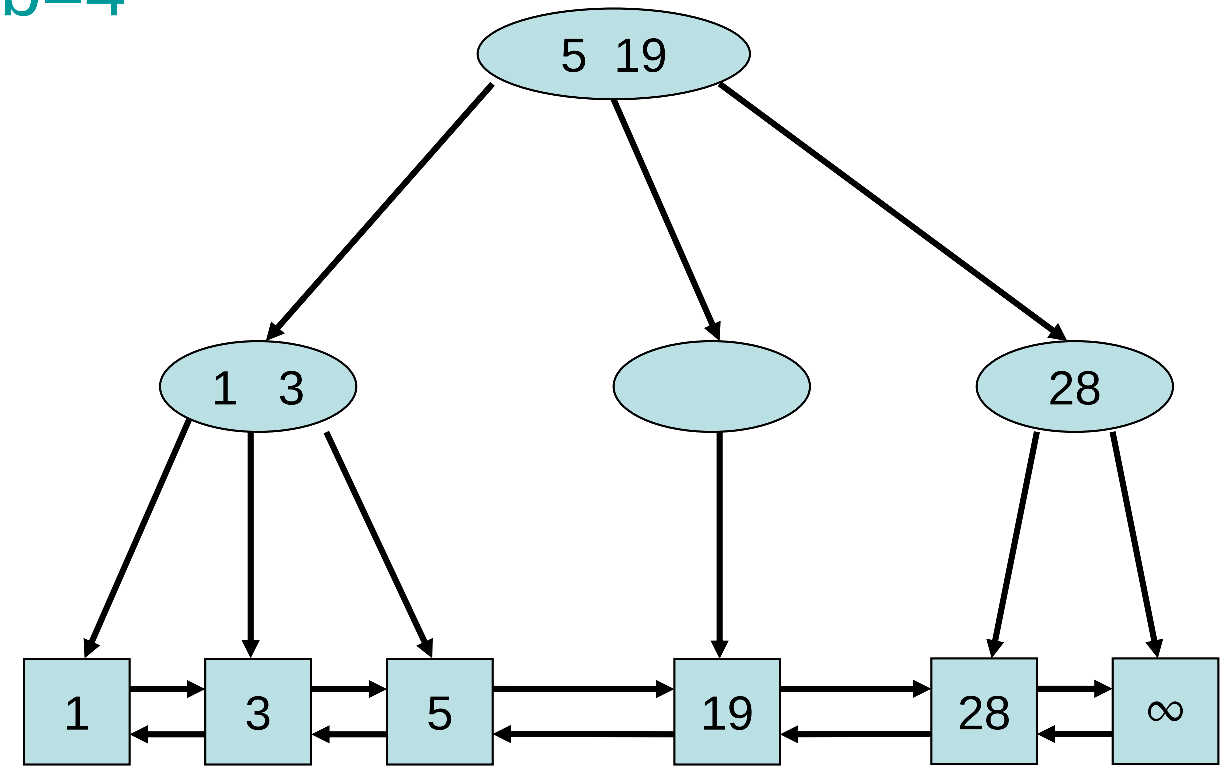
# Delete(14)

a=2, b=4



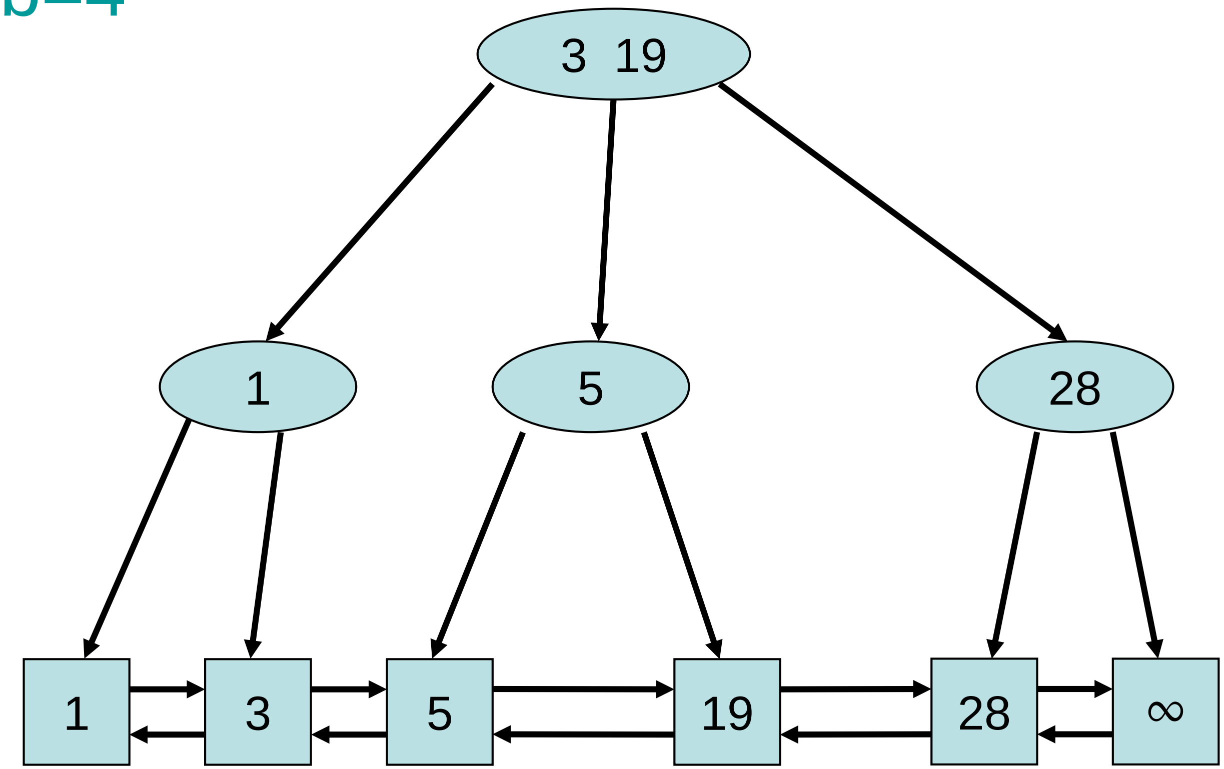
# Delete(14)

a=2, b=4



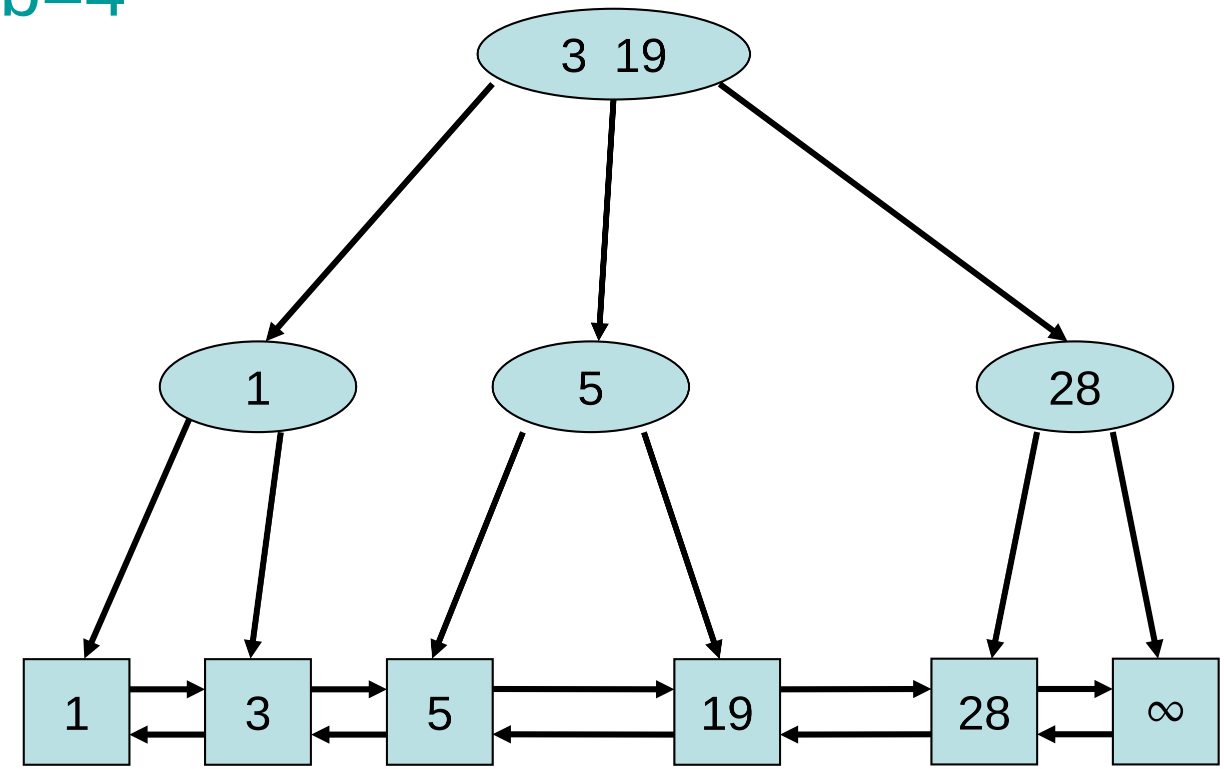
# Delete(14)

a=2, b=4



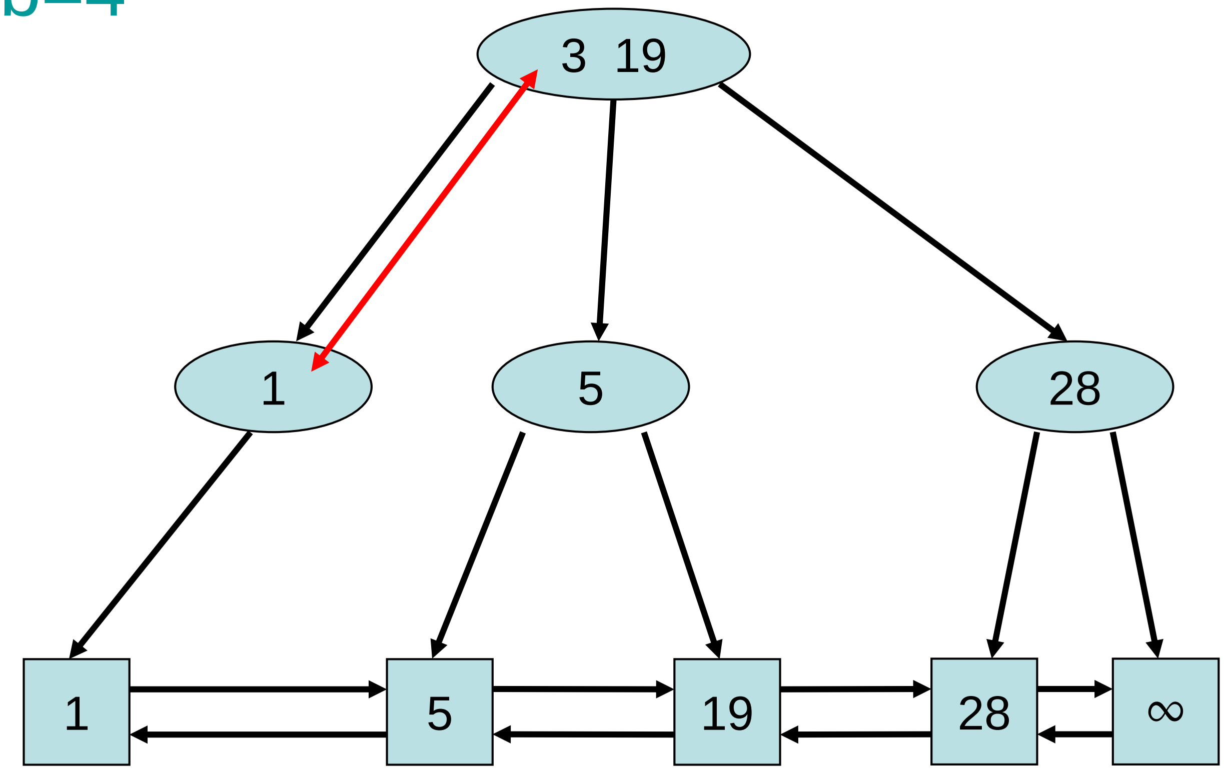
# Delete(3)

a=2, b=4



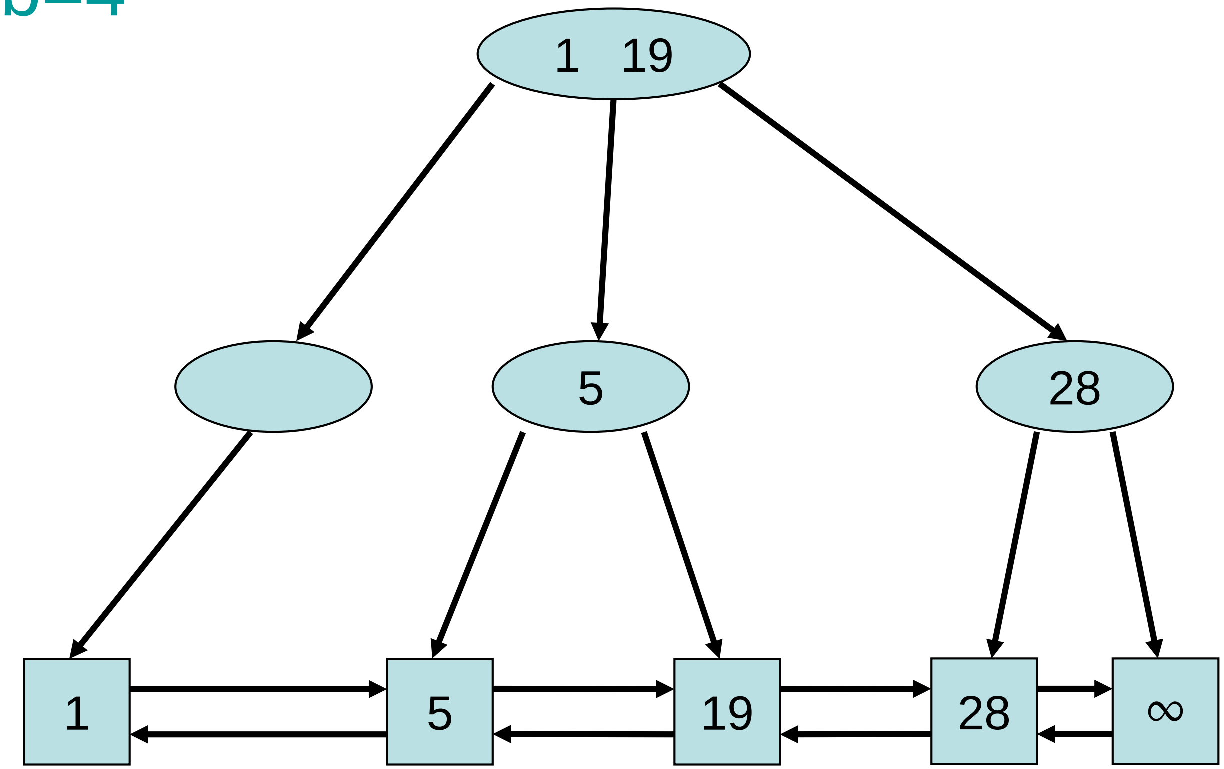
# Delete(3)

a=2, b=4



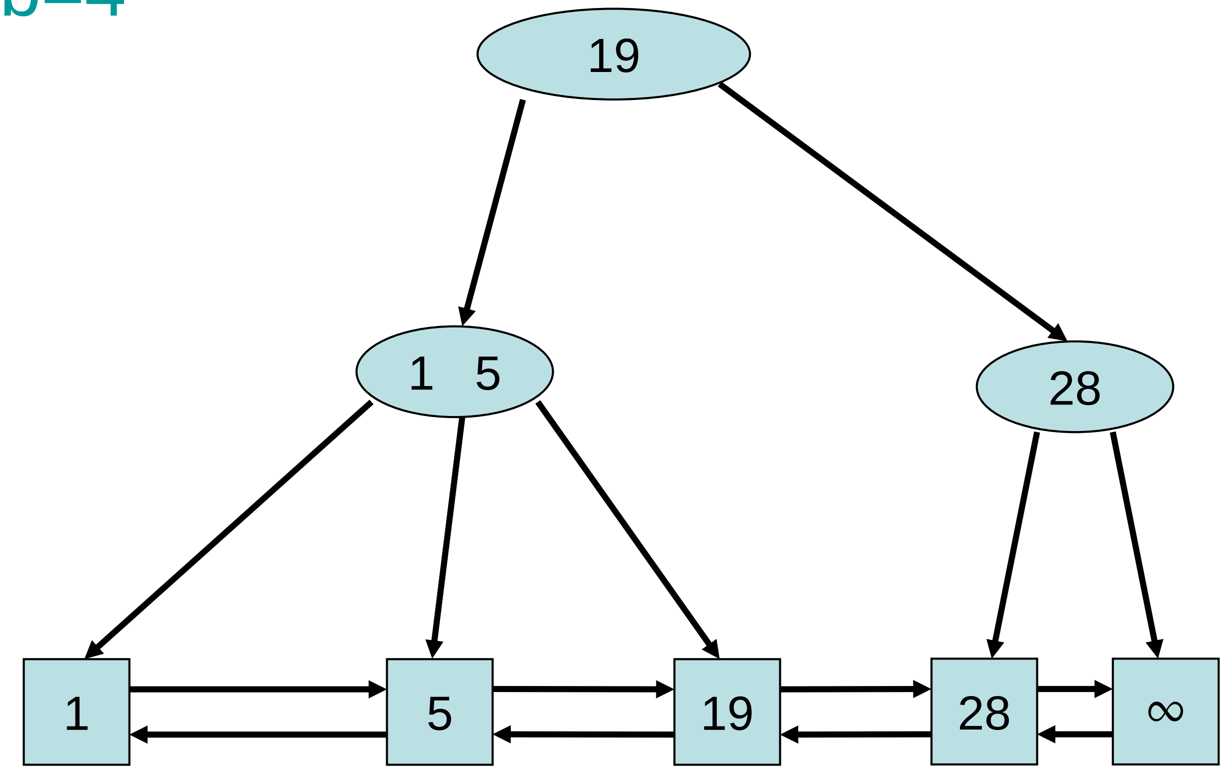
# Delete(3)

a=2, b=4



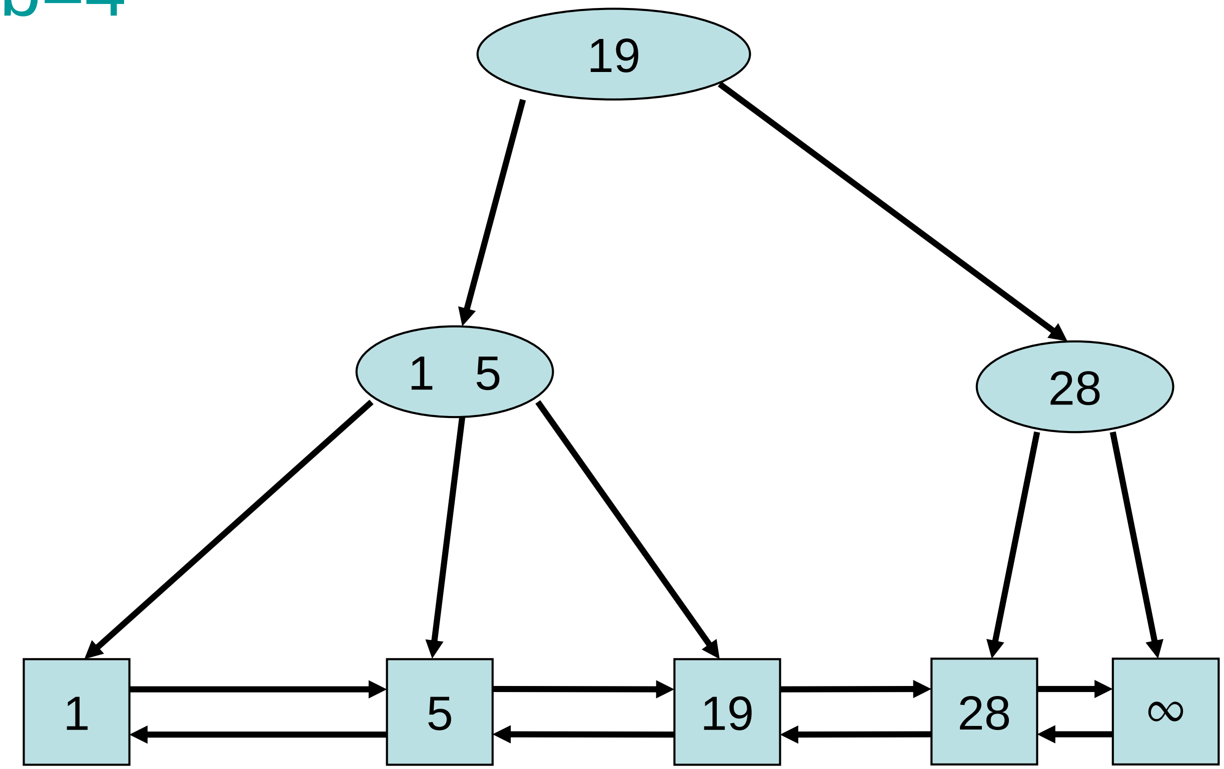
# Delete(3)

a=2, b=4



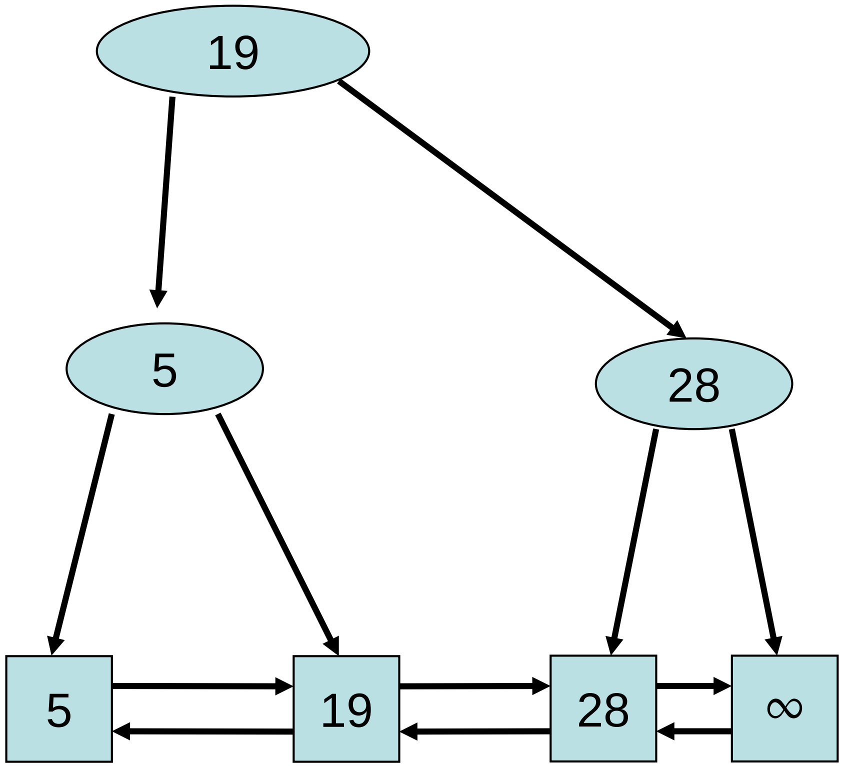
# Delete(1)

a=2, b=4



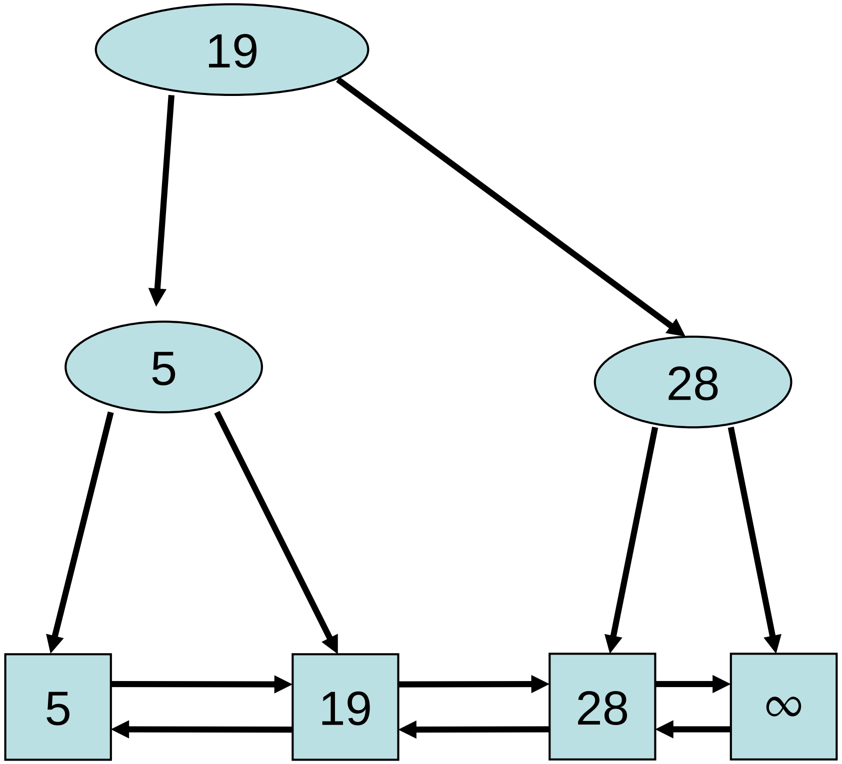
# Delete(1)

a=2, b=4



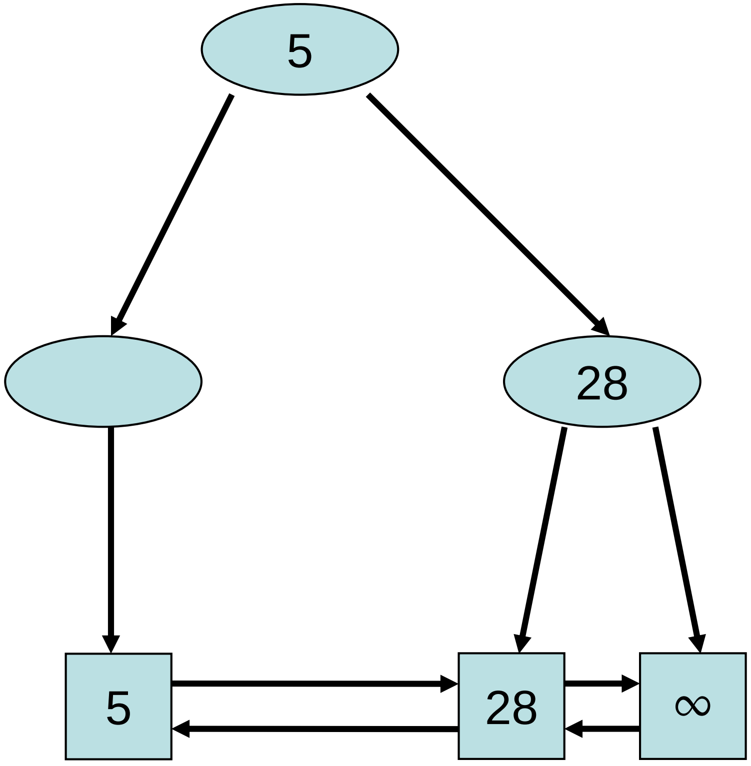
# Delete(19)

a=2, b=4



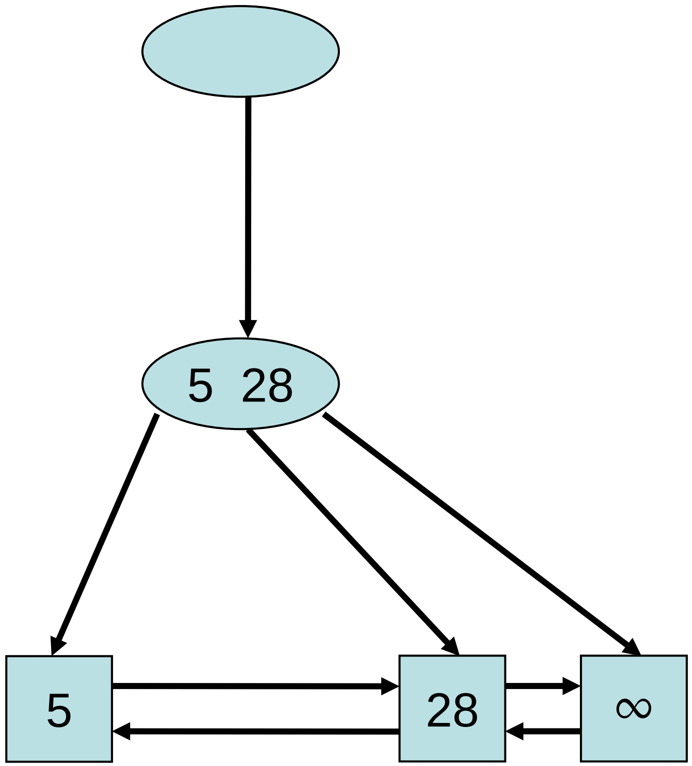
# Delete(19)

a=2, b=4



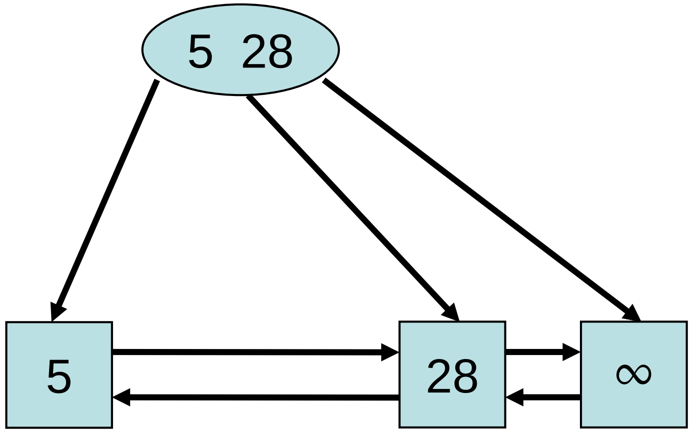
# Delete(19)

a=2, b=4



# Delete(19)

a=2, b=4



# Delete Operation

- Form Invariant:
  For all leaves $v,w$: $t(v)=t(w)$
  Satisfied by Delete!

- Degree Invariant:
  For all inner nodes $v$ except for the root: $d(v)\in[a,b]$, for root $r$: $d(r)\in[2,b]$

1) Delete merges node of degree $a-1$ with node of degree $a$. Since $b\geq 2a-1$, the resulting node has degree at most $b$.

2) Delete moves edge from a node of degree $>a$ to a node of degree $a-1$. Also OK.

3) Root deleted: children have been merged, degree of the remaining child is $\geq a$ (and also $\leq b$), so also OK.

# More Operations

- min/max Operation:
  Pointers to both ends of list: time $O(1)$.

- Range queries:
  To obtain all elements in the range $[x,y]$, perform search($x$) and go through the list till an element $>y$ is found.
  Time $O(\log n + \text{size of output})$.

# n Update Operations

Theorem 3.11: There is a sequence of n insert and delete operations in a (2,3)-tree that require $\Omega(n \log n)$ many split and merge Operations.

Proof: Exercise

# n Update Operations

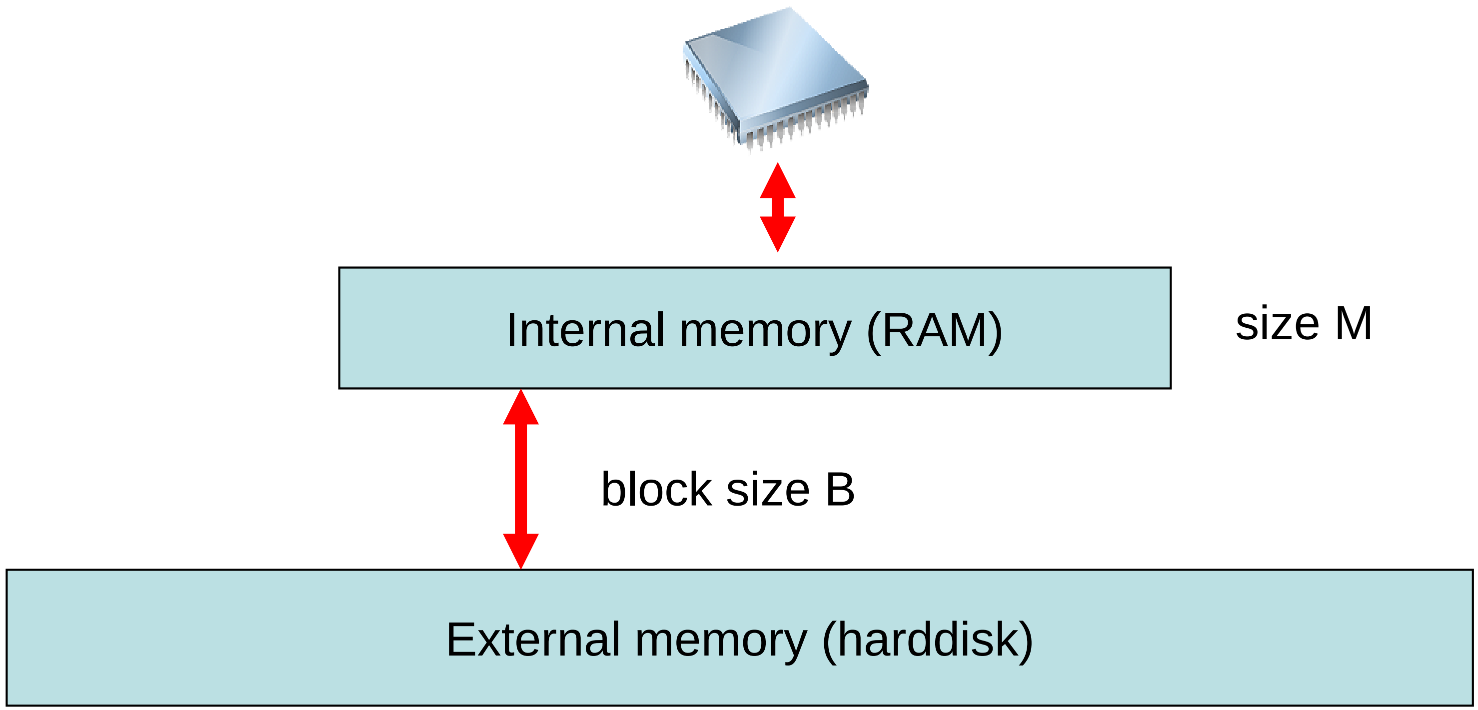
Theorem 3.12: Consider an (a,b)-tree with $b \geq 2a$ that is initially empty. For any sequence of n insert and delete operations, only $O(n)$ split and merge operations are needed.

Proof:

Amortized analysis

# External (a,b)-Tree

(a,b)-trees well suited for large amounts of data



# External (a,b)-Tree

Problem: minimize number of block transfers between internal and external memory

Solution:

- use $b=B$ (block size) and $a=b/2$
- keep highest $(1/2)\cdot\log_a(M/b)$ levels of (a,b)-tree in internal memory (storage needed $\leq M$)
- Lemma 3.10: depth of (a,b)-tree $\leq 1+\lfloor\log_a (n/2)\rfloor$
- How many levels are not in internal memory?

  $\log_a[n/2] - (1/2)\cdot\log_a(M/b) \leq \log_a[n/(2\sqrt{M})] + O(1)$ (a, b are O(1))
- Cost for insert, delete and search operations: $O(\log_B(n/\sqrt{M}))$ block transfers

# External (a,b)-Tree

Problem: minimize number of block transfers between internal and external memory

A better analysis can show (exercise):

- Cost for insert, delete and search operations: $\sim 2\log_{B/2}(n/M)+1$ block transfers (+1: list access)

Example:

- n = 100,000,000,000,000 keys
- M = 16 Gbyte (~4,000,000,000 keys)
- B = 256 Kbyte (~64,000 keys)
- $2\log_{B/2}(n/M)+1 \leq 3$

# Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- Patricia trie

# Longest Prefix Search

- All keys are encoded as binary sequence $\{0,1\}^W$
- Prefix of a key $x \in \{0,1\}^W$: arbitrary subsequence of $x$ that starts with the first bit of $x$ (example: 101 is a prefix of 10110100)

Problem: given a key $x \in \{0,1\}^W$, find a key $y \in S$ with longest common prefix

Solution: Trie Hashing

# Trie

A trie is a search tree over some alphabet $\Sigma$ that has the following properties:

- Every edge is associated with a symbol $c \in \Sigma$
- Every key $x \in \Sigma^k$ that has been inserted into the trie can be reached from the root of the trie by following the unique path of length $k$ whose edge labels result in $x$.

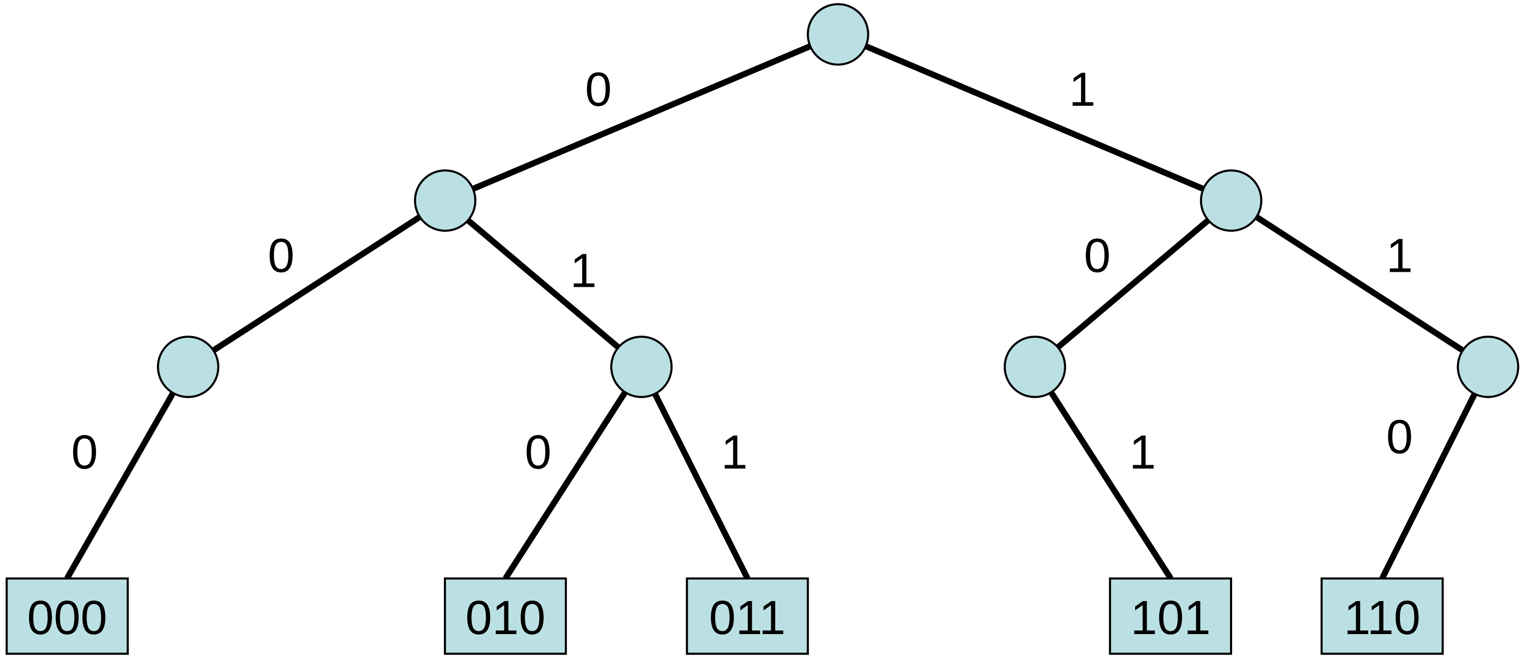
For simplicity: all keys from $\{0,1\}^W$ for some $W \in \mathbb{N}$.

Example:

$(0,2,3,5,6)$ with $W=3$ results in $(000,010,011,101,110)$

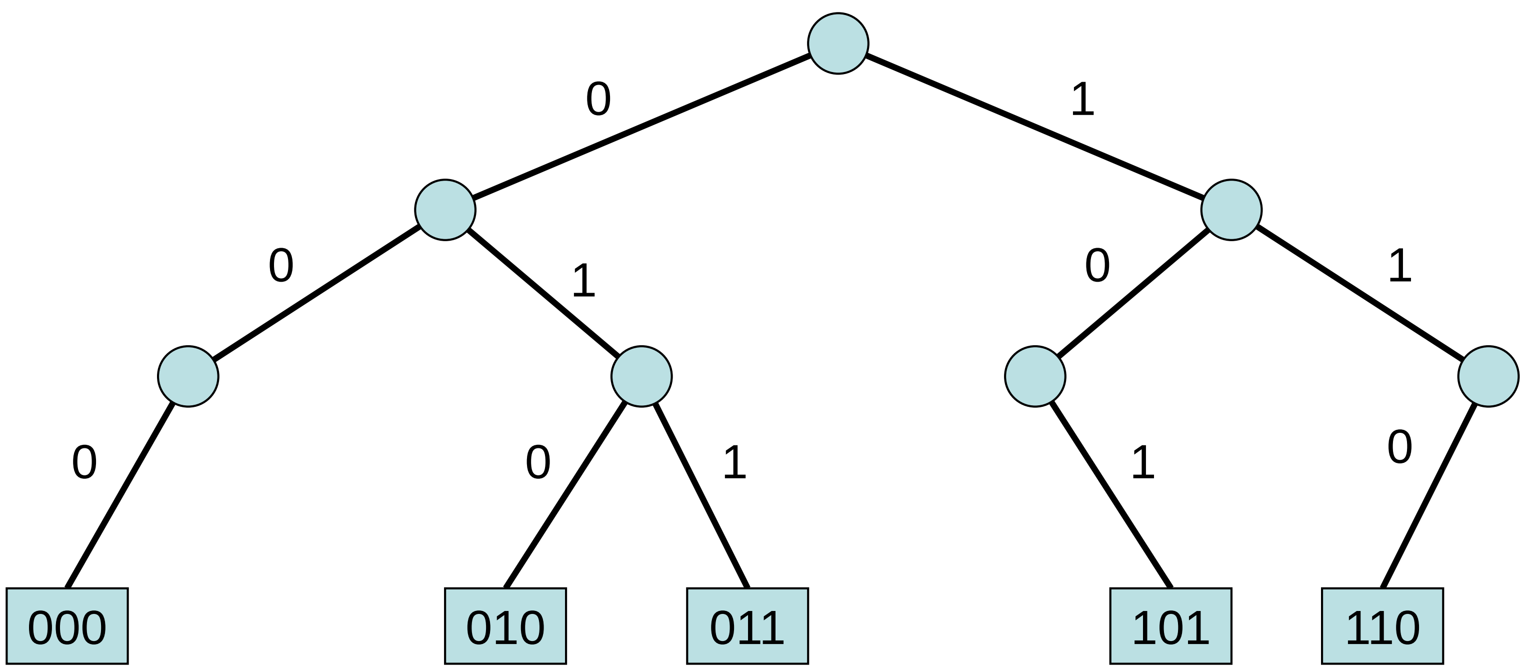
# Trie

Example: (without list at bottom)



# Trie

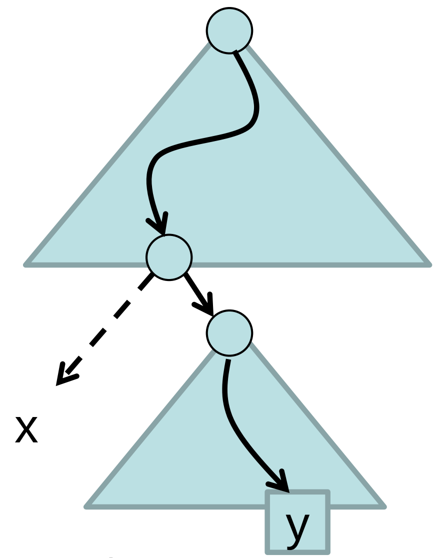
search(4) (4 corresponds to 100):




Output: 5 (longest common prefix)

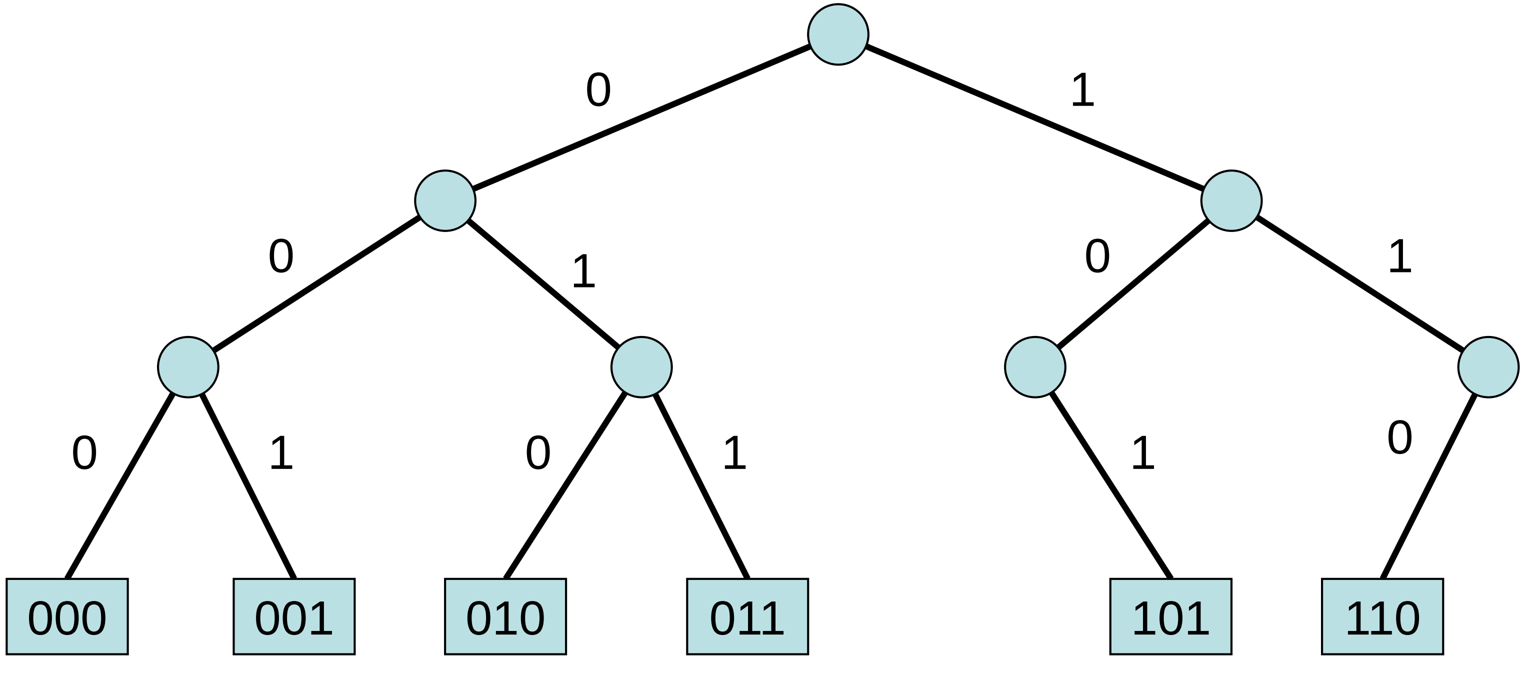
# Trie

In general: a search(x) request follows the edges in the trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, the request may be forwarded to any leaf y in the subtrie below since all of them have the same longest prefix match with x.



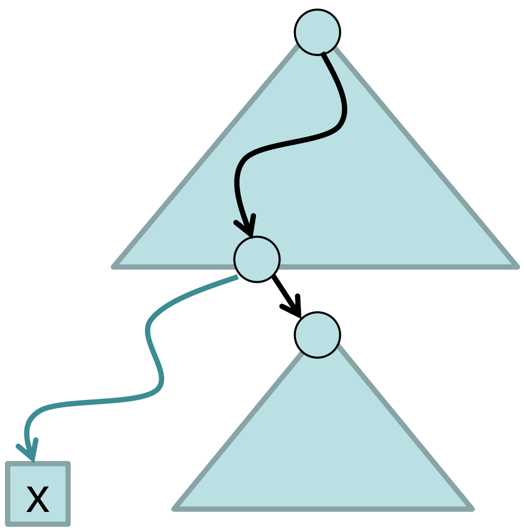
# Trie

insert(1)  (1 corresponds to 001):



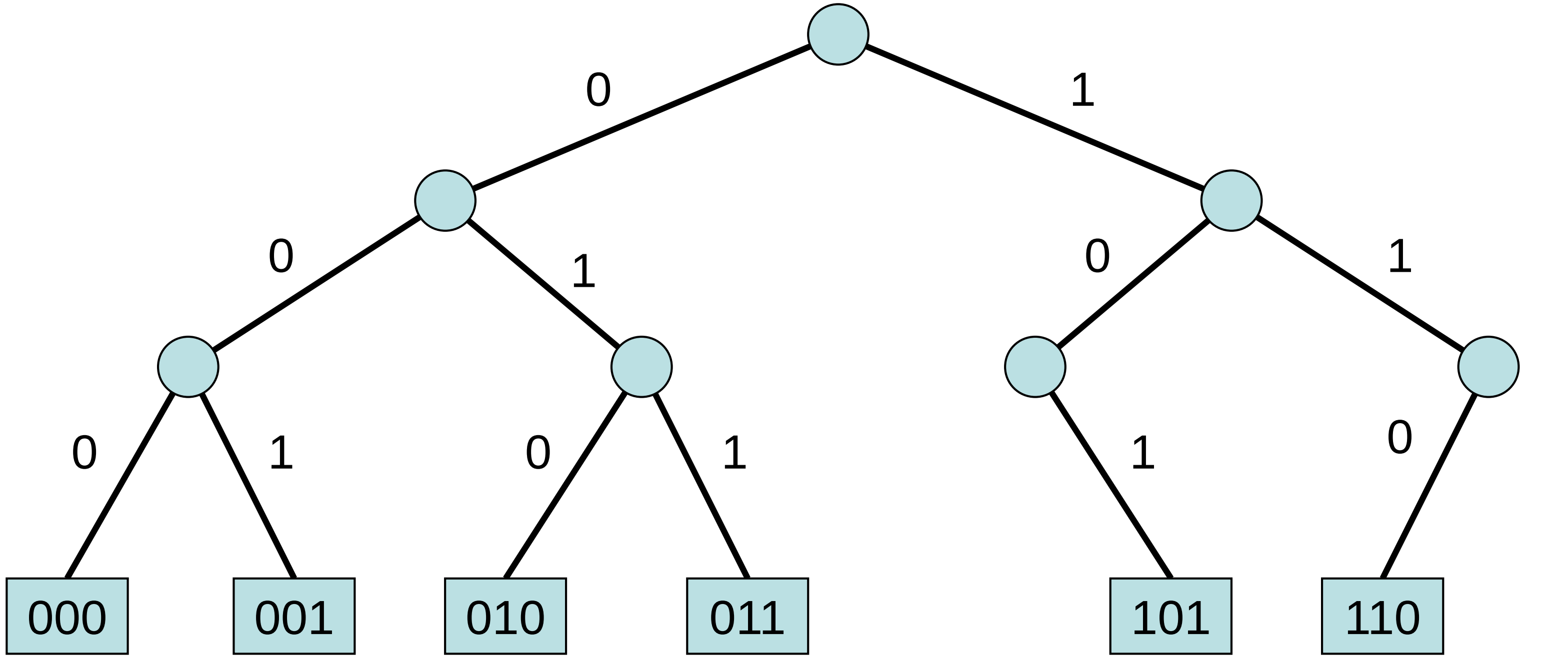
# Trie

In general: an insert(x) request follows the edges in the trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, a new path (of length the remaining bits in x) is created that leads to the new leaf x.



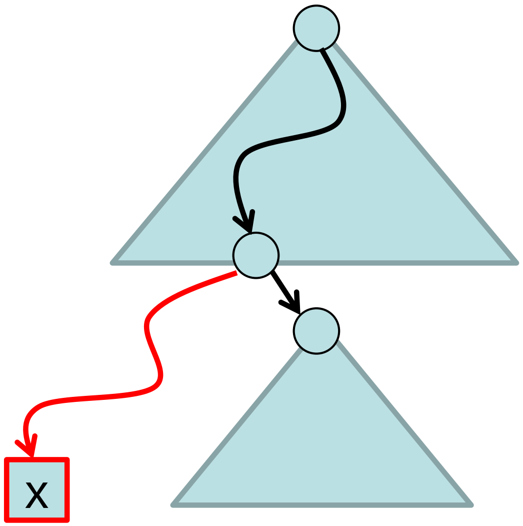
# Trie

delete(5):



# Trie

In general: a delete(x) request follows the edges in the trie down to the leaf x. If x does not exist, the delete operation terminates. Otherwise, x as well as the chain of nodes upwards till the first node with at least two children is deleted.



# Patricia Trie

Problem:

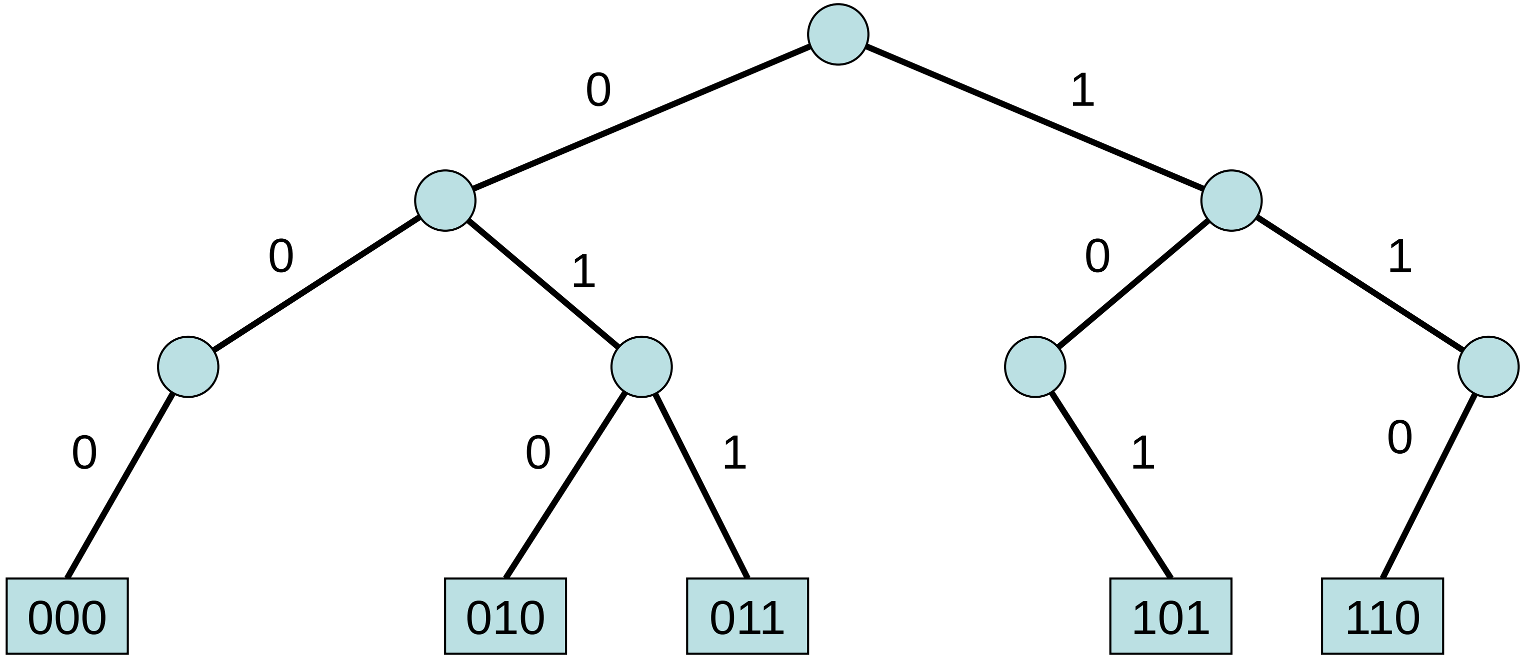
- Longest common prefix search for some $x \in \{0,1\}^W$ can take $\Theta(W)$ time.
- Insert and delete may require $\Theta(W)$ structural changes in the trie.

Improvement: use Patricia trie

A Patricia trie is a compressed trie in which all chains (i.e., maximal sequences of nodes of degree 1) are merged into a single edge whose label is equal to the concatenation of the labels of the merged trie edges.

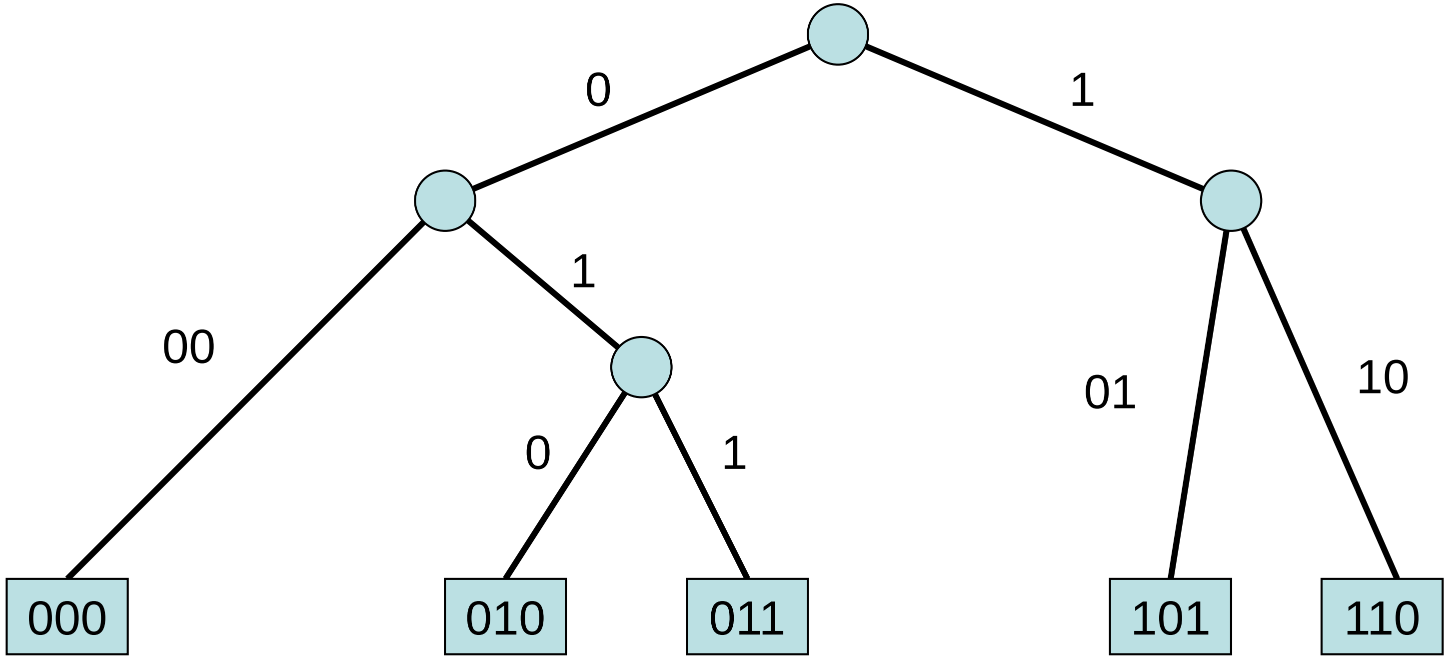
# Trie

## Example 1:



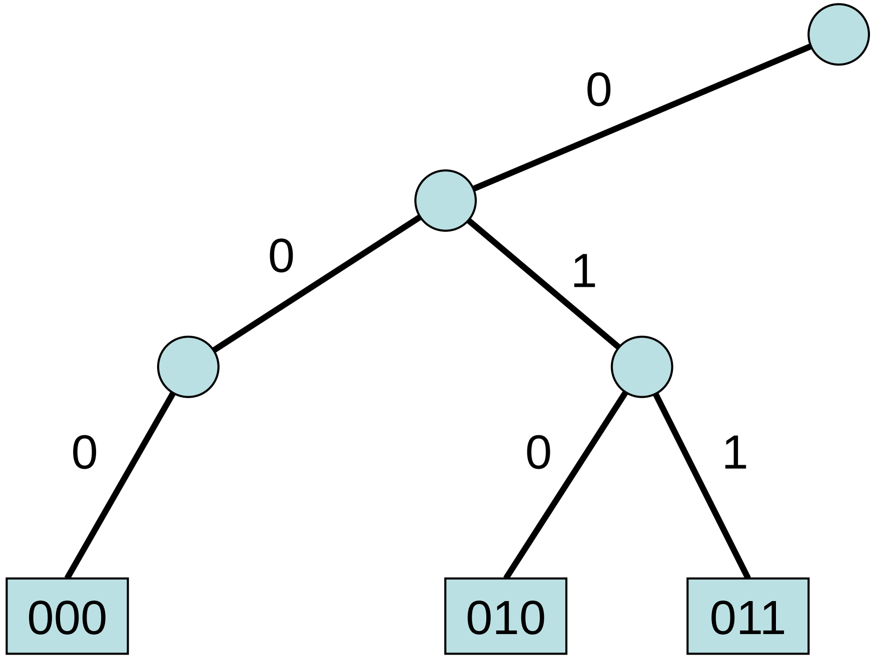
# Patricia Trie

Example 1:



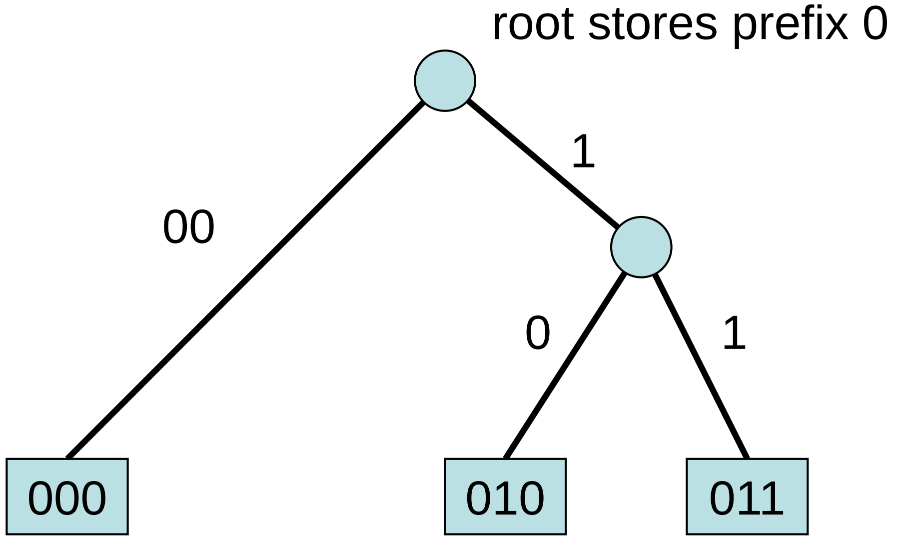
# Trie

Example 2:



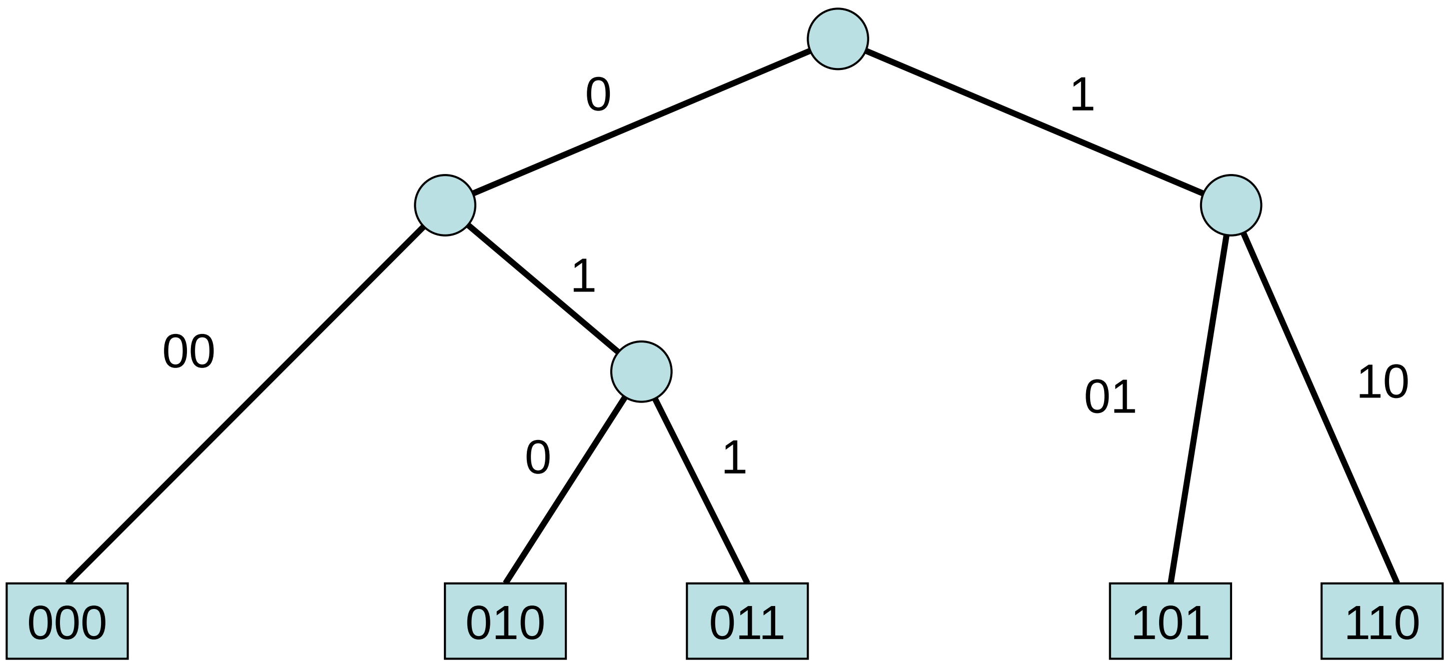
# Patricia Trie

## Example 2:



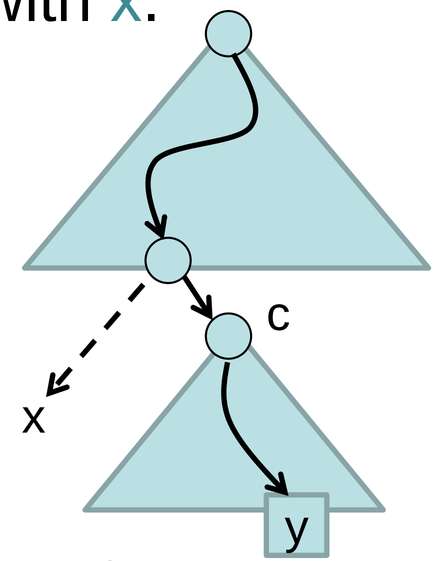
# Patricia Trie

search(4):



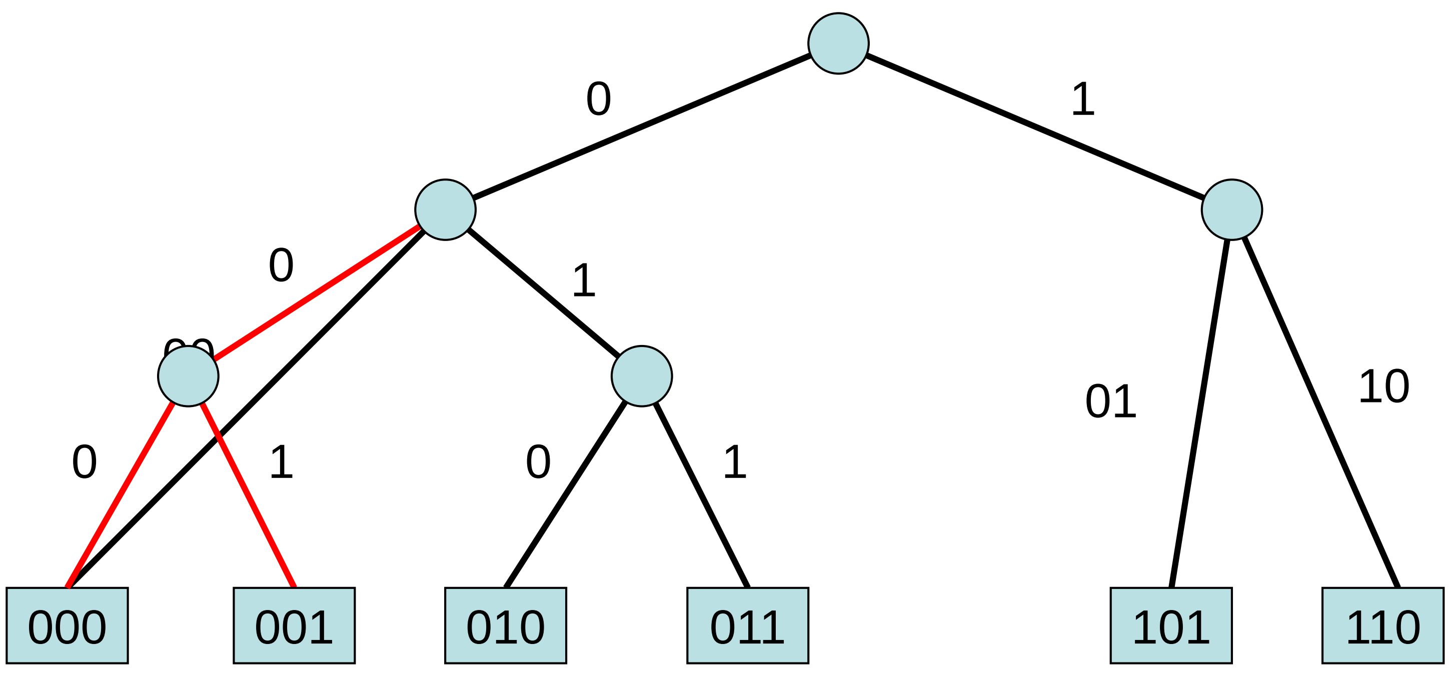
# Patricia Trie

In general: a search(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, choose the current child c with longest common prefix. Then, the request may be forwarded to any leaf y in the subtrie rooted c at below since all of them have the same longest prefix match with x.



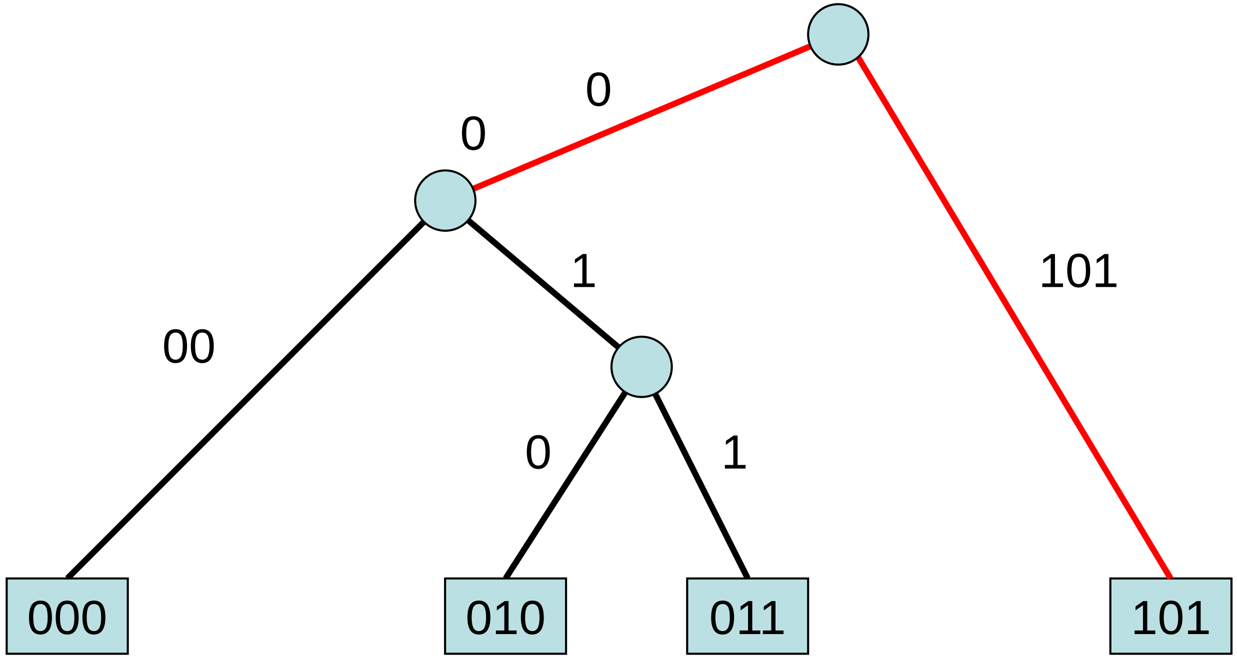
# Patricia Trie

## insert(1):



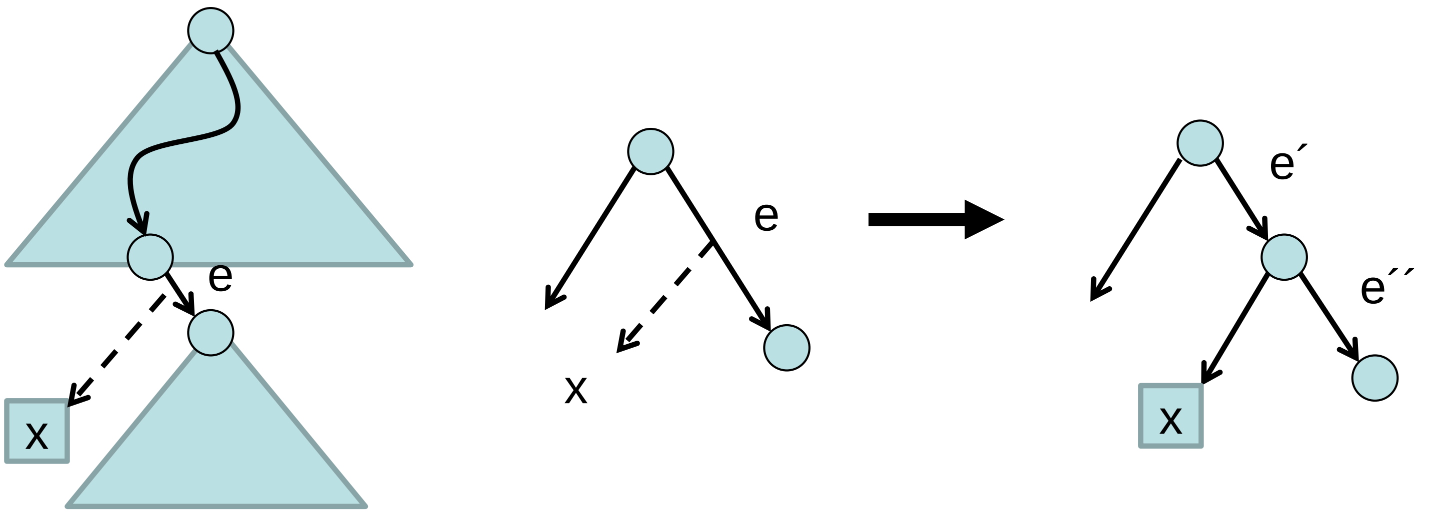
# Patricia Trie

Insert(5):



# Patricia Trie

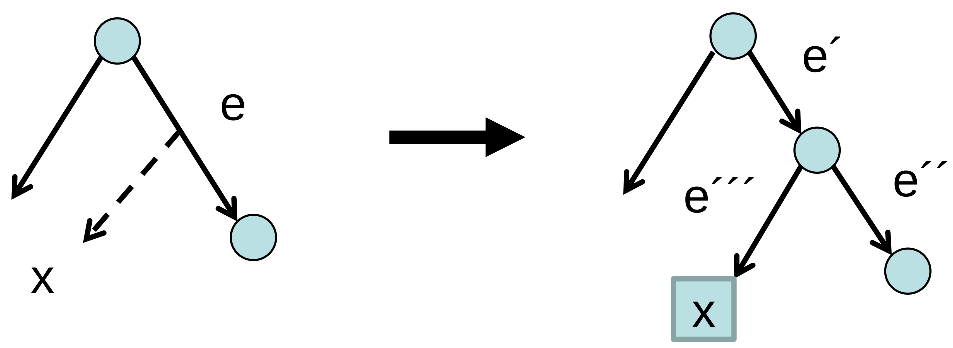
In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.



# Patricia Trie

In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.

Example: l(e)=10010, x=…10110100

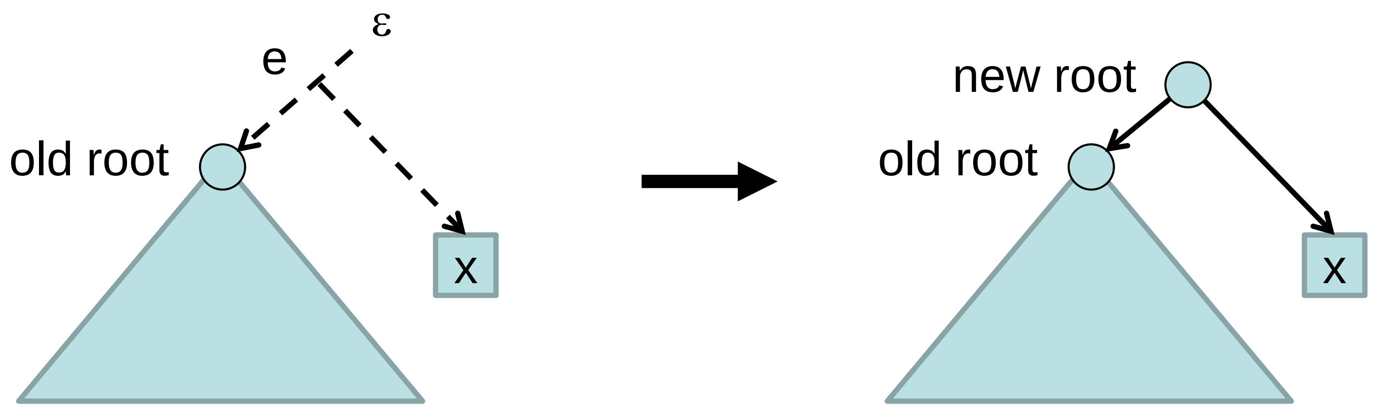



l(e´)=10
l(e´´)=010
l(e´´´)=110100

# Patricia Trie

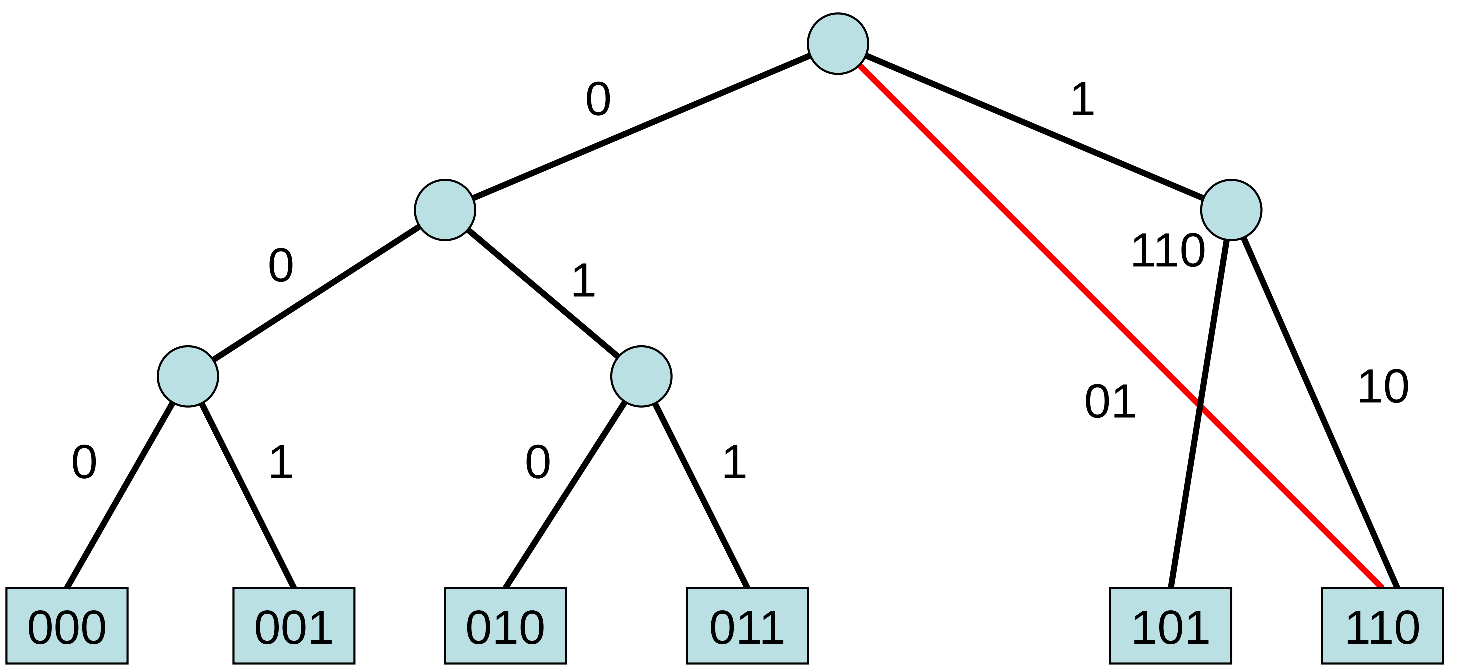
In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.

Special case:



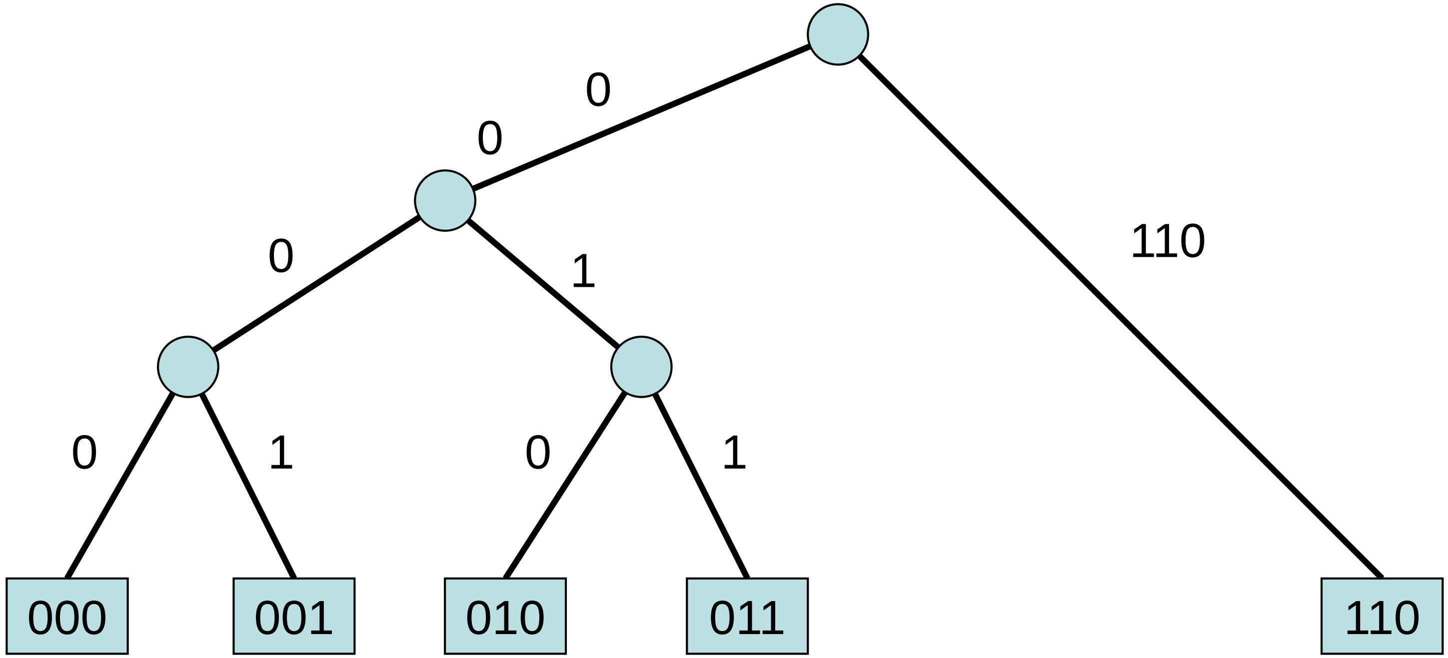
# Patricia Trie

delete(5):



# Patricia Trie

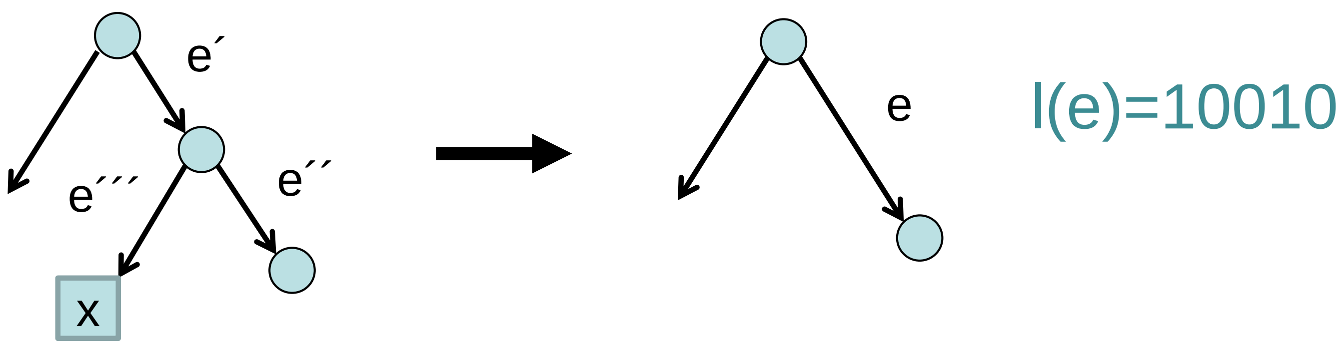
delete(6):



# Patricia Trie

In general: a delete(x) request follows the edges in the Patricia trie down to the leaf x. If x does not exist, the delete operation terminates. Otherwise, x as well as its parent are deleted.

Example: l(e´)=10, l(e´´)=010, l(e´´´)=110100, x=…10110100



# Patricia Trie

- Search, insert, and delete like in an ordinary binary tree, with the difference that we have labels at the edges.
- Search time still O(W) in the worst case, but just O(1) structural changes.

# Patricia Trie

- History:
  - Invented independently by D. R. Morrison (1968) and G. Gwehenberger (1968).
  - Morrison called them „Patricia trees“, where PATRICIA stands for Practical Algorithm To Retrieve Information Coded in Alphanumeric.
  - Patricia trees are also referred to as *radix* trees (with radix 2).

Idea (Kniesburges and Scheideler, 2011):

- Can improve search time to O(log W) using „hashed Patricia tries“. (Will not cover this here.)