Fundamental Algorithms

Chapter 3: Advanced Search Structures

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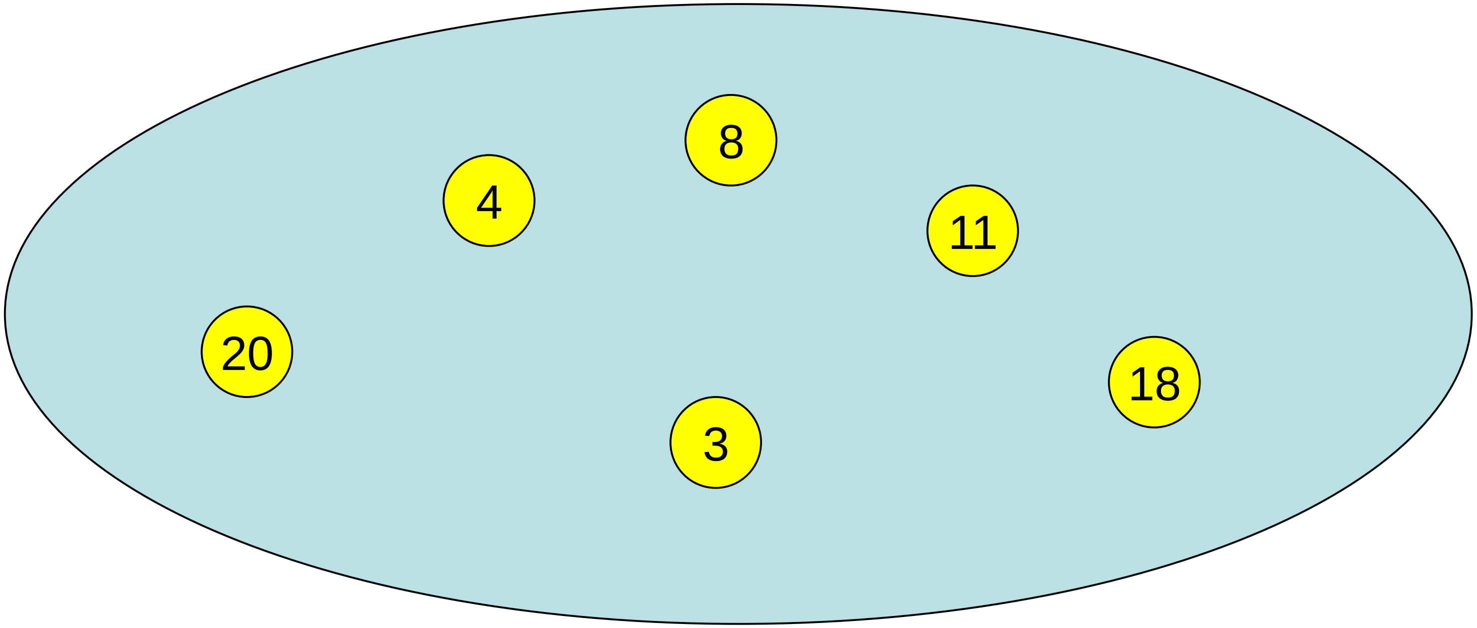
(based on slides of Christian Scheideler)

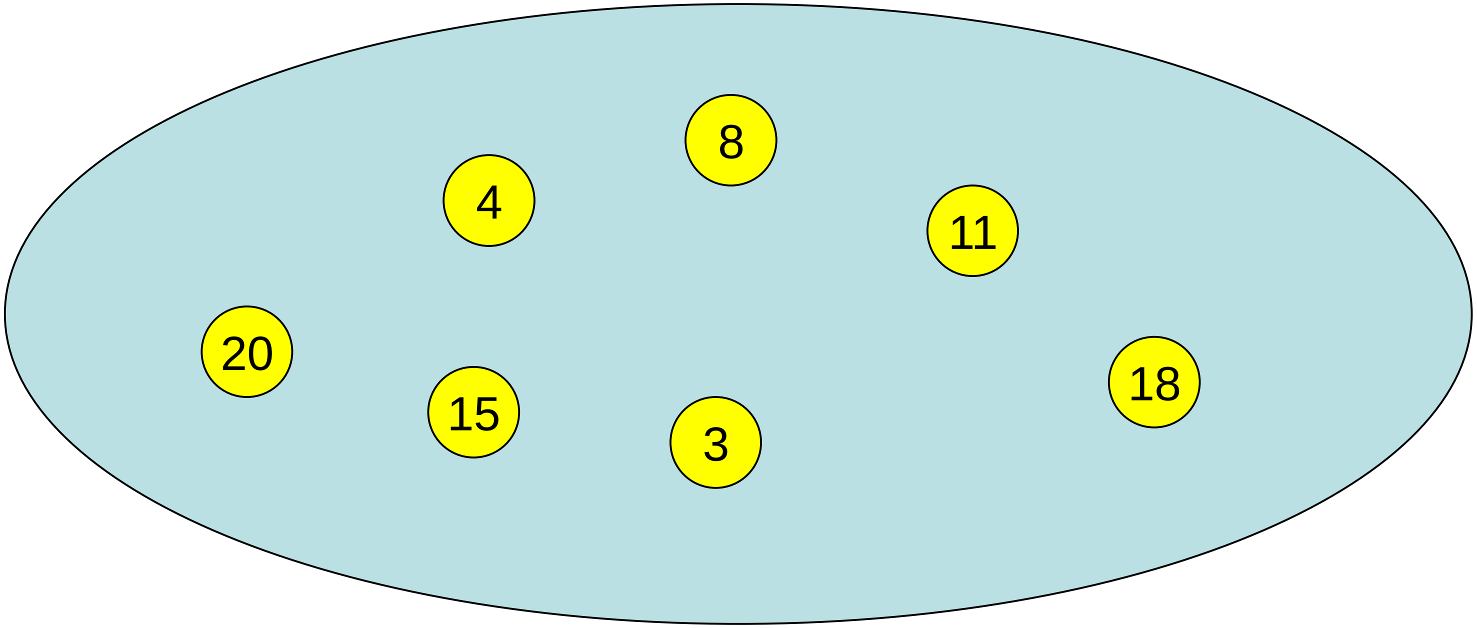
WS 2018

Search Structure



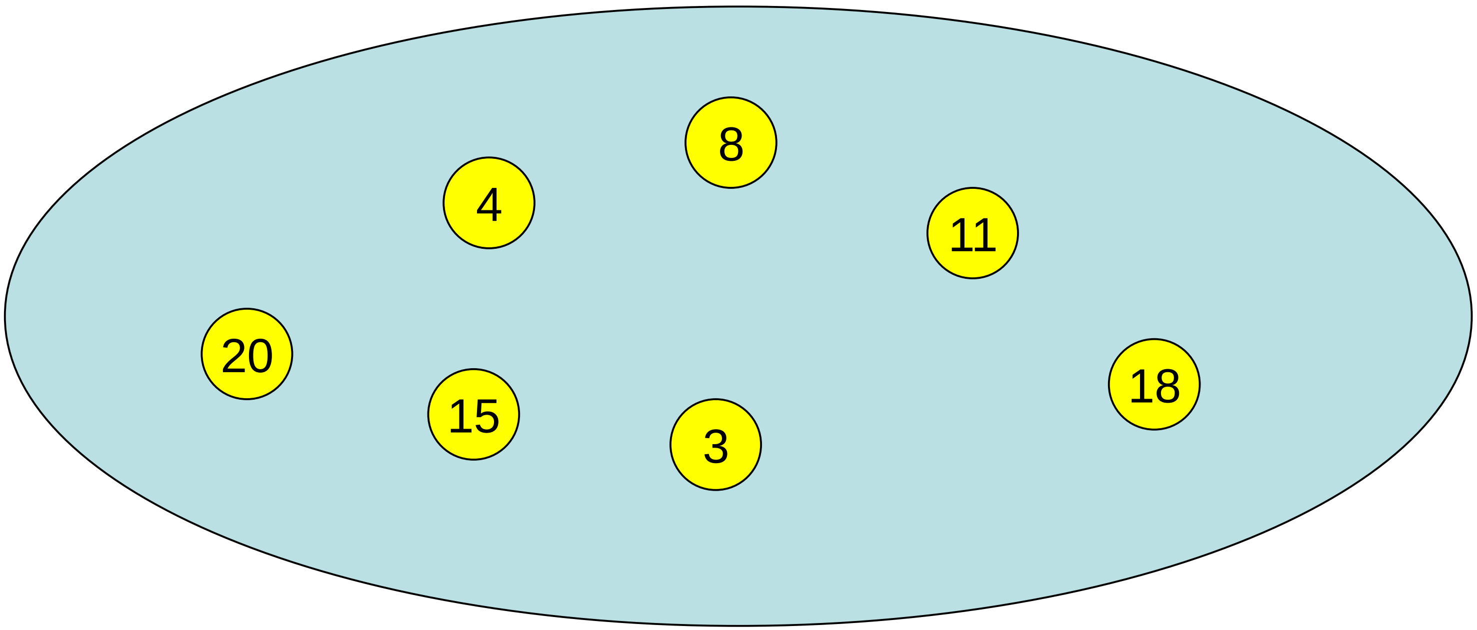
Search Structure

insert(15)



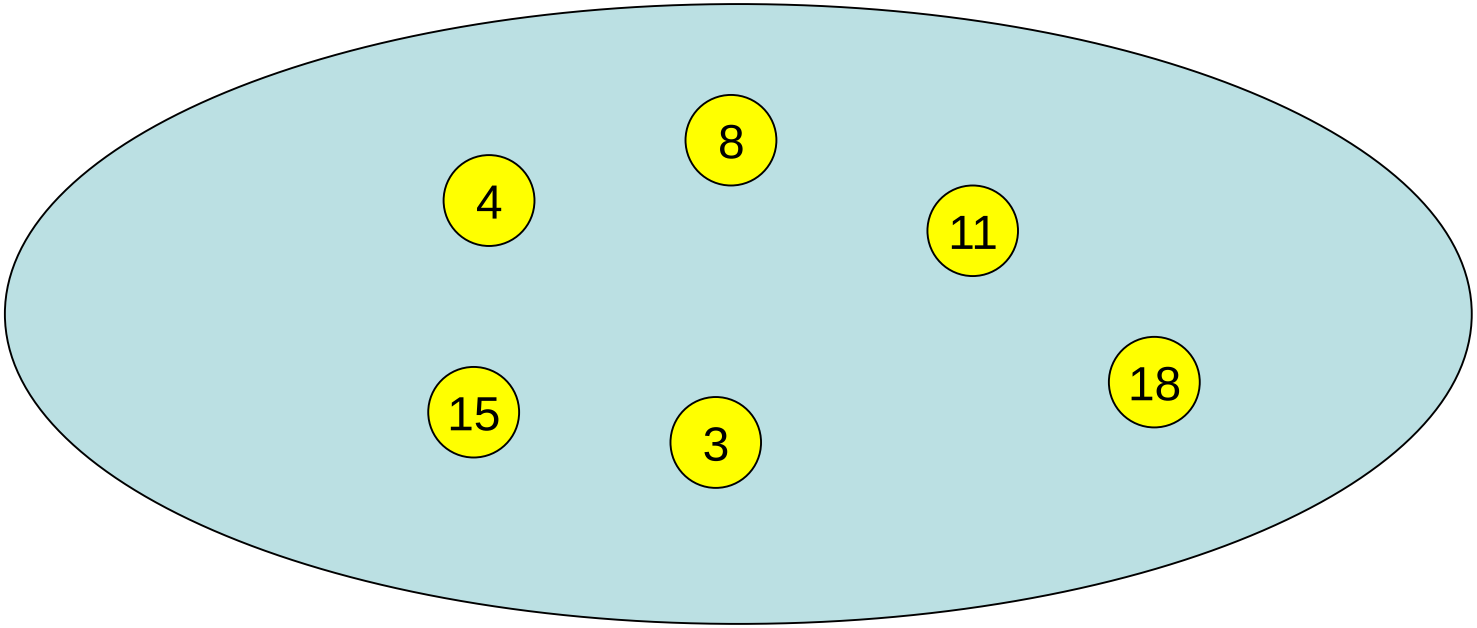
Search Structure

delete(20)



Search Structure

search(7) gives 8 (closest successor)



Search Structure

S: set of elements

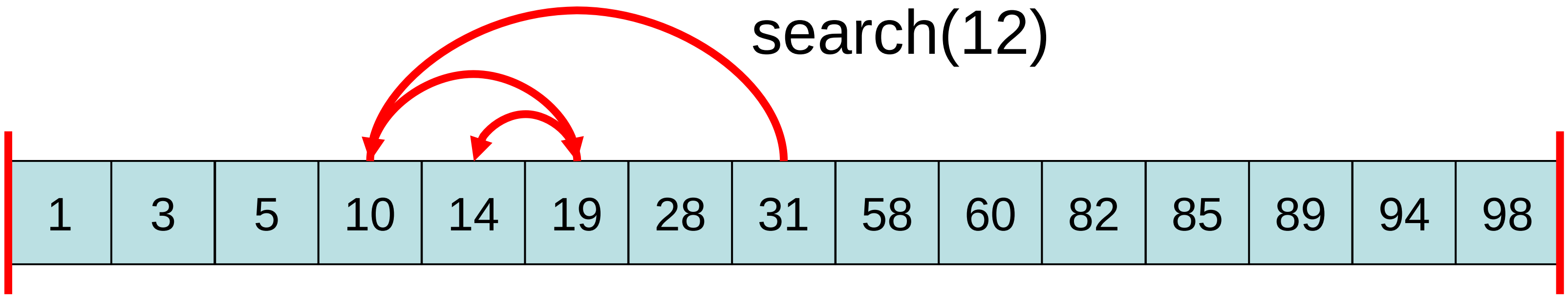
Every element e identified by key(e).

Operations:

- S.insert(e: Element): $S := S \cup \{e\}$
- S.delete(k: Key): $S := S \setminus \{e\}$, where e is the element with key(e)=k (note: now given *key*, not *pointer* to e!)
- S.search(k: Key): outputs $e \in S$ with minimal key(e) so that $key(e) \geq k$

Static Search Structure

1. Store elements in sorted array.



search: via binary search (in O(log n) time)

Binary Search

Input: number x and sorted array A[1],…,A[n]

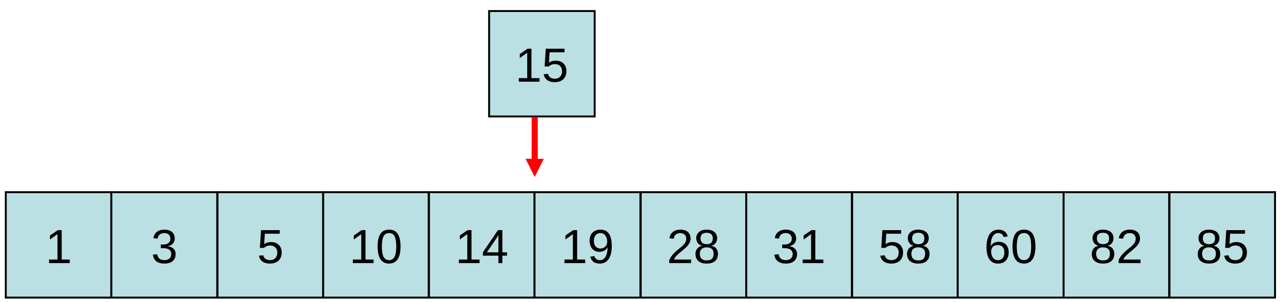
Algorithm BinarySearch:

```
l:=1; r:=n
while l < r do
    m:=(r+l) div 2
    if A[m] = x then return m
    if A[m] < x then l:=m+1
                 else r:=m
return l
```

Dynamic Search Structure

insert und delete Operations:

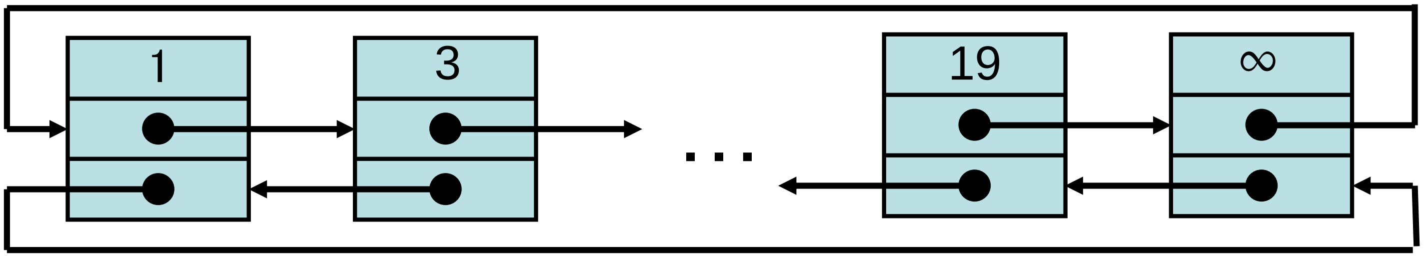
Sorted array difficult to update!



Worst case: $\Theta(n)$ time

Search Structure

2. Sorted List (with an ∞-Element)

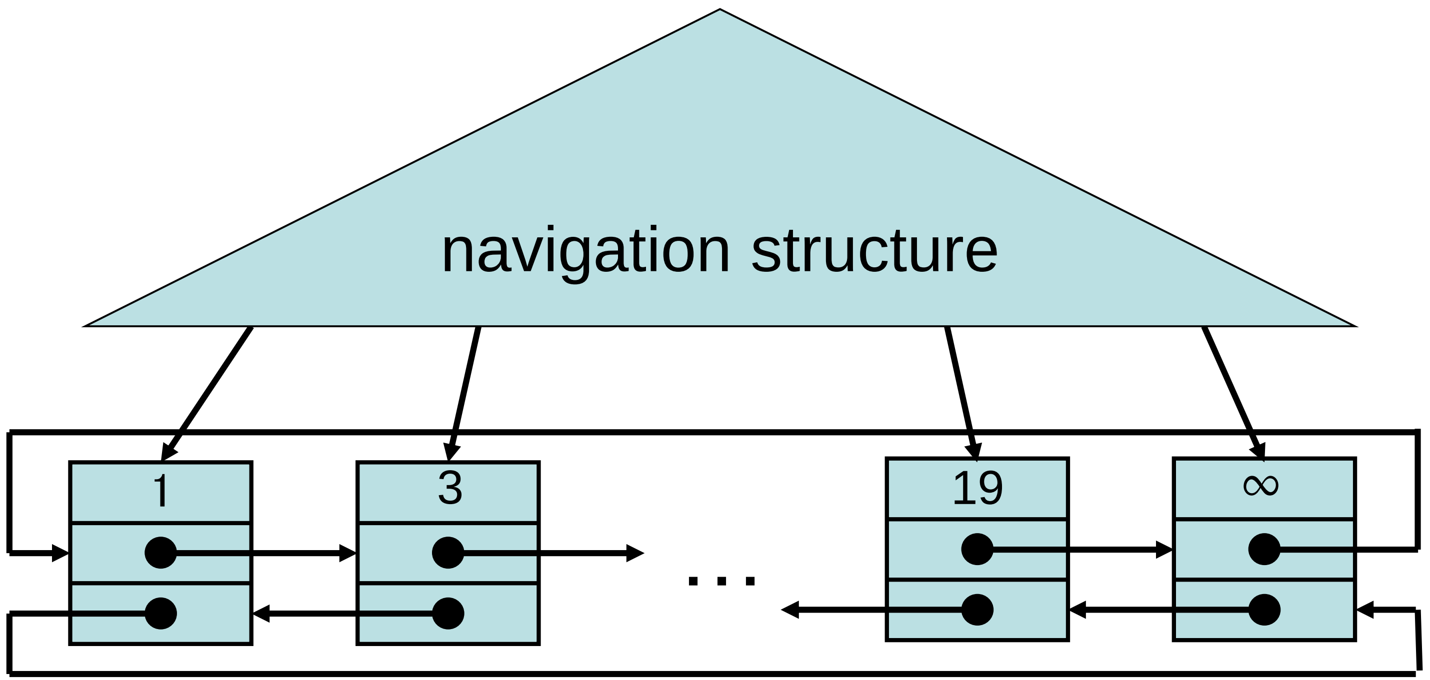


Problem: insert, delete and search take $\Theta(n)$ time in the worst case (why for insert/delete?)

Observation: If search could be implemented efficiently, then also all other operations

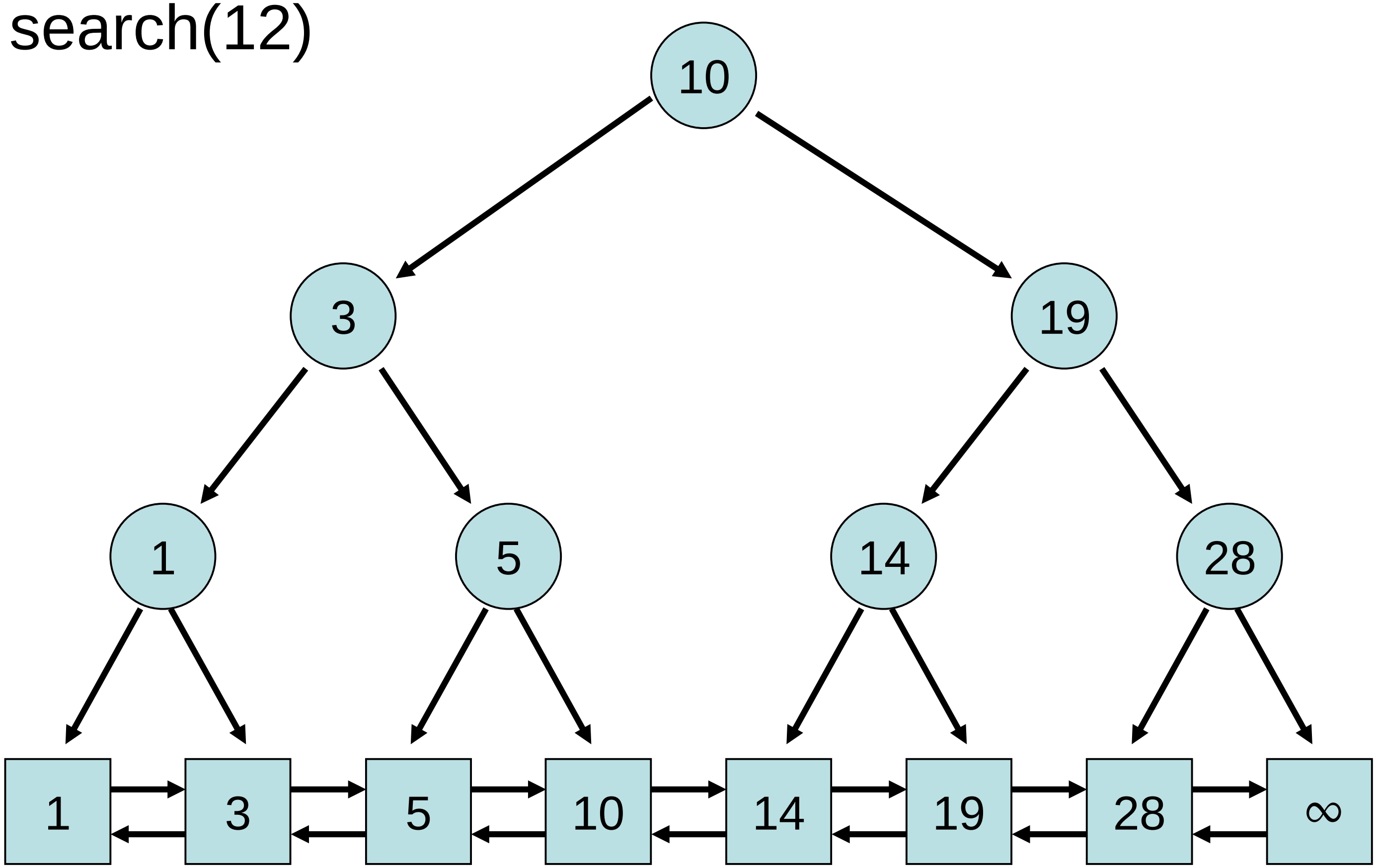
Search Structure

Idea: add navigation structure that allows search to run efficiently



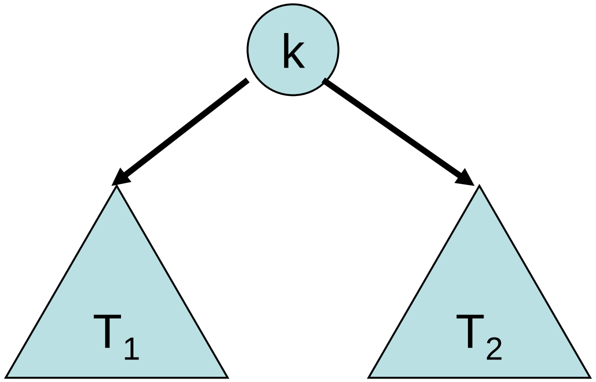
Binary Search Tree (ideal)

search(12)



Binary Search Tree

Search tree invariant:



For **all** keys k' in T_1 and k'' in T_2: $k' \leq k < k''$

Binary Search Tree

Formally: for every tree node v let

- key(v) be the key stored at v
- d(v) the number of children (degree) of v

- Search tree invariant: (as above)

- Degree invariant:
  All tree nodes have exactly two children
  (as long as the number of elements in the list is >0, recall presense of ∞ node)

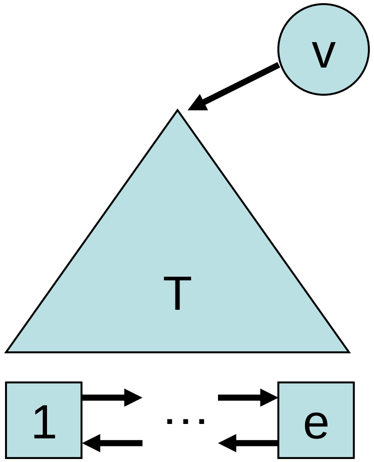
- Key invariant:
  For every element e in the list there is exactly one tree node v with key(v)=key(e).

Binary Search Tree

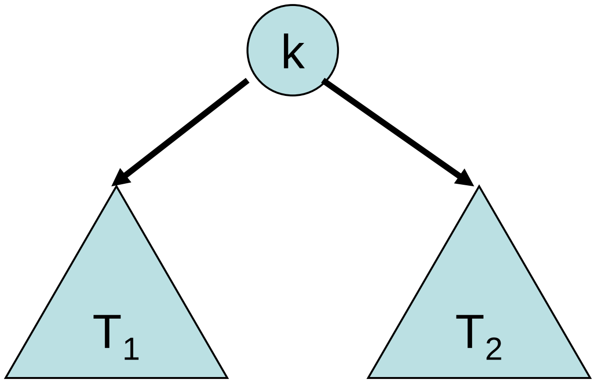
- Search tree invariant: (as before)
- Degree invariant:
  All tree nodes have exactly two children
  (as long as the number of elements is >0)
- Key invariant:
  For every element e in the list there is exactly one tree node v with key(v)=key(e).

From the search tree and key invariants it follows that for every left subtree T of a node v, the rightmost list element e under T satisfies key(v)=key(e).

(Why?)



search(x) Operation

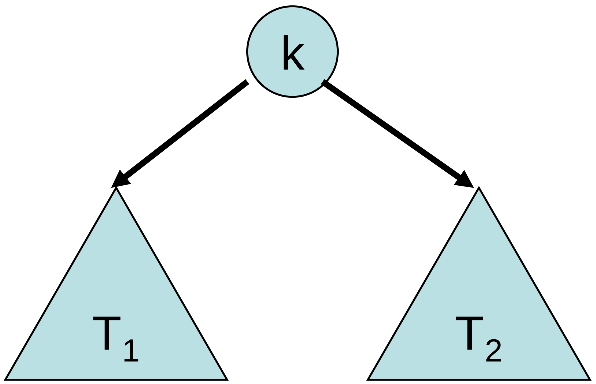


For all keys k' in T_1 and k'' in T_2: $k' \leq k < k''$

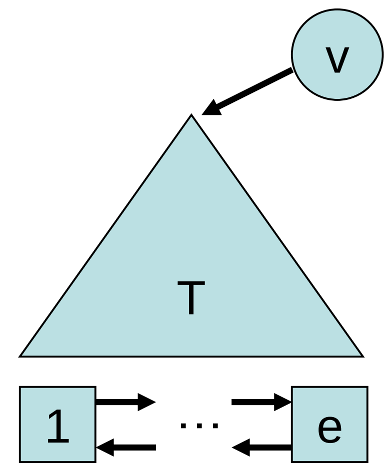
Search strategy:

- Start at the root, v, of the search tree
- while v is a tree node:
  - if $x \leq key(v)$ then let v be the left child of v, otherwise let v be the right child of v
- Output (list node) v

search(x) Operation



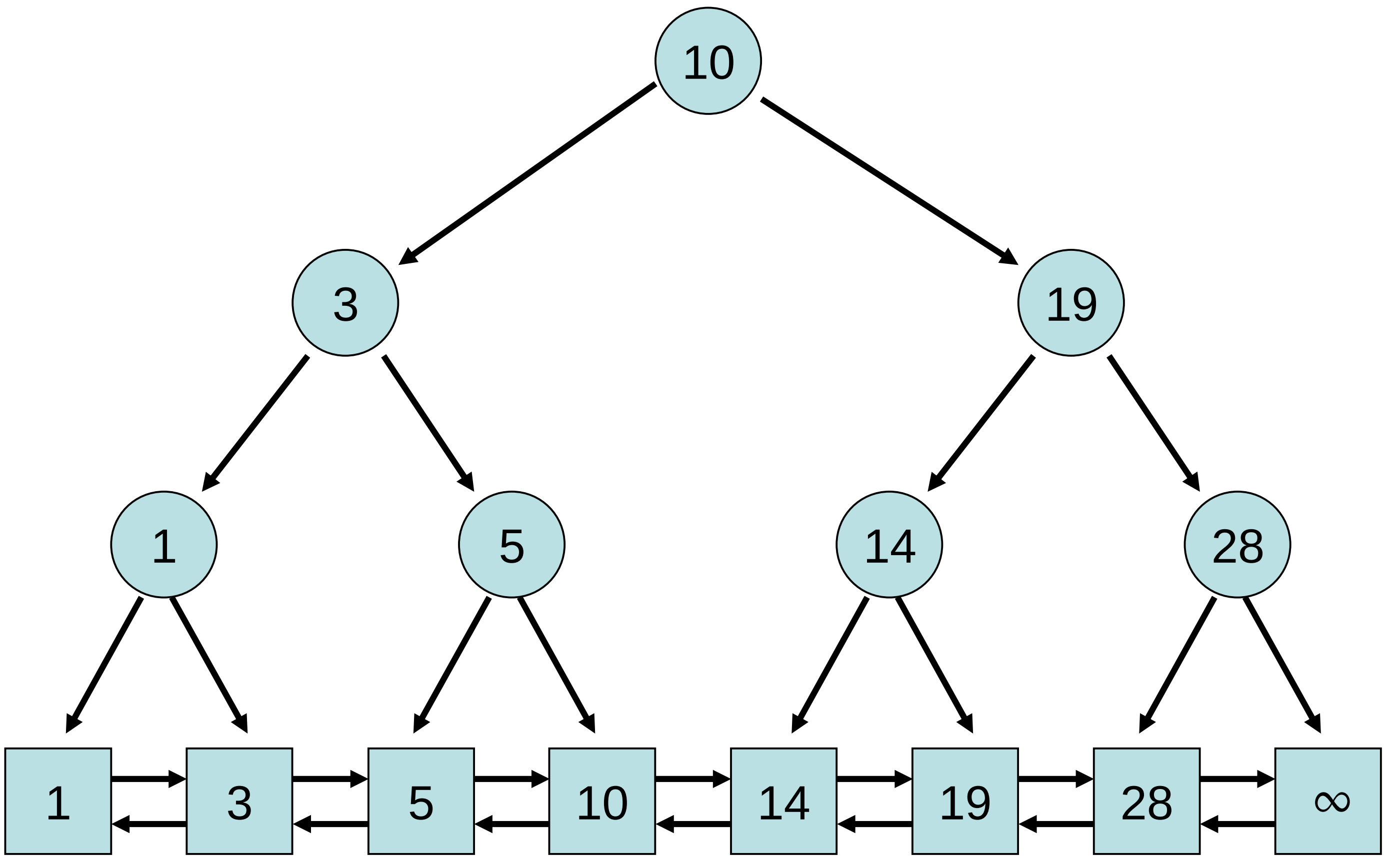
For all keys k' in T_1 and k'' in T_2: $k' \leq k < k''$



Correctness of search strategy:

- For every left subtree T of a node v, the rightmost list element e under T satisfies key(v)=key(e).
- If search(x) enters T, since key(v)≥x, there is an element e in the list below T with key(e)≥x.

Search(9)



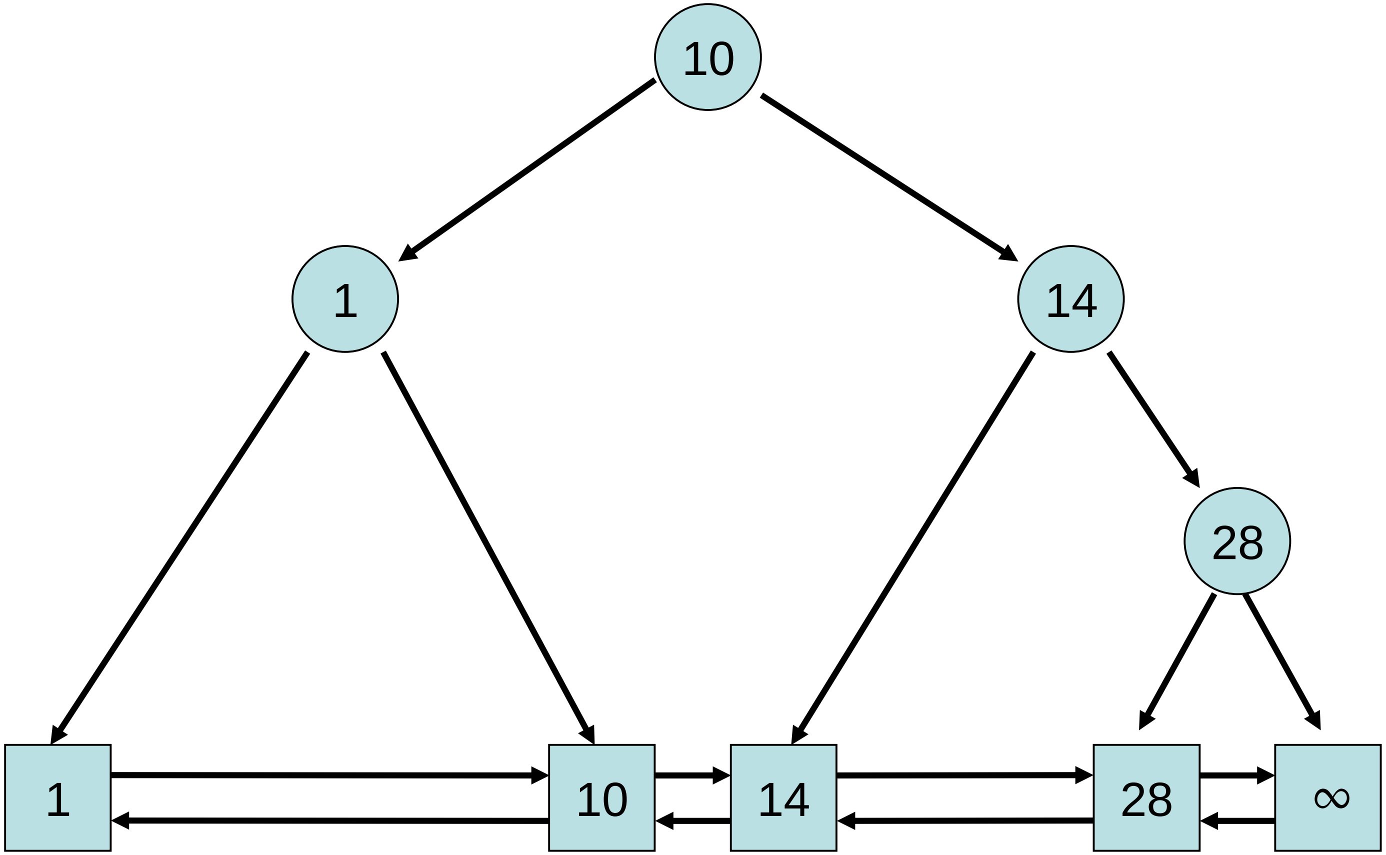
Insert and Delete Operations

Strategy:

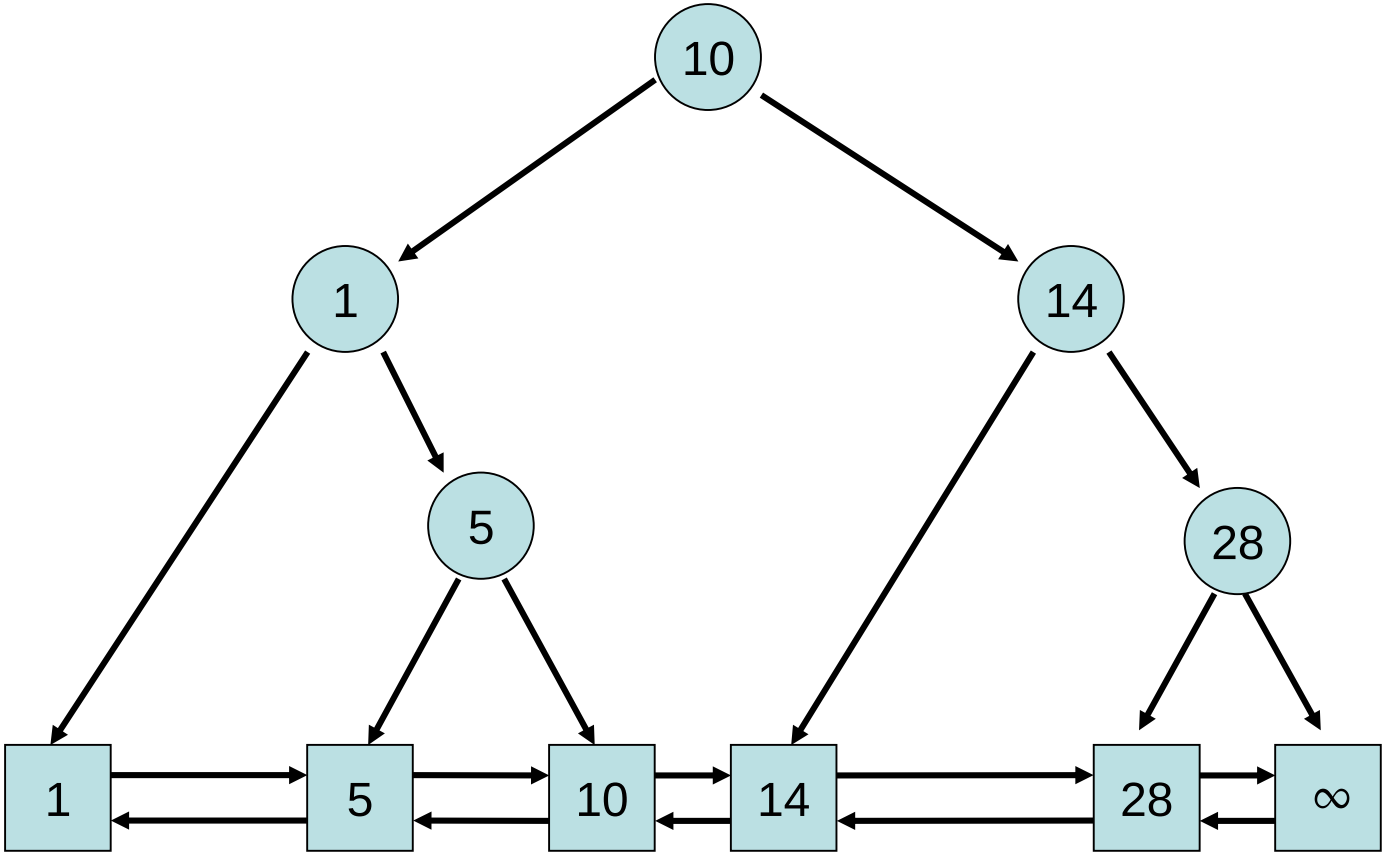
- insert(e):
  First, execute search(key(e)) to obtain a list element e´. If key(e)=key(e´), replace e´ by e, otherwise insert e between e´ and its predecessor in the list and add a new search tree leaf leading to e (left) and e´ (right) with key key(e).

- delete(k):
  First, execute search(k) to obtain a list element e. If key(e)=k, then delete e from the list and the parent v of e from the search tree, and relabel tree node w with key(w)=k as key(w):=key(v).

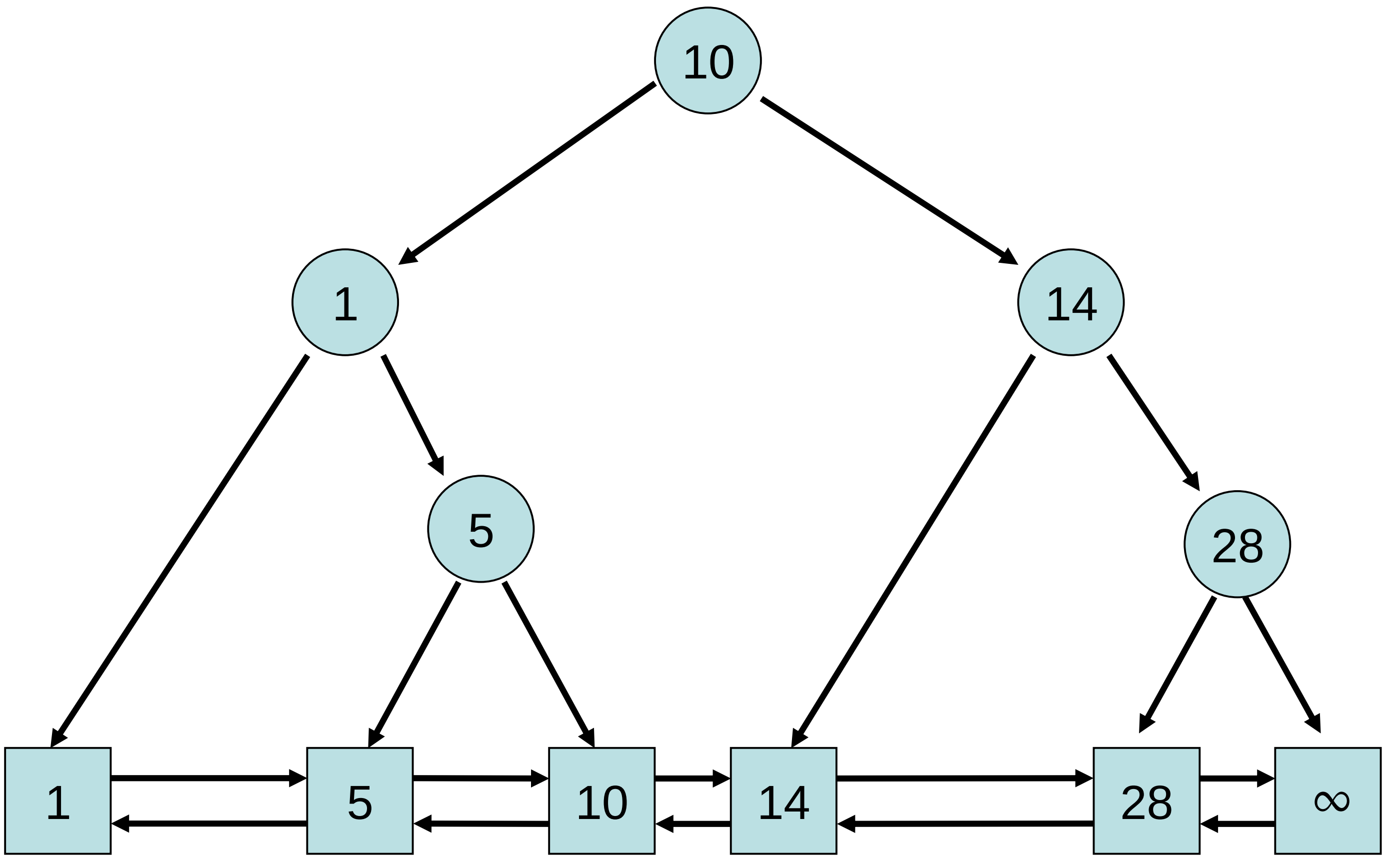
Insert(5)



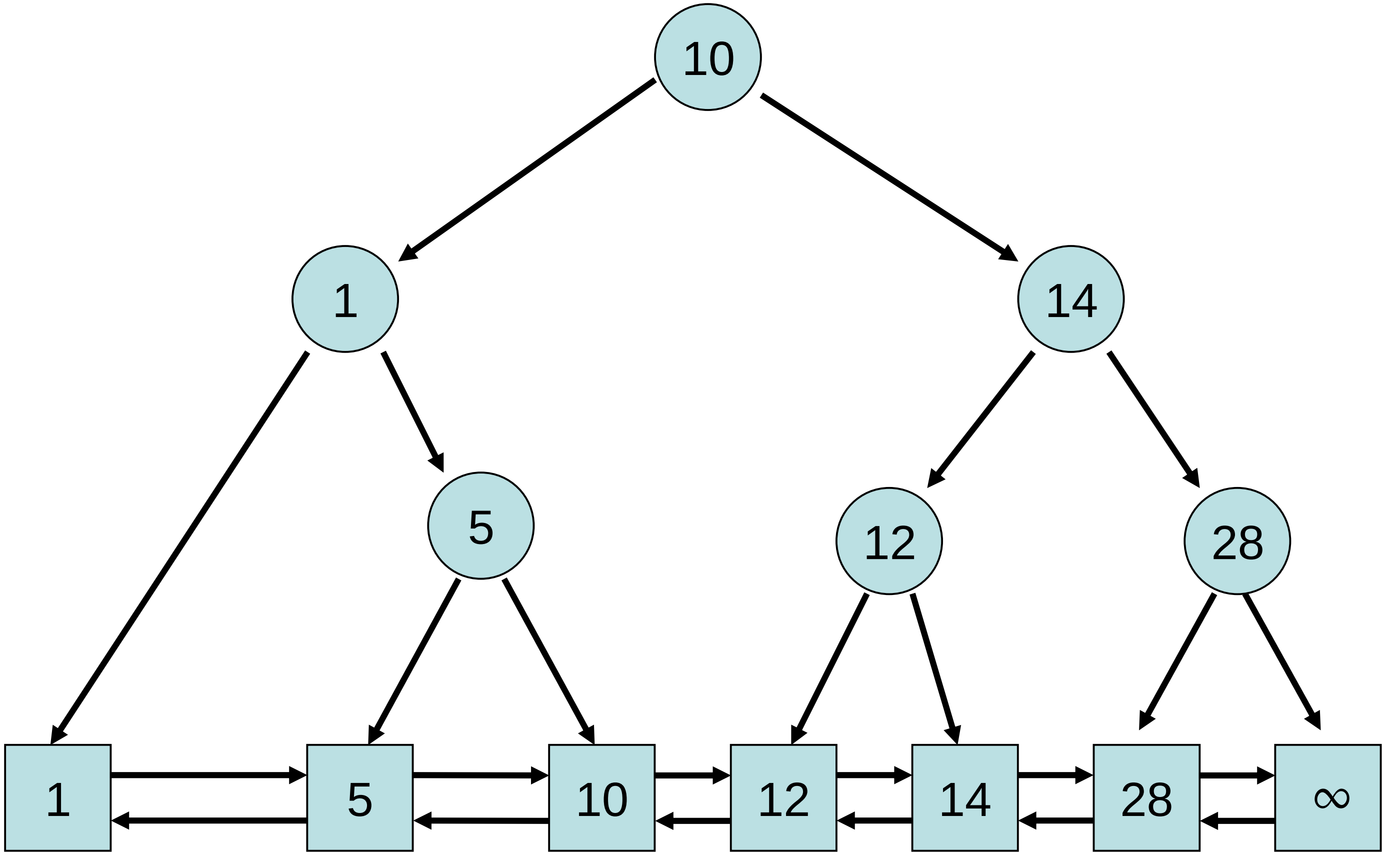
Insert(5)



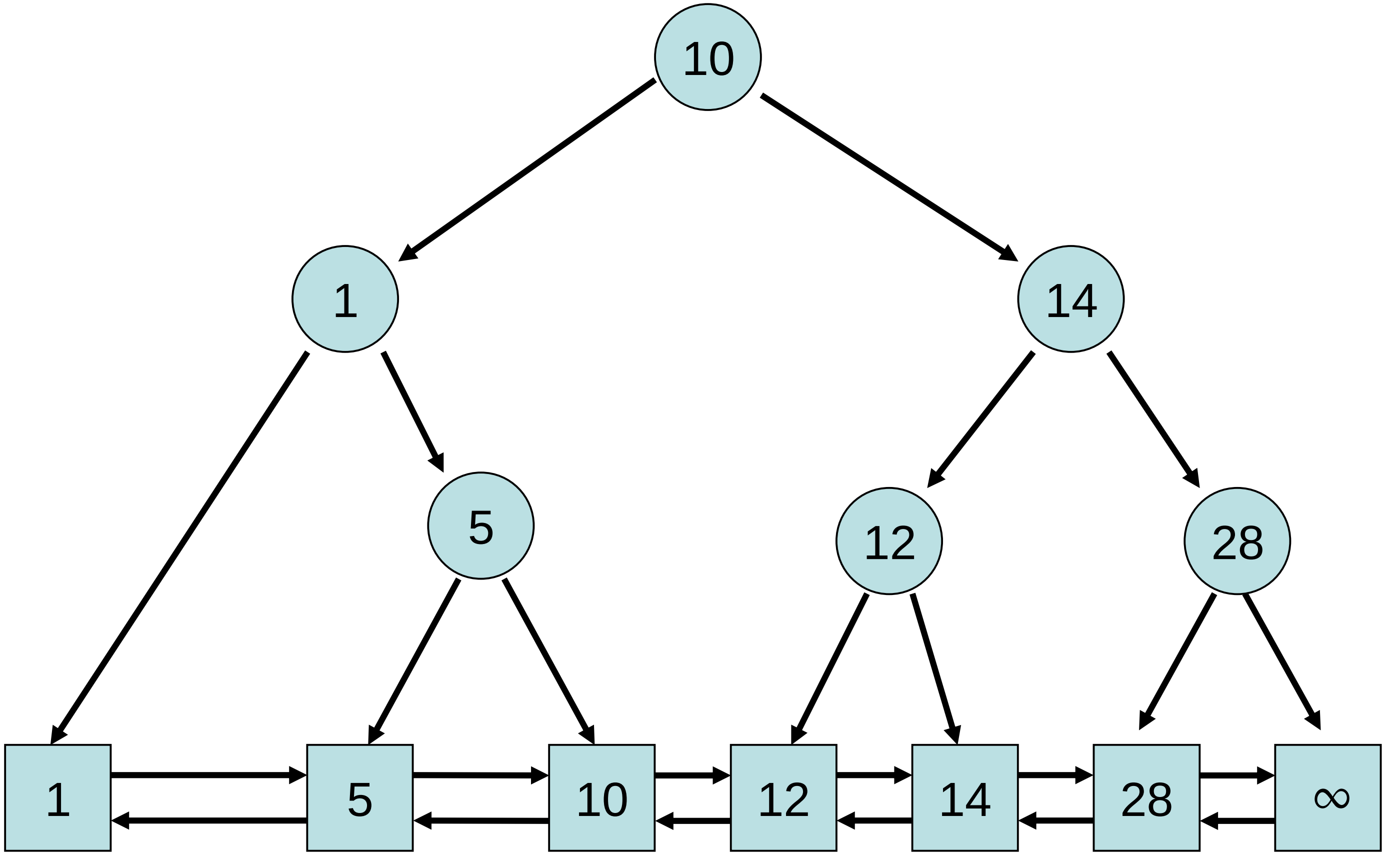
Insert(12)



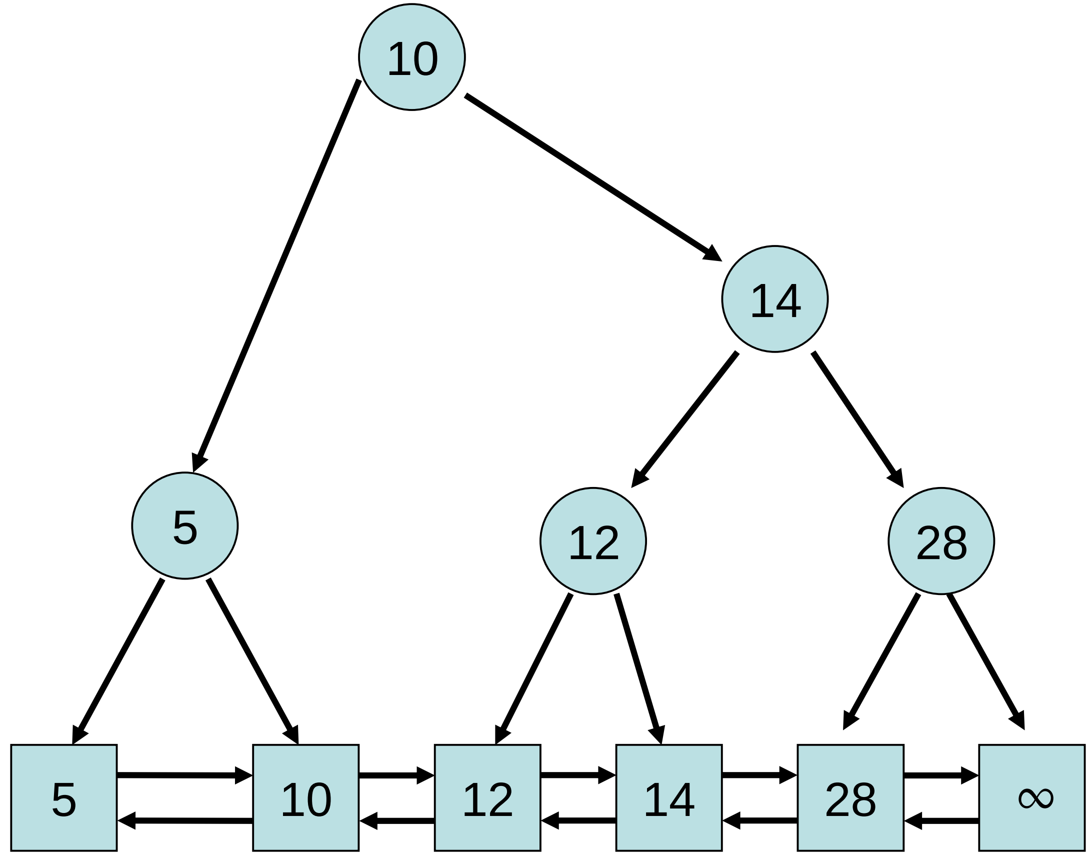
Insert(12)



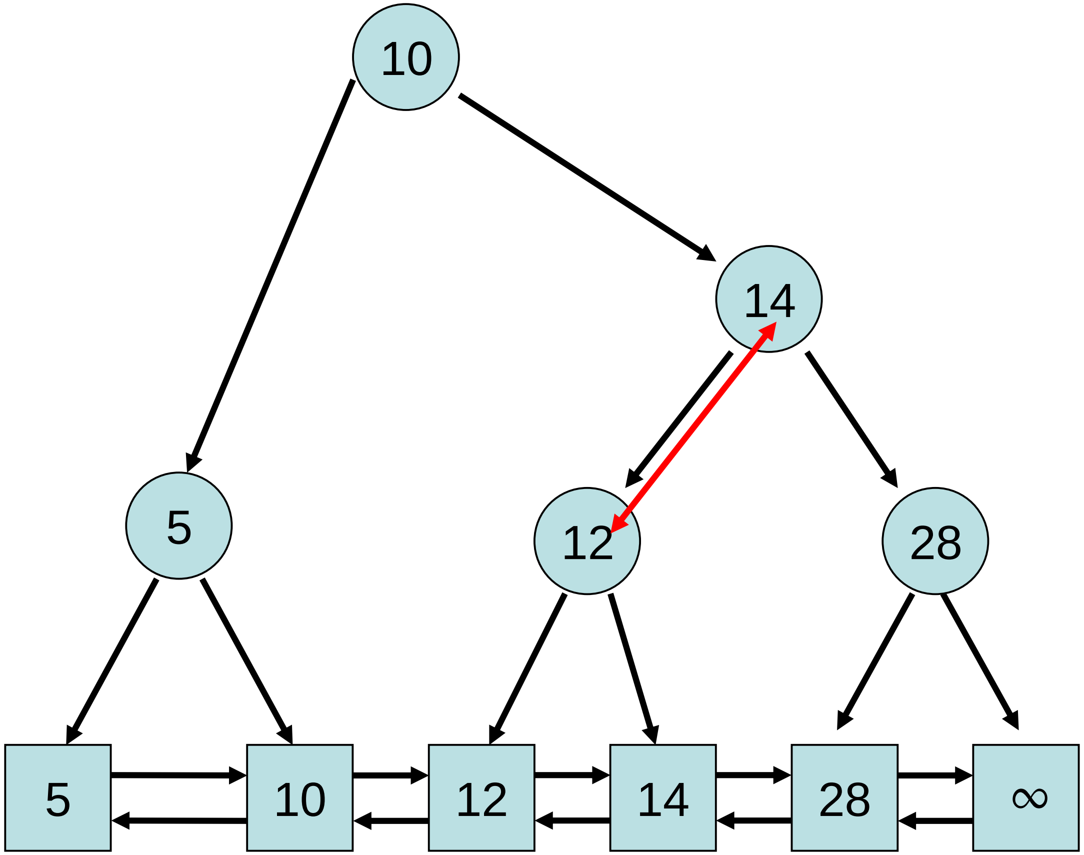
Delete(1)



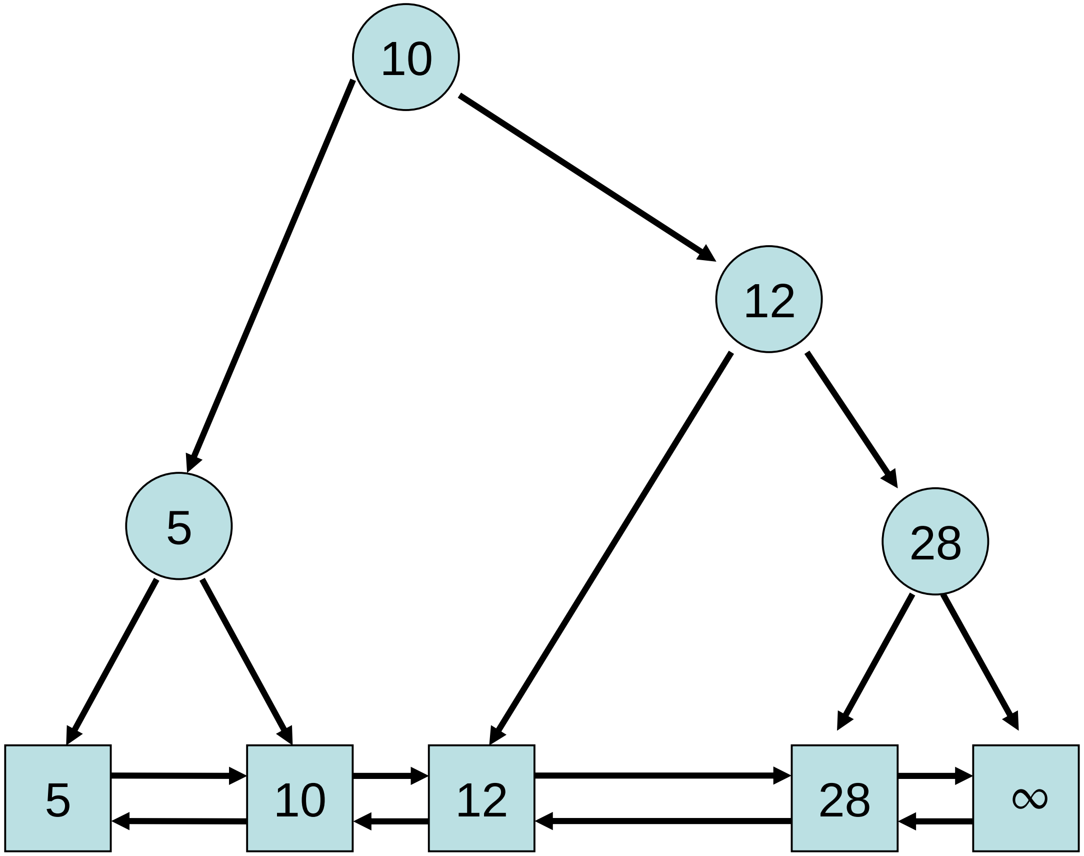
Delete(1)



Delete(14)



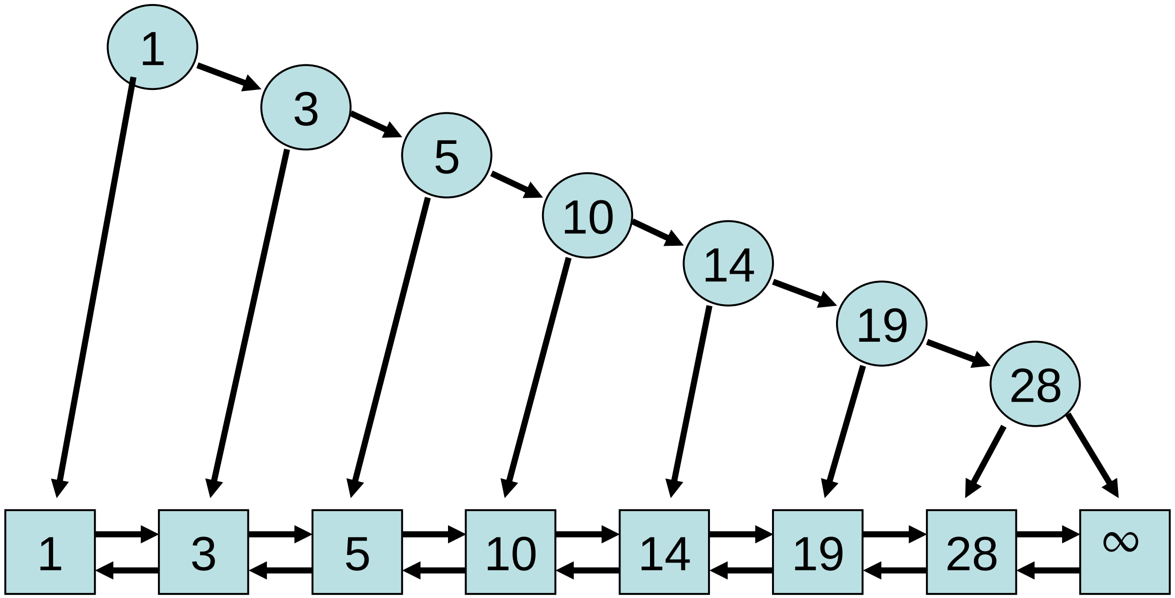
Delete(14)



Binary Search Tree

Problem: binary tree can degenerate!

Example: numbers are inserted in sorted order



Pop quiz

Q1: What is the worst case runtime for binary search on a sorted array?

O(logn).

Q2: What is the worst case runtime for searching in a binary search tree?

O(n)! (see e.g. previous slide)

Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- hashed Patricia trie
  (loglog-search time)

Applications

Splay Tree

Usually: Implementation as internal search tree (i.e., elements directly integrated into tree and not in an extra list)

Here: Implementation as external search tree (like for the binary search tree above)

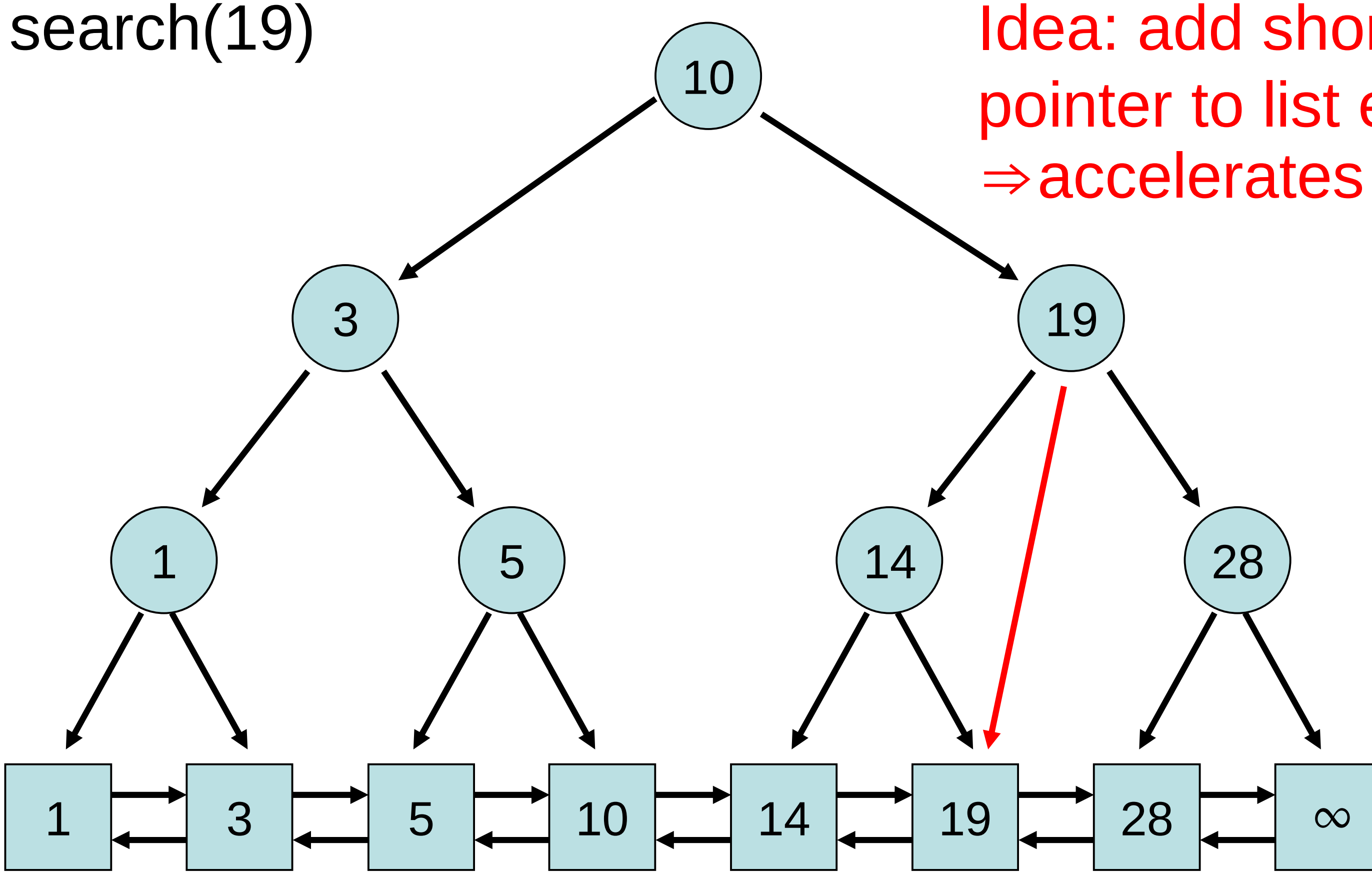
Why Splay Trees?

- Self-adjusting binary search tree
- Invented by Sleator and Tarjan (1985)
- Pros:
  - Recently accessed elements quick to access again. (Great for caches, garbage collection!)
  - Low amortized costs
- Cons:
  - Can still have highly unbalanced trees, hence worst-case linear time search.

Splay Tree

search(19)

Idea: add shortcut pointer to list element
⇒accelerates search



Splay Tree

Ideas:

1. Add shortcut pointers in tree to list elements
2. For every search(k) operation, move pred(k) (the closest predecessor of k in T) to the root (why?)

Movement for (2): via Splay operation

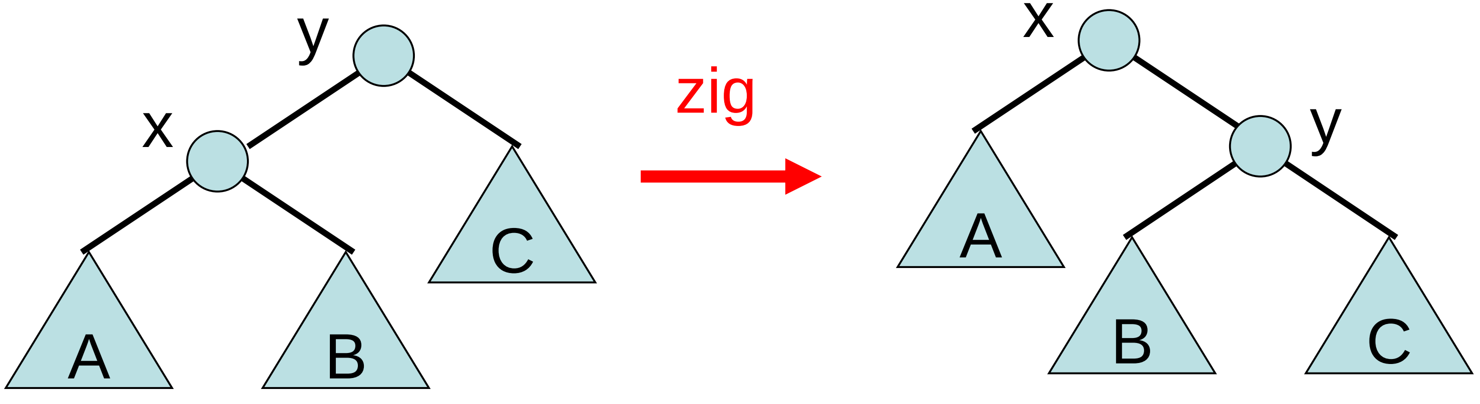
For simplicity: we focus on search(k) for keys k *already in the search tree*.

Splay Operation

Movement of key x to the root: 3 cases.

Case 1:

1a. x is a *left* child of the root:

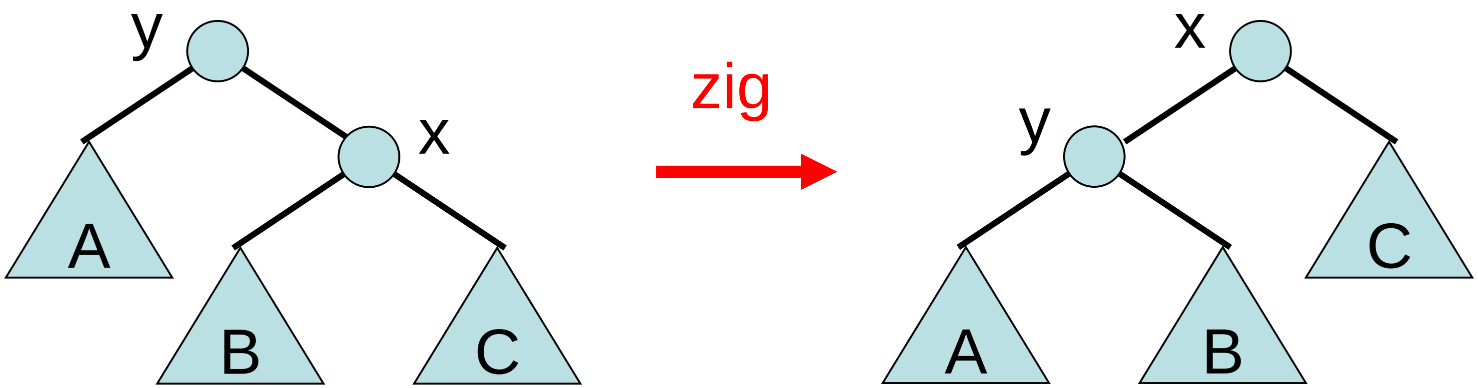


Splay Operation

Movement of key x to the root: 3 cases

Case 1:

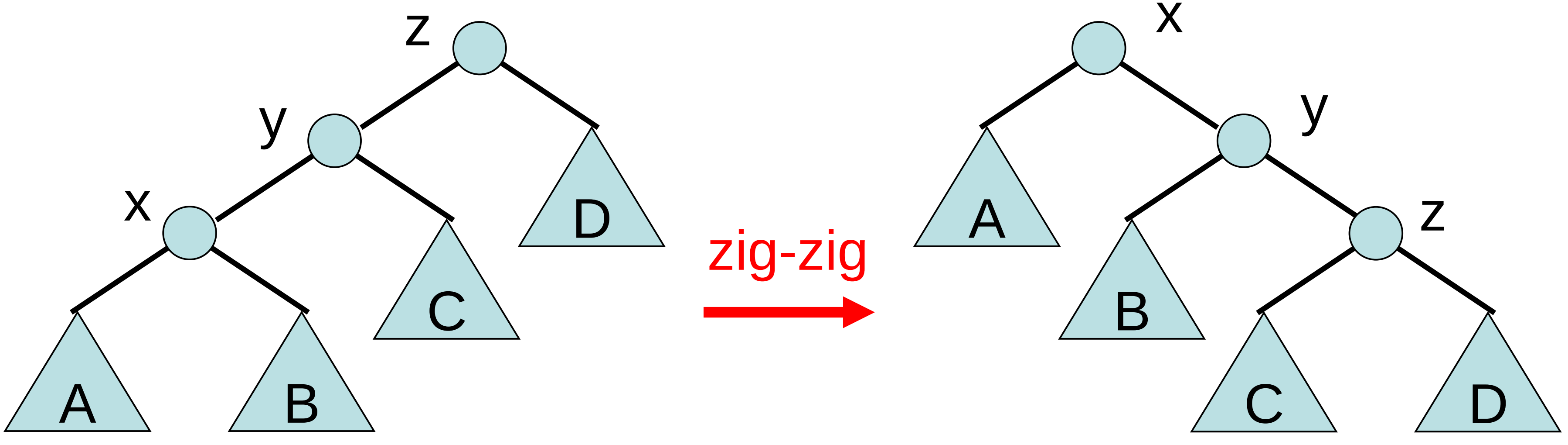
1b. x is a *right* child of the root:



Splay Operation

Case 2:

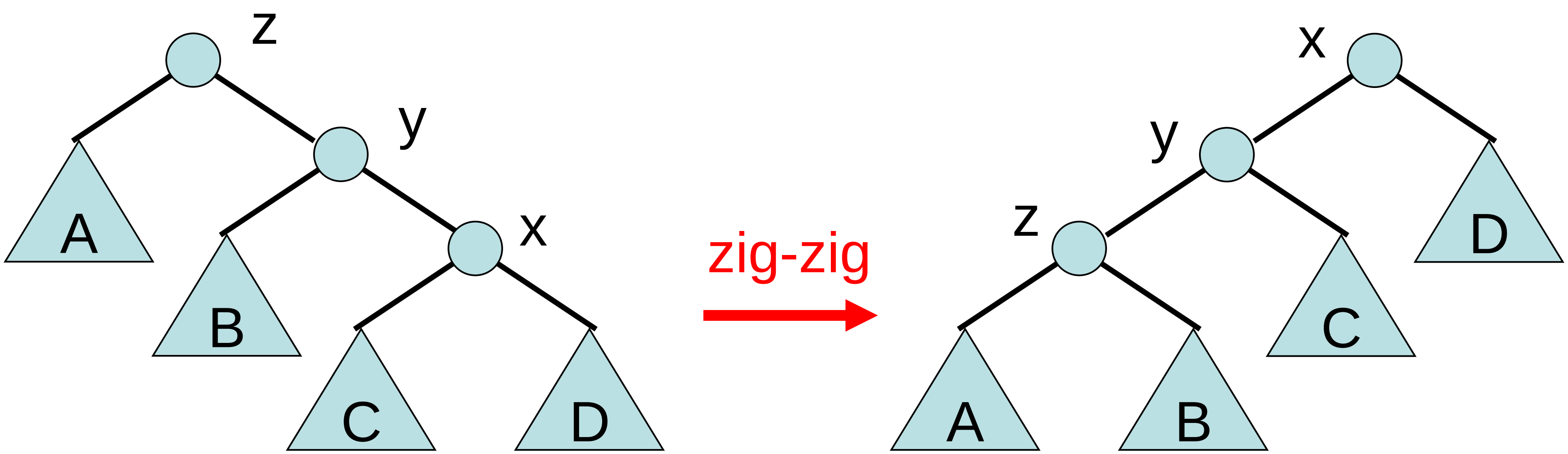
2a. x has father and grand father to the *right*



Splay Operation

Case 2:

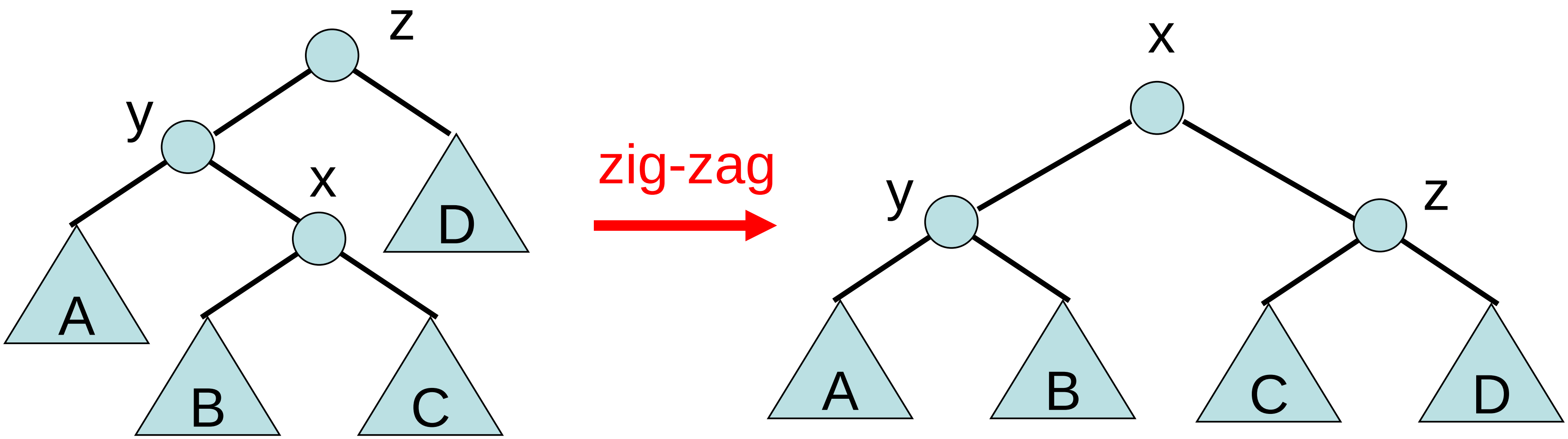
2b. x has father and grand father to the *left*



Splay Operation

Case 3:

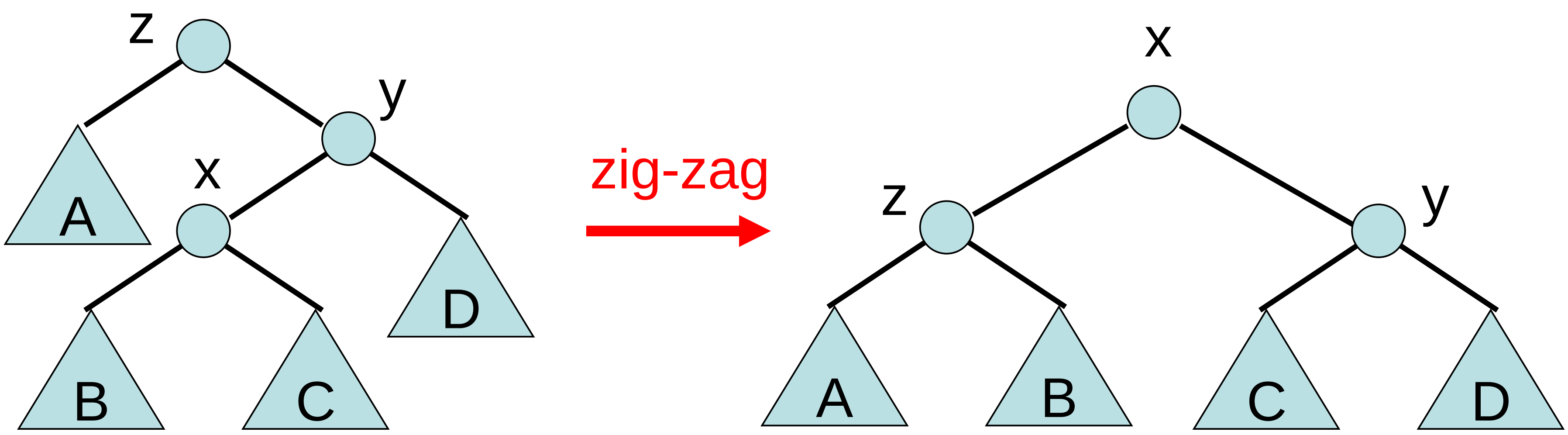
3a. x: father *left*, grand father *right*



Splay Operation

Case 3:

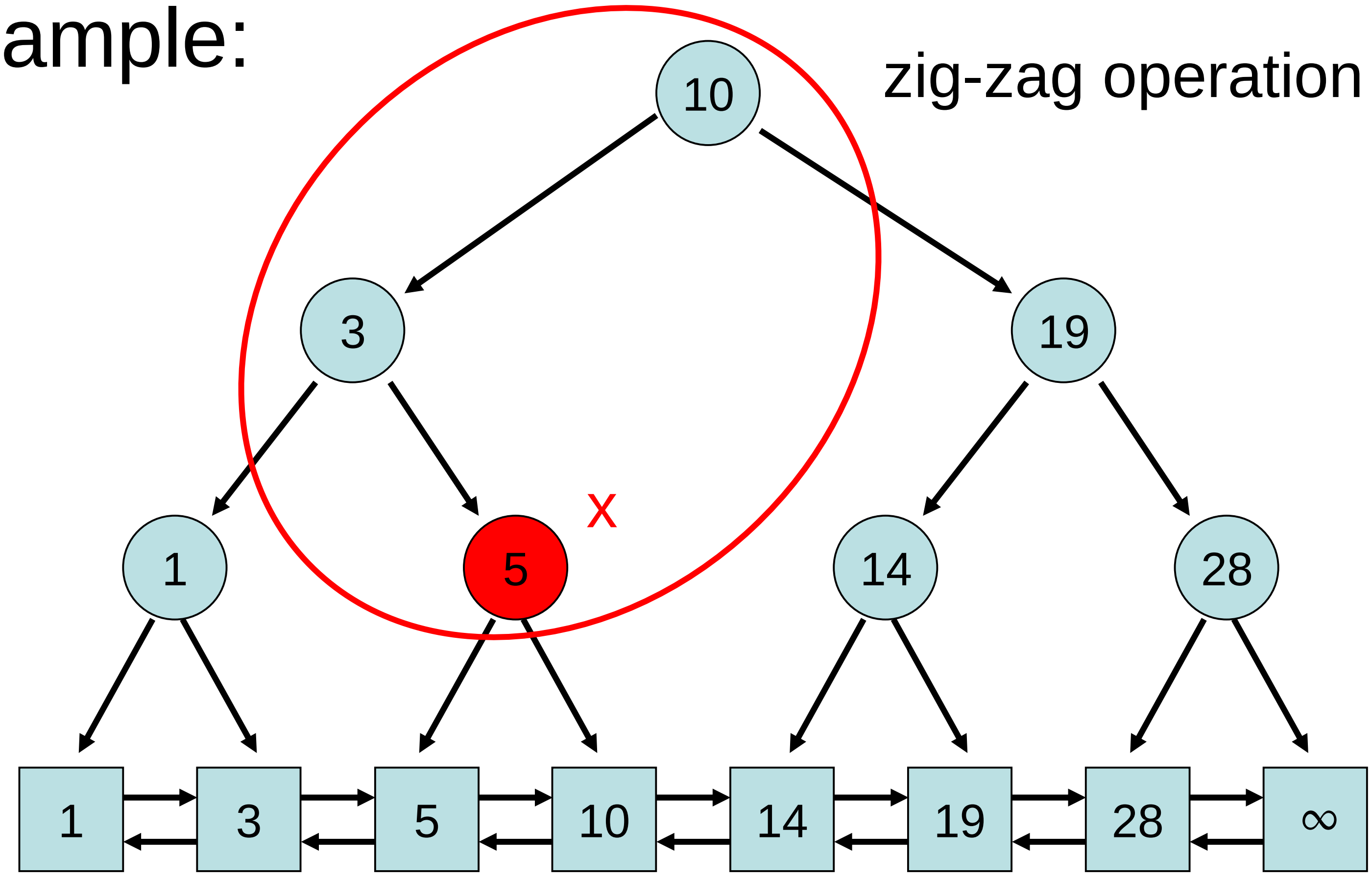
3b. x: father *right*, grand father *left*



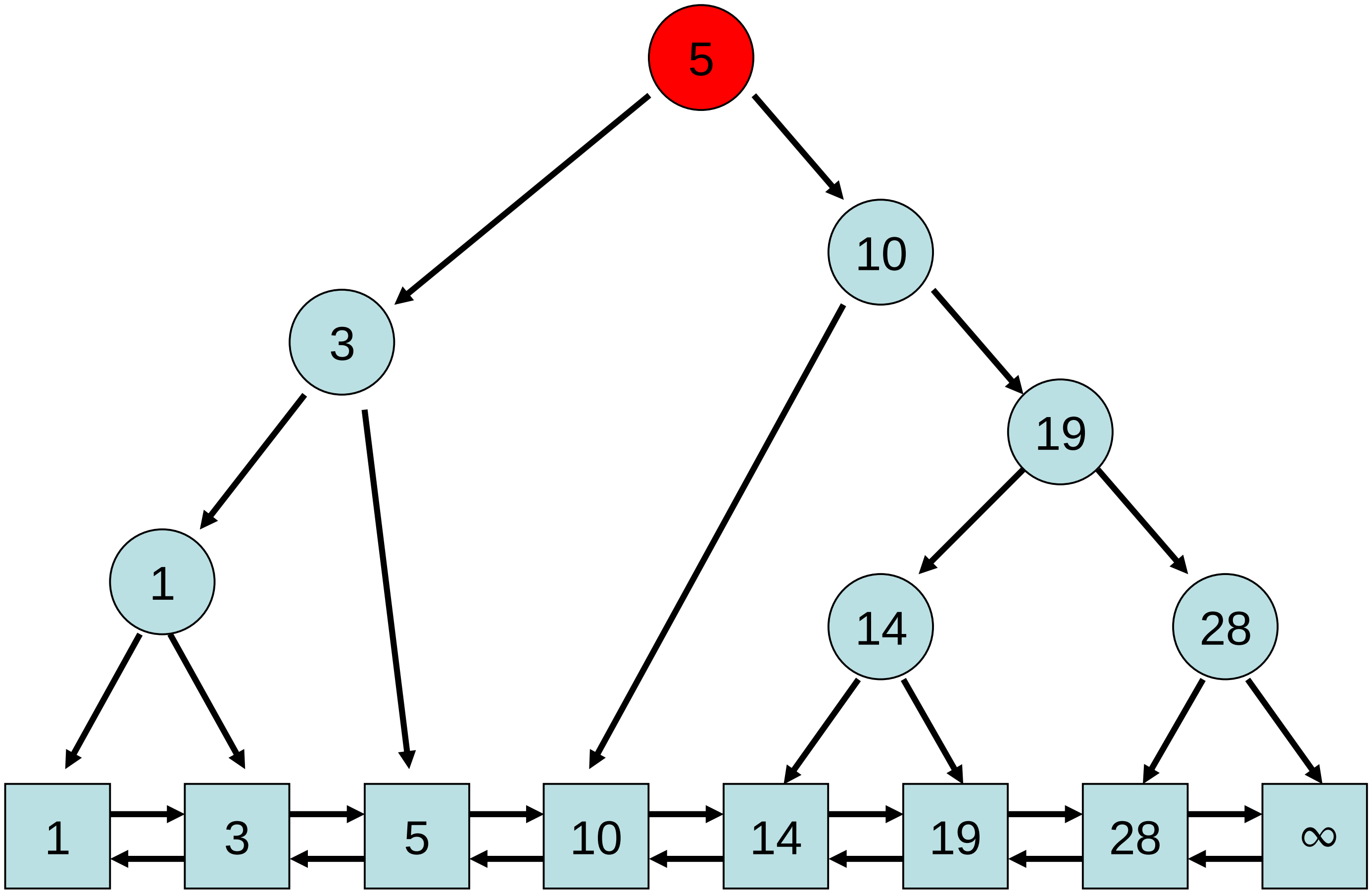
Splay Operation

Example:

zig-zag operation (3a)

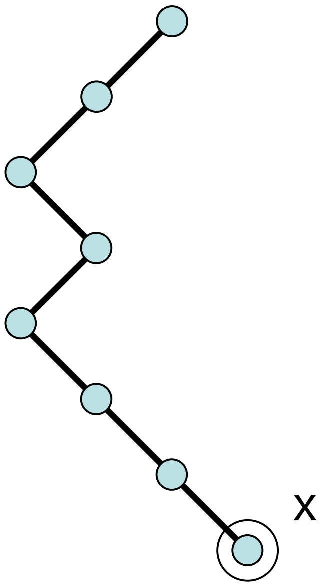


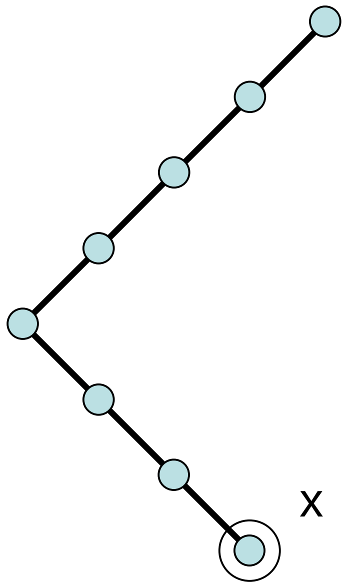
Splay Operation



Splay Operation

Examples:



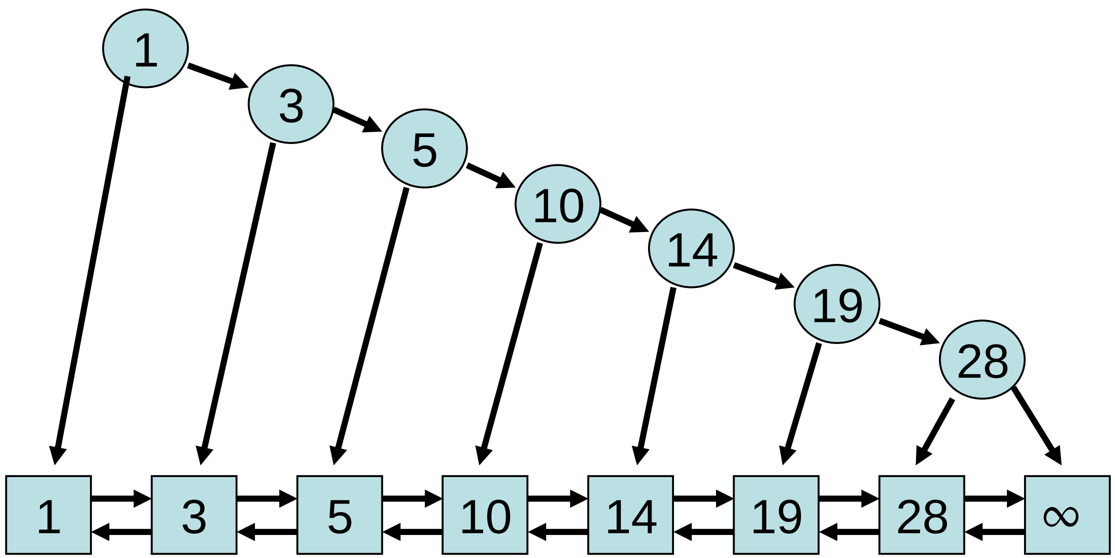


zig-zig, zig-zag, zig-zag, zig

zig-zig, zig-zag, zig-zig, zig

Splay Operation

Observation: Tree can still be highly imbalanced! But amortized costs are low.



Splay Operation

search(k)-operation:

- Move downwards from the root (as in standard binary tree) till pred(k) found in search tree (which can be checked via shortcut to the list) or the list is reached
- call splay(pred(k)), output next successor, succ(k) (recall we assume k exists in tree for simplicity: pred(k)=succ(k)=k)

Amortized Analysis:

- Note: runtime of search(k) is O(runtime of splay(pred(k))).
- Our goal: bound runtime of m Splay operations on arbitrary binary search tree with n elements (m>n)

Splay Operation

- Weight of node x: $w(x)>0$
- Tree weight of tree T with root x: $tw(x)=\sum_{y\in T} w(y)$
- Rank of node x: $r(x) = \log(tw(x))$
- Potential of tree T: $\phi(T) = \sum_{x\in T} r(x)$

Lemma 3.1: Let T be a Splay tree with root x and u be a node in T. The amortized cost for splay(u,T) is at most $1+3(r(x)-r(u))$.

Splay Operation

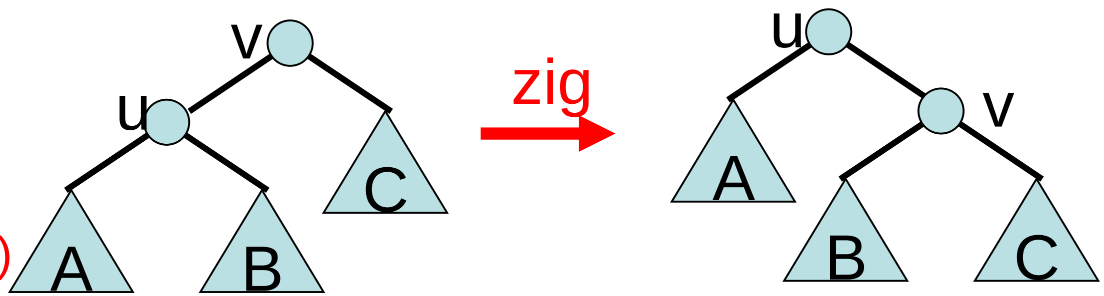
(Recall: Amortized cost $A_X(s) := T_X(s) + (\phi(s') - \phi(s))$)

Proof of Lemma 3.1:

Induction over the sequence of rotations.

- r and tw : rank and weight before the rotation
- r' and tw': rank and weight after the rotation

Case 1:



Runtime (# rotations)

Amortized cost:

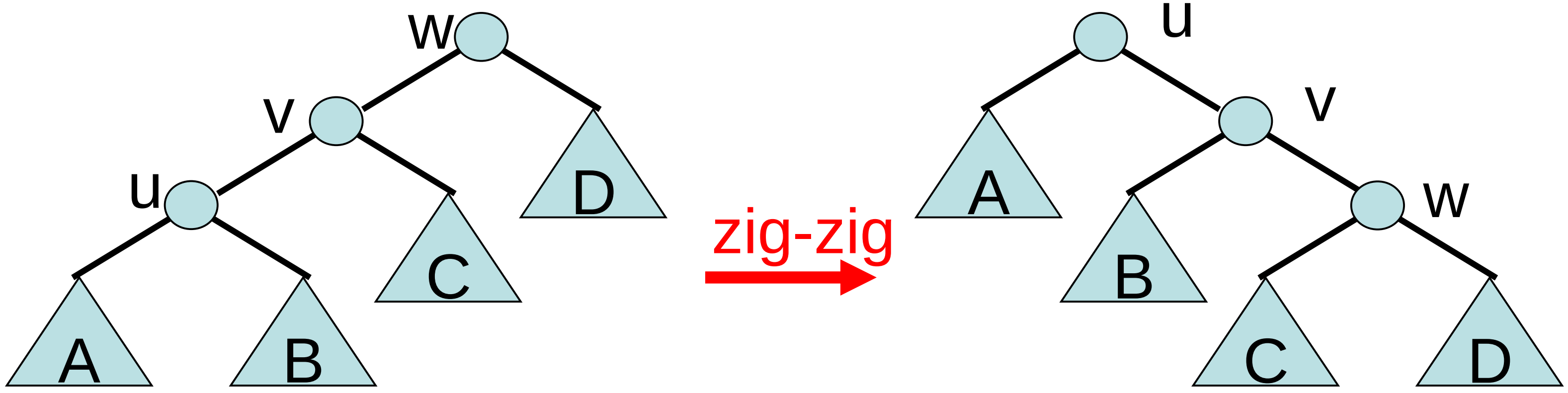
$\leq 1+r'(u)+r'(v)-r(u)-r(v) \leq 1+r'(u)-r(u)$ since $r'(v)\leq r(v)$

$\leq 1+3(r'(u)-r(u))$ since $r'(u)\geq r(u)$

Change in ϕ

Splay Operation

Case 2:



Amortized cost:

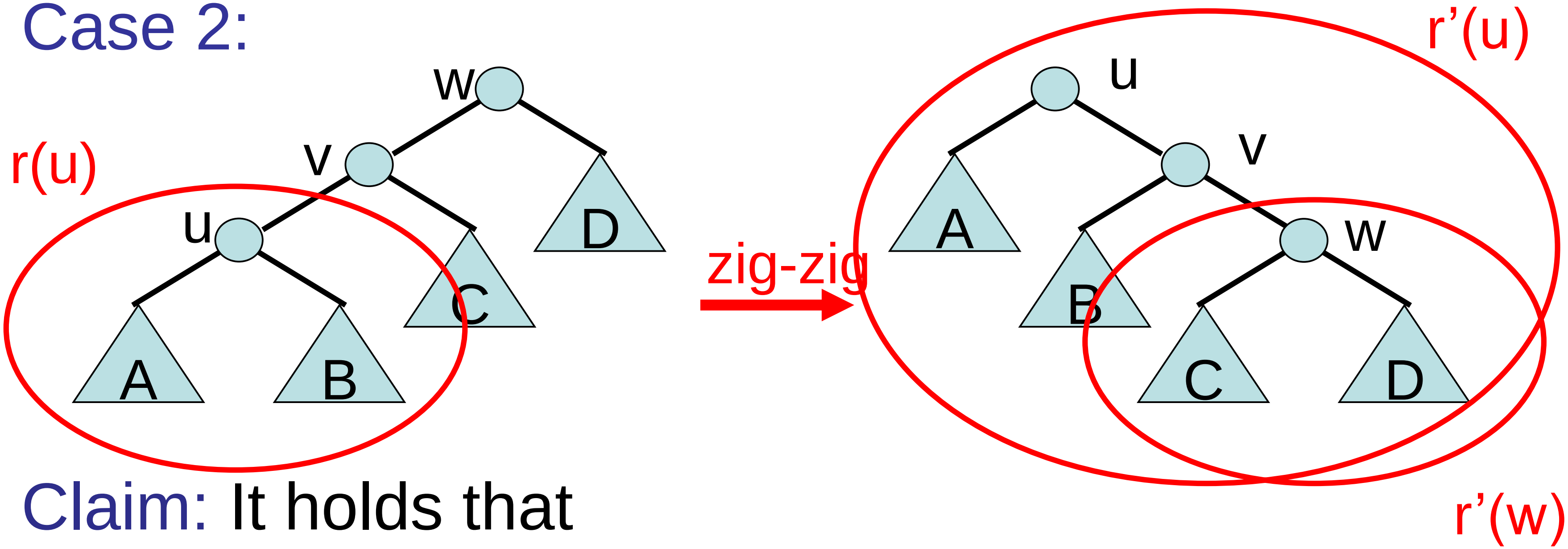
$\leq 2+r'(u)+r'(v)+r'(w)-r(u)-r(v)-r(w)$

$= 2+r'(v)+r'(w)-r(u)-r(v)$ since $r'(u)=r(w)$

$\leq 2+r'(u)+r'(w)-2r(u)$ since $r'(u)\geq r'(v)$ and $r(v)\geq r(u)$

Splay Operation

Case 2:



Claim: It holds that

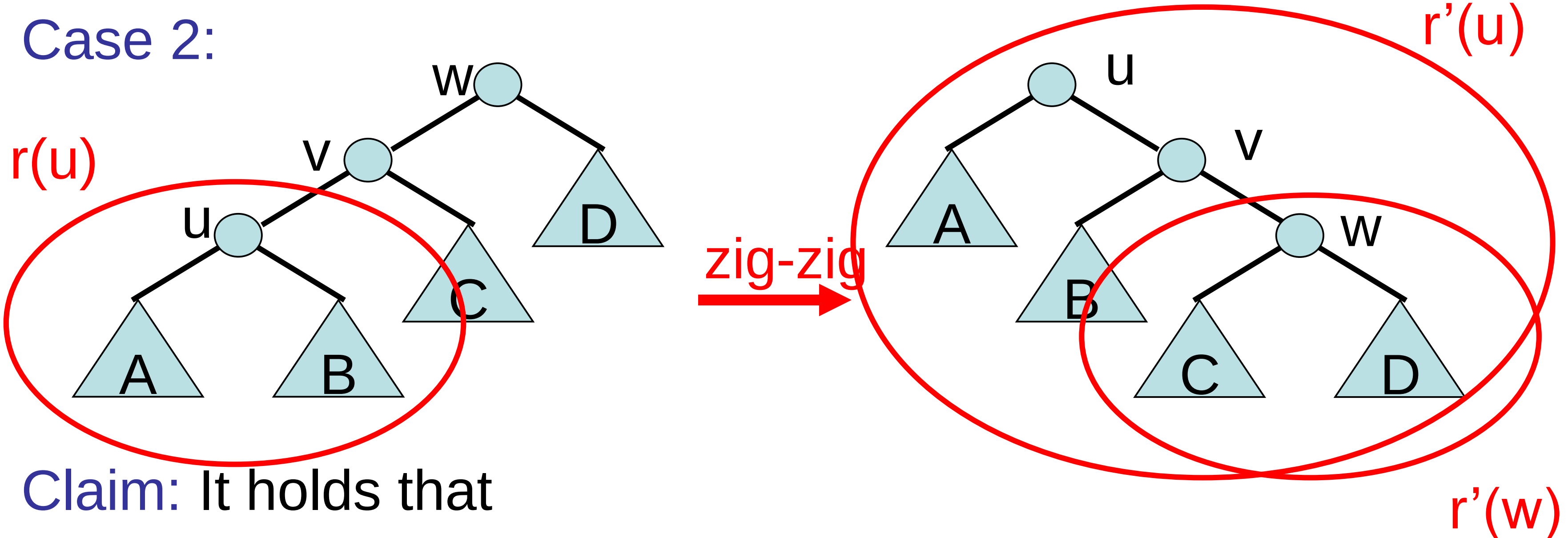
$$2+r'(u)+r'(w)-2r(u) \leq 3(r'(u)-r(u))$$

i.e.

$$r(u)+r'(w) \leq 2(r'(u)-1)$$

Splay Operation

Case 2:



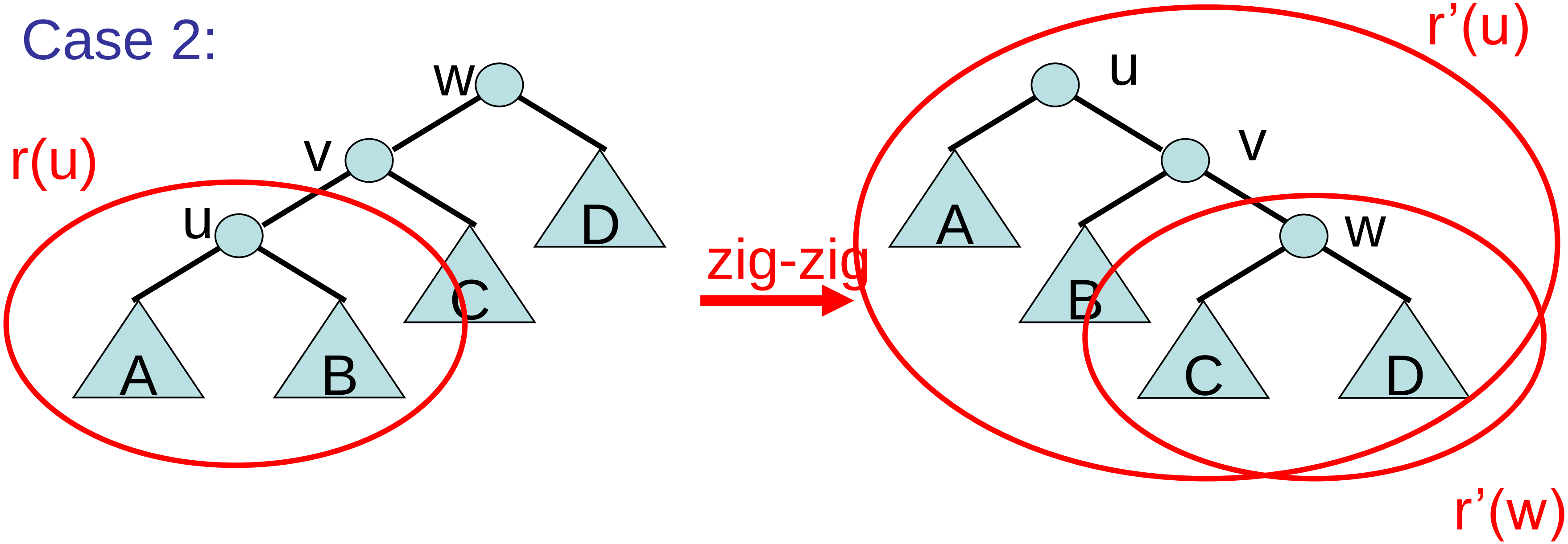
Claim: It holds that

$$r(u)+r'(w) \leq 2(r'(u)-1)$$

- Observe: There exist $0<x,y<1$ and scaling factor $c>0$ with $r(u)=\log(c\cdot x)$, $r'(w)=\log(c\cdot y)$, and $r'(u)\geq\log(c(x+y))$.
- Hence, the claim holds if $\log(c\cdot x)+\log(c\cdot y) \leq 2(\log(c(x+y))-1)$ for all $0<x,y<1$ and $c>0$.

Splay Operation

Case 2:



- For all $0<x,y<1$ and $c>0$ holds:

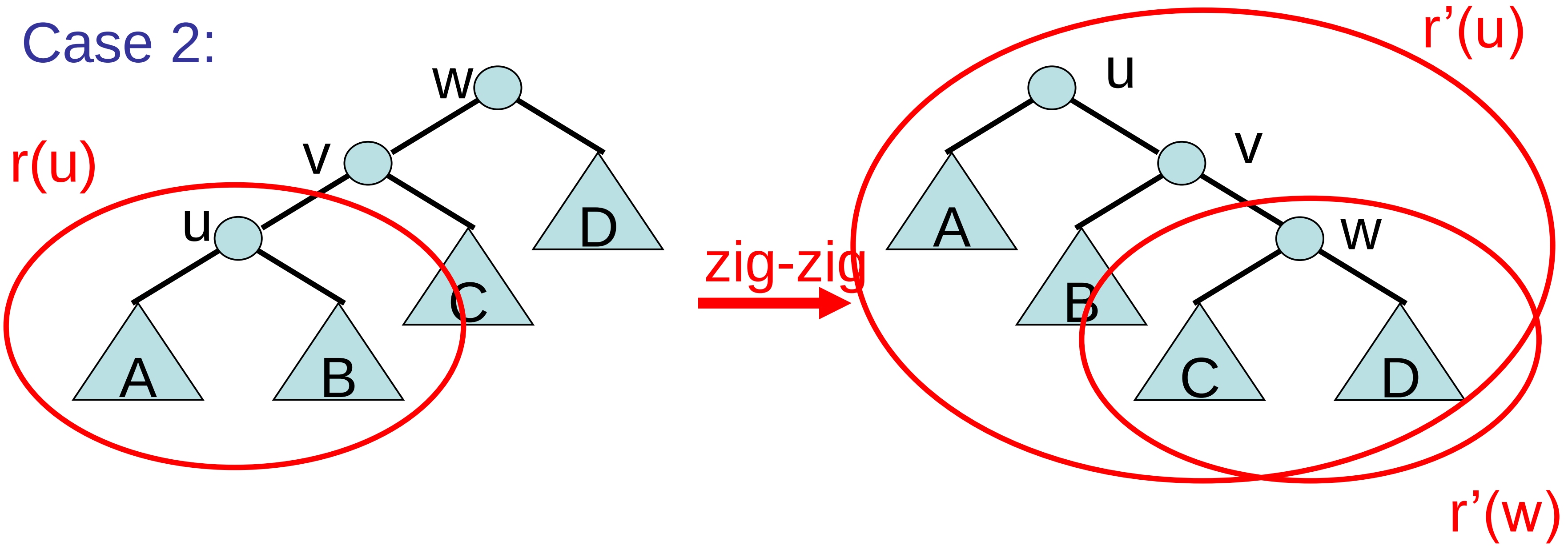
  $\log(c\cdot x)+\log(c\cdot y) \le 2(\log(c(x+y))-1)$

  $\Leftrightarrow \log(x)+\log(y) \le 2(\log(x+y)-1)$

- WLOG set c so that $c(x+y)=1$. Let $x'=c\cdot x$ and $y'=c\cdot y$.

Splay Operation

Case 2:



- To show: for all 0<x',y'≤1,with x'+y'=1:

  $\log(x')+\log(y') \leq 2(\log(1)-1) = -2$

- Or more generally: show for $f(x,y)=\log(x)+\log(y)$ that

  $f(x,y)\leq -2$ for all $x,y>0$ with $x+y\leq 1$

Splay Operation

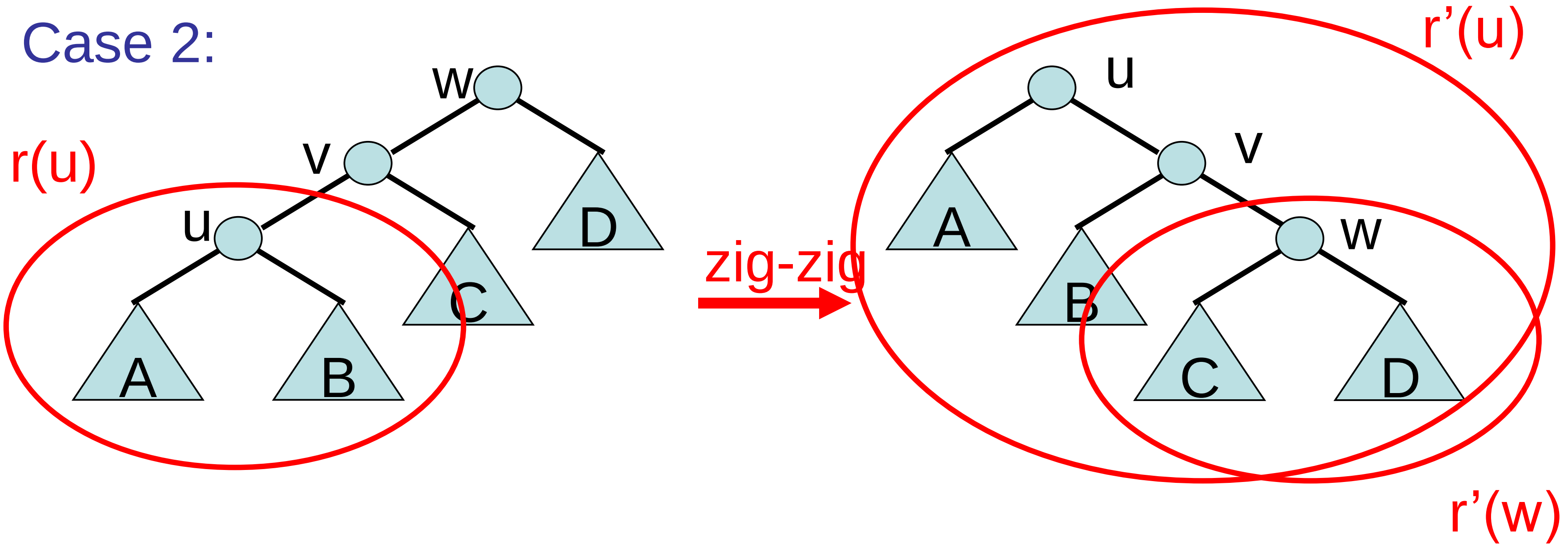
Lemma 3.2: In the area $x,y>0$ with $x+y\leq 1$, the function $f(x,y)=\log x + \log y$ has its maximum at $(½,½)$.

Proof:

- Reduce to univariate problem:
  - $\log x$ is monotonically increasing. Hence, WLOG maximum satisfies $x+y=1$, $x,y>0$.
  - Consider determining the maximum for $g(x) = \log x + \log (1-x)$
- High school calculus: (note base of log WLOG is e)
  - The only root of $g'(x) = 1/x - 1/(1-x)$ is at $x=1/2$.
  - For $g''(x)= -(1/x^2 + 1/(1-x)^2))$ it holds that $g''(1/2)<0$.
- Hence, f has its maximum at $(½,½)$.

Splay Operation

Case 2:

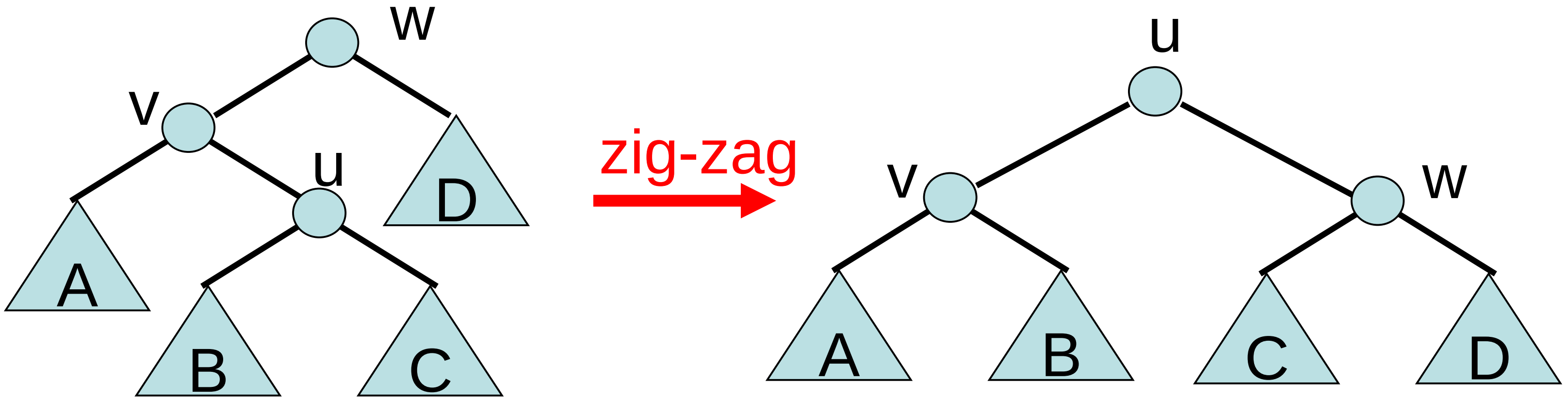


Hence, it holds that $f(x,y) \leq -2$ for all $x,y>0$ with $x+y\leq 1$, which implies the claim that $r(u)+r'(w) \leq 2(r'(u)-1)$, which was equivalent to obtaining upper bound

$$3(r'(u)-r(u)).$$

Splay Operation

Case 3:



Amortized cost:

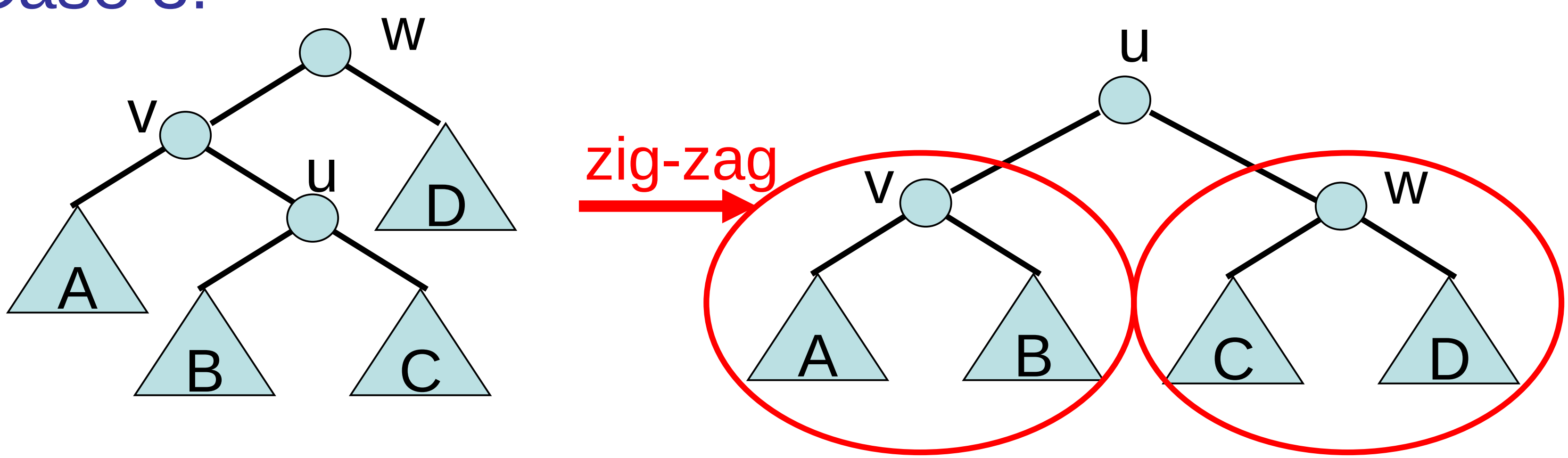
$\leq$ 2+r'(u)+r'(v)+r'(w)-r(u)-r(v)-r(w)

$\leq$ 2+r'(v)+r'(w)-2r(u) since r'(u)=r(w) and r(u)$\leq$r(v)

$\leq$ 2(r'(u)-r(u)) because…

Splay Operation

Case 3:



…it holds that:

$$2+r'(v)+r'(w)-2r(u) \leq 2(r'(u)-r(u))$$

$$\Leftrightarrow \quad 2r'(u)-r'(v)-r'(w) \geq 2$$

$$\Leftrightarrow \quad r'(v)+r'(w) \leq 2(r'(u)-1)$$, which can be shown to hold

Splay Operation

Proof of Lemma 3.1: (Follow-up)

Induction over the sequence of rotations.

- r and tw : rank and weight before the rotation
- r' und tw': rank and weight after the rotation
- For every rotation (i.e. zig, zig-zig, or zig-zag), the amortized cost is <= $1+3(r'(u)-r(u))$ (case 1) resp. $3(r'(u)-r(u))$ (cases 2 and 3)
- Summation of the costs gives at most (x: root)
  $1 + \sum_{Rotations} 3(r'(u)-r(u)) = 1+3(r(x)-r(u))$
  - 1. Why do we only add 1 before the summation?
  - 2. Why do we get a telescoping series above?

Splay Operation

- Tree weight of tree T with root x: $tw(x) = \sum_{y \in T} w(y)$
- Rank of node x: $r(x) = \log(tw(x))$
- Potential of tree T: $\phi(T) = \sum_{x \in T} r(x)$

Lemma 3.1: Let T be a Splay tree with root x and u be a node in T. The amortized cost for $splay(u,T)$ is at most $1+3(r(x)-r(u)) = 1+3 \cdot \log(tw(x)/tw(u))$.

Corollary 3.3: Let $W = \sum_x w(x)$ and w_i be the weight of key k_i in the i-th search call (recall we assume k_i is in tree). For m search operations, the amortized cost is $O(m + \sum_{i=1}^{m} \log (W/w_i))$.

Splay Tree

Theorem 3.4: The runtime for m successful search operations in a Splay tree T with n elements is at most

$$O(m+(m+n)\log n).$$

Proof:

- Let $w(x) = 1$ for all nodes x in T.
- Then $W=n$ and $r(x) \leq \log W = \log n$ for all x in T.
- For sequence F of operations, total runtime satisfies $T(F) \leq A(F) + \phi(s_0)$ for any amortized cost function A and any initial state s_0 (Recall: $A_X(s) := T_X(s) + (\phi(s') - \phi(s))$)
- $\phi(s_0) = \sum_{x\in T} r_0(x) \leq n \log n$
- Hence, Corollary 3.3 implies Theorem 3.4.

Splay Tree

Suppose we have a probability distribution for the search requests, where each key in tree is searched for at least once.

- $p(x)$: probability of searching for key x
- $H(p) = \sum_x p(x) \cdot \log(1/p(x))$: entropy of p

Theorem 3.5: The expected runtime for m successful search operations in a Splay tree T with n elements is at most

$$O(m \cdot (1+H(p))).$$

Proof: Follows from proof of Theorem 3.4 with $w(x) = p(x)$ for all x, and assuming each item x is searched for $m \cdot p(x)$ times.

Note: This proof requires us to relax our requirement that the potential function ϕ is non-negative. Why?

Splay Tree

Something amazing:

For a *fixed* optimal Binary Search Tree where each key x in tree is searched for with probability p(x), one can show expected cost of a successful search is

$\Omega(H(p))$ (*entropy bound)*.

Our Theorem 3.5 says Splay Trees are almost optimal, in that the cost per search scales as $O(1+H(p))$!

Note: $0 \le H(p) \le \log n$

Question: How does this O(1+H(p)) support the idea that Splay trees would be good for applications like caching?

Splay Tree

So far, we assumed all searches were successful, i.e. the key we were searching for was in the tree.

Q1: Where in our analysis did this assumption play a role?

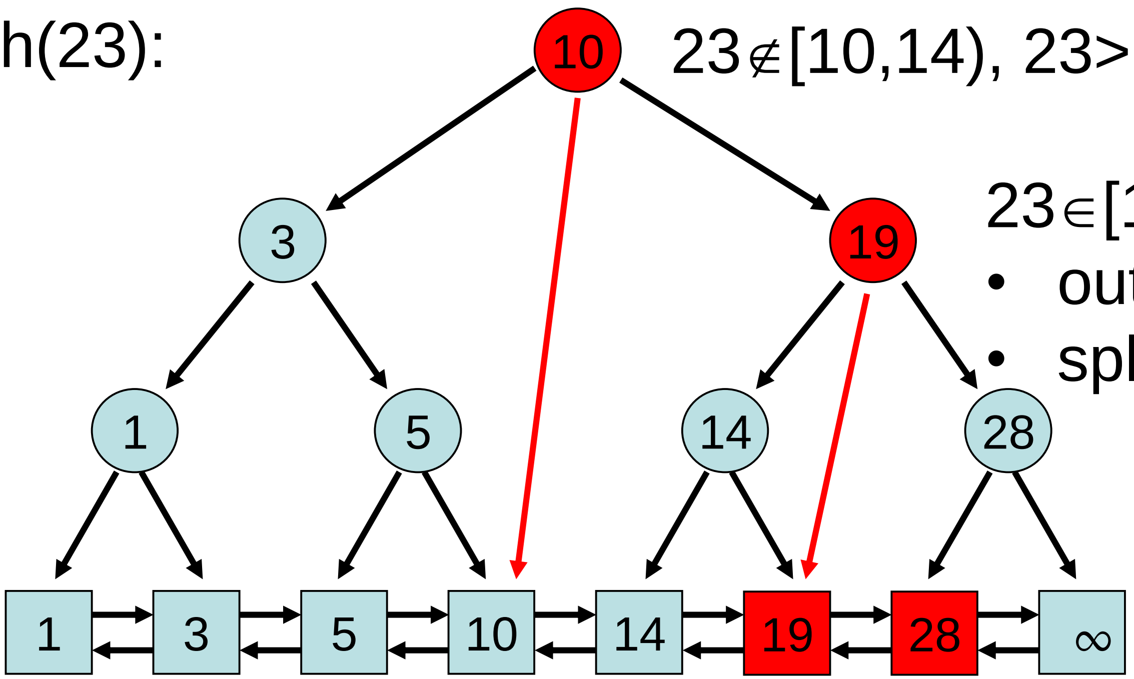
Q2: What if we consider the more general case of allowing unsuccessful searches?

Splay Tree – Unsuccessful Searches

- Instead of just successful searches, the Splay tree T should also support the search for the closest successor.

search(23):

$23 \notin [10,14)$, $23>10$



$23 \in [19,28)$:

- output 28
- splay(19)

Splay Tree – Unsuccessful Searches

- To obtain a low amortized time bound, we associate with a key x in T the search range $[x,x_+)$ (including x but excluding x_+), where x_+ is closest successor of x in T.
- Each search range $[x,x_+)$ is associated with a weight $w([x,x_+))$. Using that, we can revise Corollary 3.3 to:

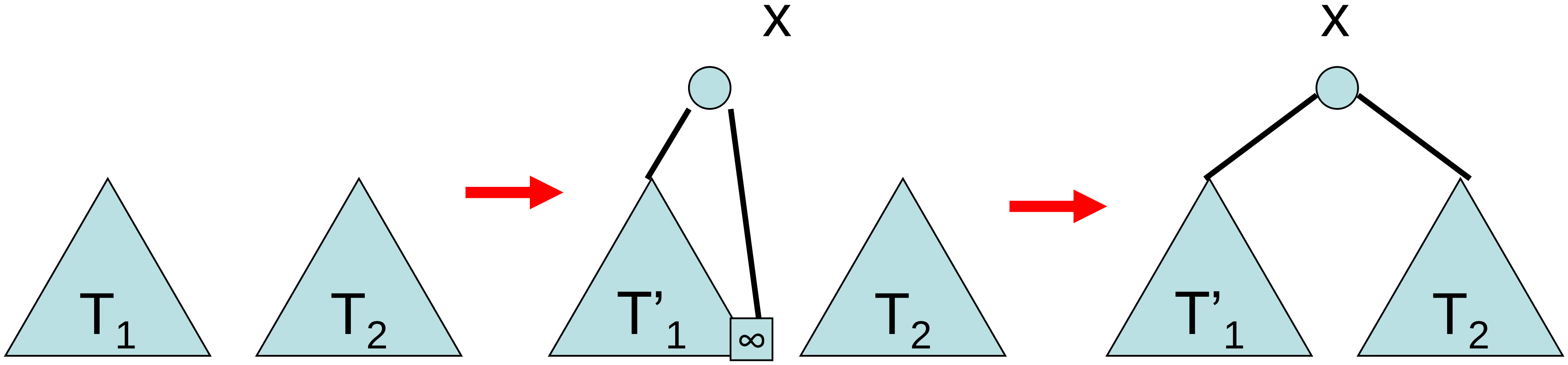
Corollary 3.3': Let $W=\sum_x w(x)$ and w_i be the weight of the range $[x,x_+)$ containing the i-th search key. For m search operations, the amortized cost is

$$O(m + \sum_{i=1}^{m} \log (W/w_i)).$$

Splay Tree Operations

Let T_1 and T_2 be two Splay trees with key(x)<key(y) for all $x \in T_1$ und $y \in T_2$.

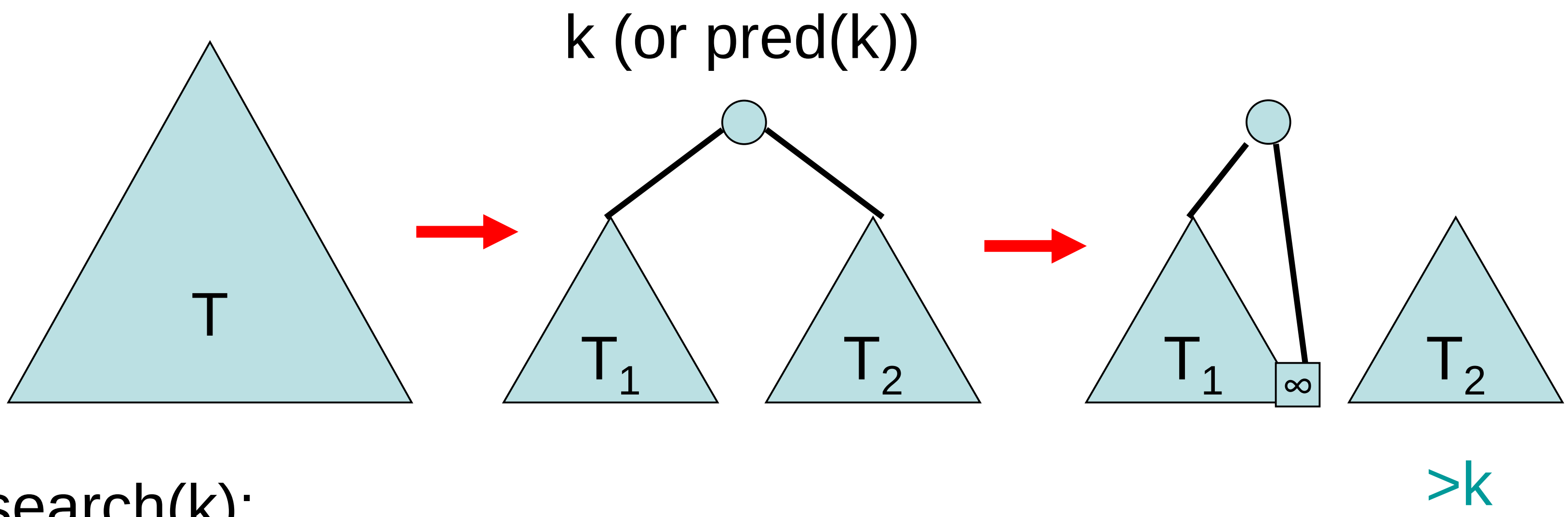
merge(T_1,T_2):



Take max. element $x<\infty$ in T_1 and splay it up to root

Splay Tree Operations

split(k,T): returns two trees as follows



search(k):
causes splay(k)
or splay(pred(k))

Splay Tree Operations

insert(e):

- insert like in binary search tree
- Splay operation to move key(e) to the root

delete(k):

- execute search(k) (splays k to the root)
- remove root and execute merge(T_1,T_2) of the two resulting subtrees

Splay Operations

- k_-: closest predecessor $\leq k$ in T
- k_+: closest successor $>k$ in T

Theorem 3.6: The amortized cost of the following operations in the Splay tree are:

- search(k): $O(1+\log(W/w([k_-,k_+])))$
- insert(e): $O(1+\log(W/w([key(e),key(e)_+])))$
- delete(k): $O(1+\log(W/w([k,k_+])) + \log((W-w([k,k_+]))/w([k_-,k])))$

Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- hashed Patricia trie
  (loglog-search time)

Applications

(a,b)-Trees

Problem: how to maintain balanced search tree

Idea:

- All nodes v (except for the root) have degree $d(v)$ with $a \leq d(v) \leq b$, where $a \geq 2$ and $b \geq 2a-1$ (otherwise this cannot be enforced)
- All leaves have the same depth

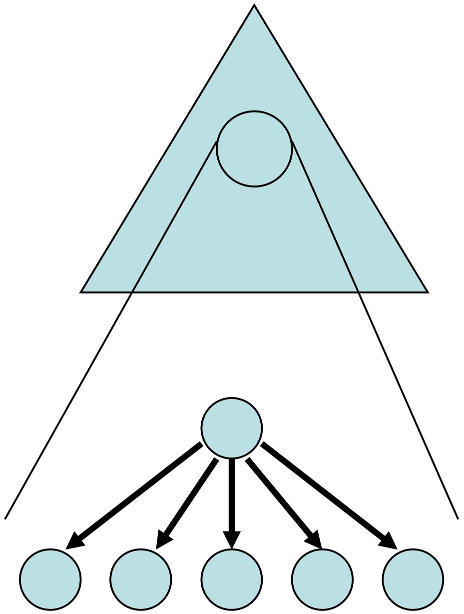
(a,b)-Trees

Formally: for a tree node v let

- d(v) be the number of children of v
- t(v) be the depth of v (root has depth 0)

- Form Invariant:
  For all leaves v,w: t(v)=t(w)

- Degree Invariant:
  For all inner nodes v
  except for root: $d(v) \in [a,b]$,
  for root r: $d(r) \in [2,b]$
  (as long as #elements >1)



(a,b)-Trees

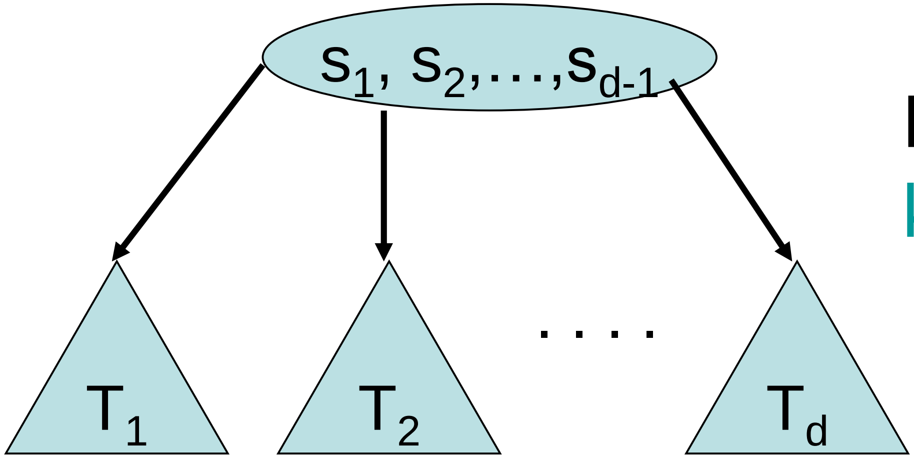
Lemma 3.10: An (a,b)-tree with n elements has depth at most $1+\lfloor \log_a (n/2) \rfloor$

Proof:

- The root has degree ≥ 2 and every other inner node has degree $\geq a$.
- At depth t there are at least $2a^{t-1}$ nodes
- $n \geq 2a^{t-1} \Leftrightarrow t \leq 1+\lfloor \log_a(n/2) \rfloor$

(a,b)-Trees

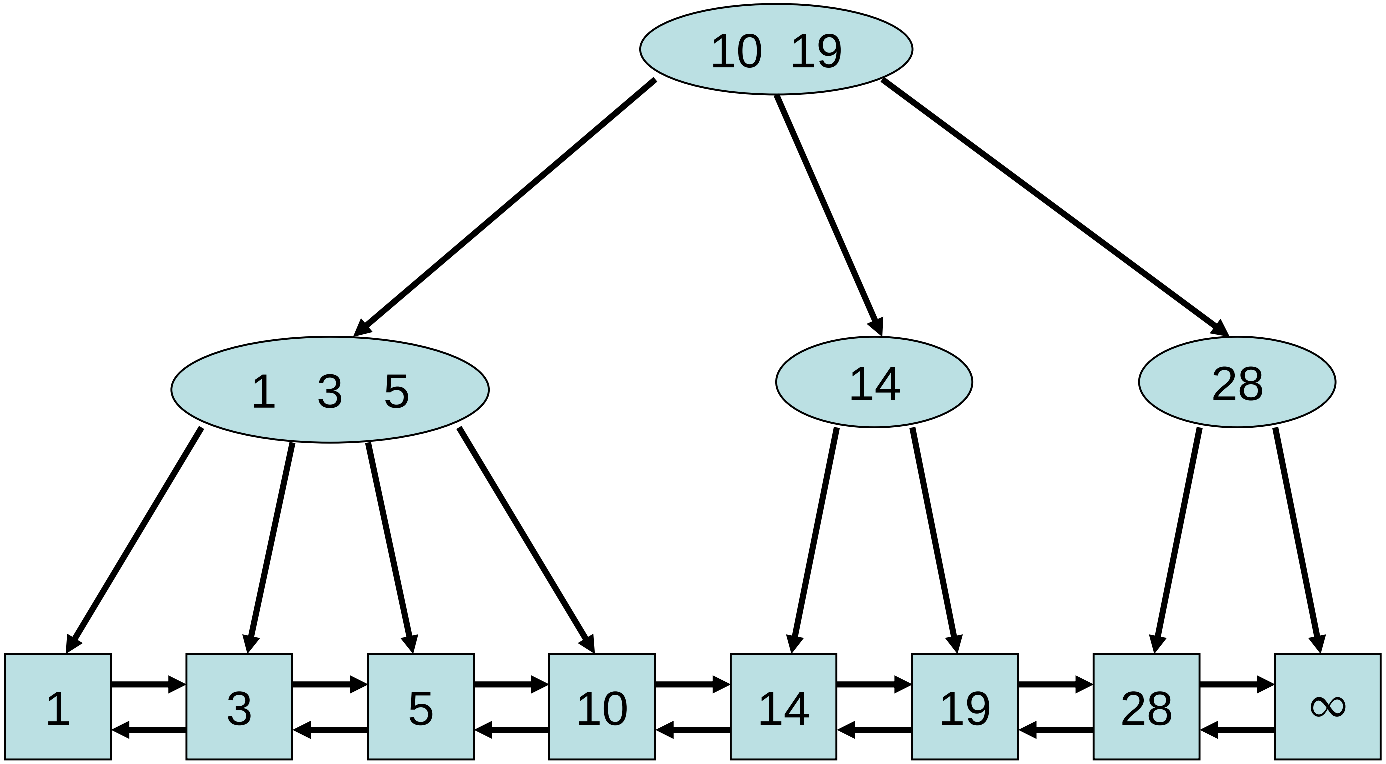
(a,b)-Tree-Rule:



For all keys k in T_i and k' in T_{i+1}: $k \leq s_i < k'$

Then search operation easy to implement.

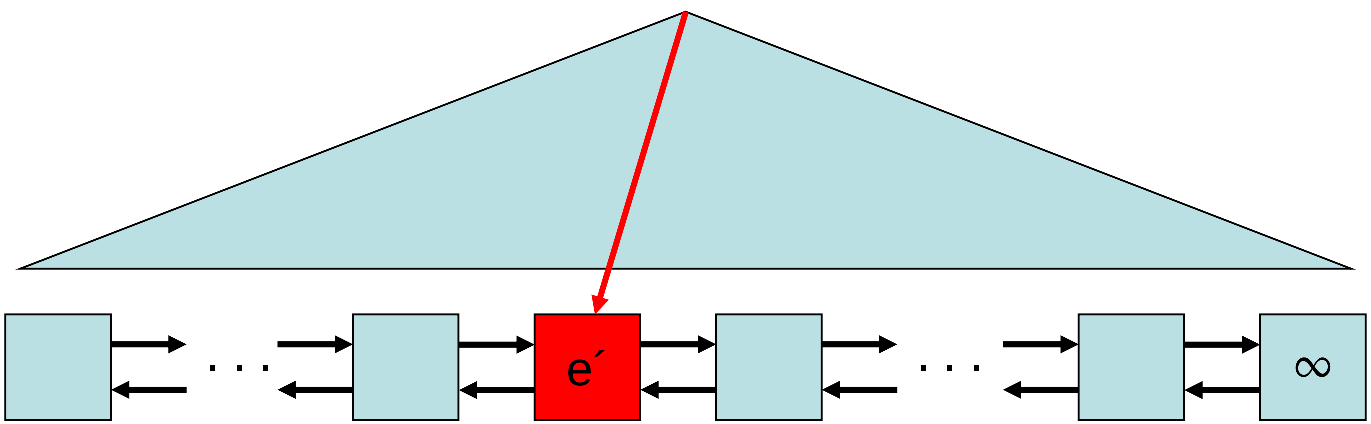
Search(9)



Insert(e) Operation

Strategy:

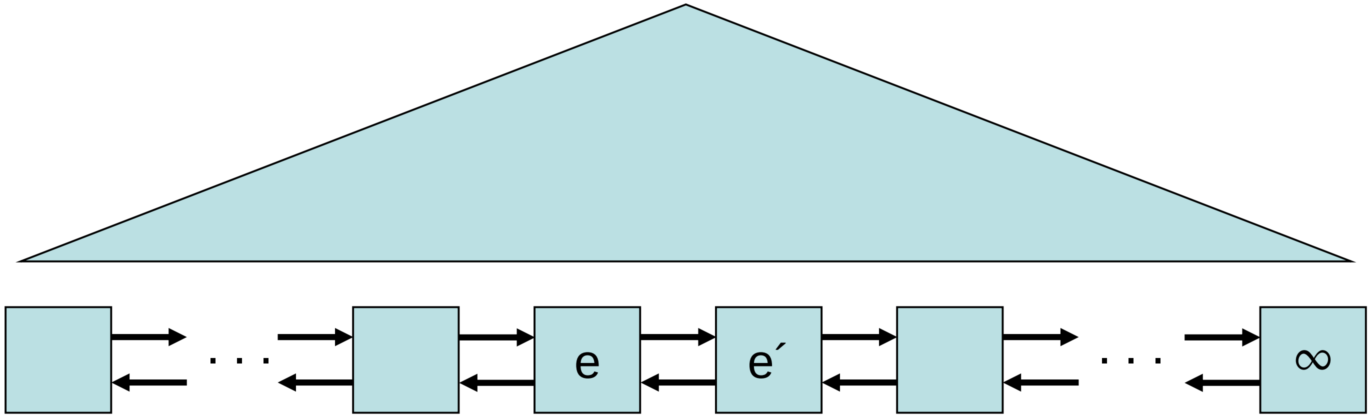
- First search(key(e)) until some e´ found in the list. If key(e´)>key(e), insert e in front of e´, otherwise replace e´ by e.



Insert(e) Operation

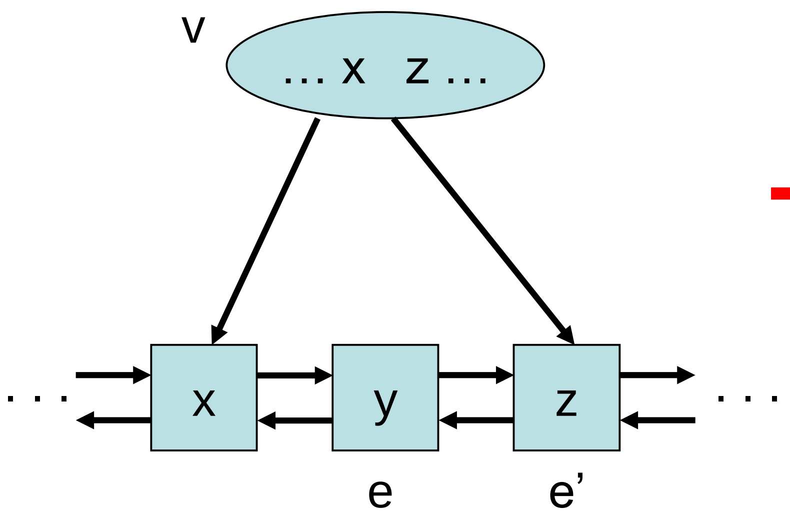
Strategy:

- First search(key(e)) until some e´ found in the list. If key(e´)>key(e), insert e in front of e´, otherwise replace e´ by e.



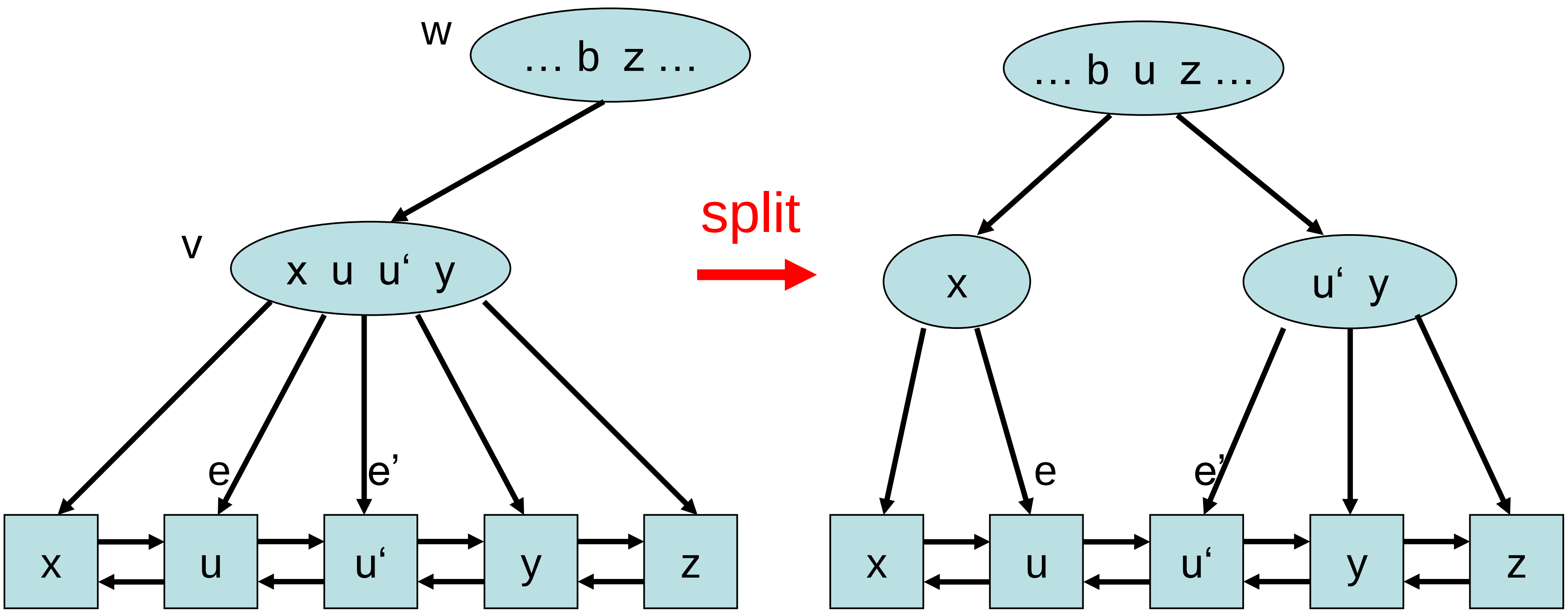
Insert(e) Operation

- Add key(e) and pointer to e in tree node v which is parent of e´. If we still have d(v)∈[a,b] after-wards, then we are done.



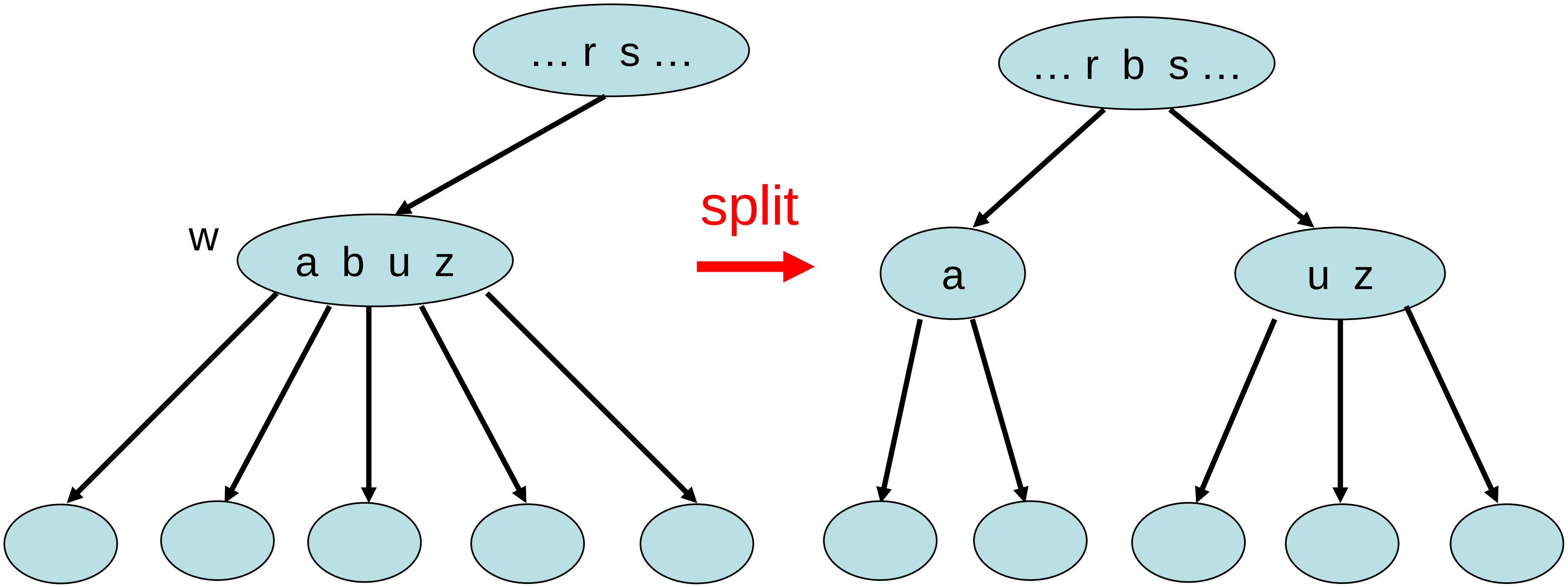
Insert(e) Operation

- If d(v)>b, then cut v into two nodes. (Example: a=2, b=4)



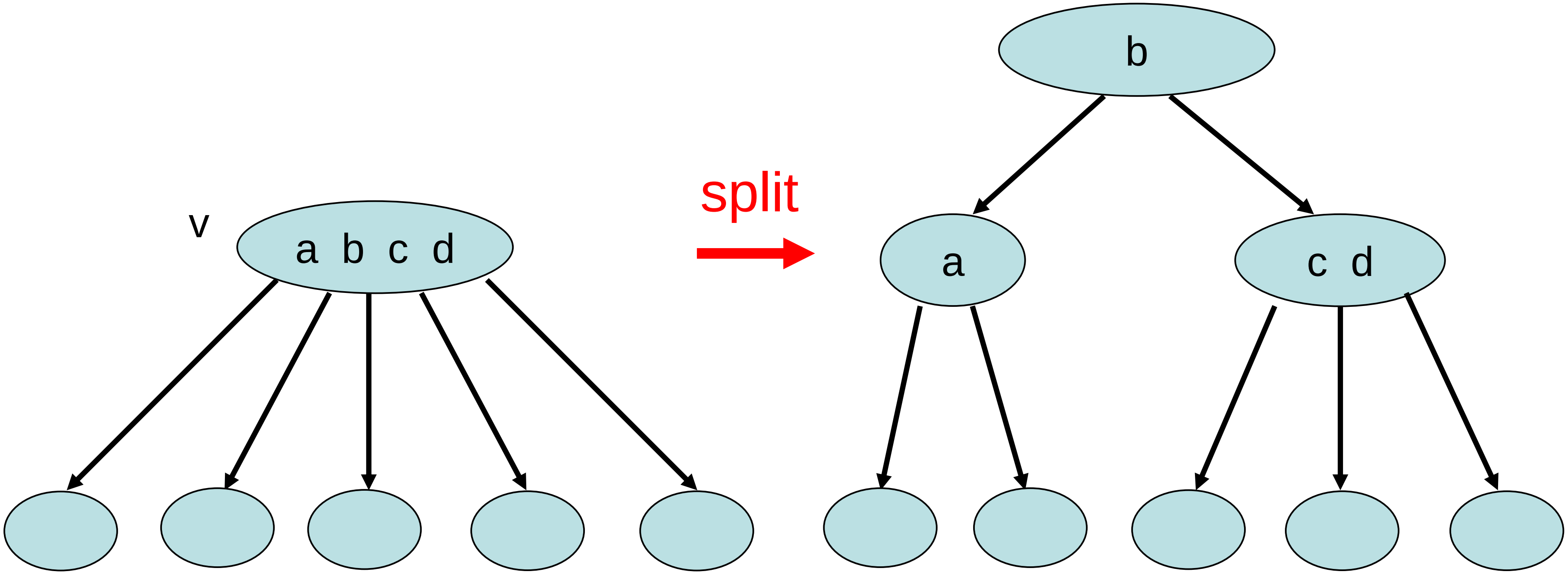
Insert(e) Operation

- If after splitting v, d(w)>b, then cut w into two nodes (and so on, until all nodes have degree ≤b or we reached the root)



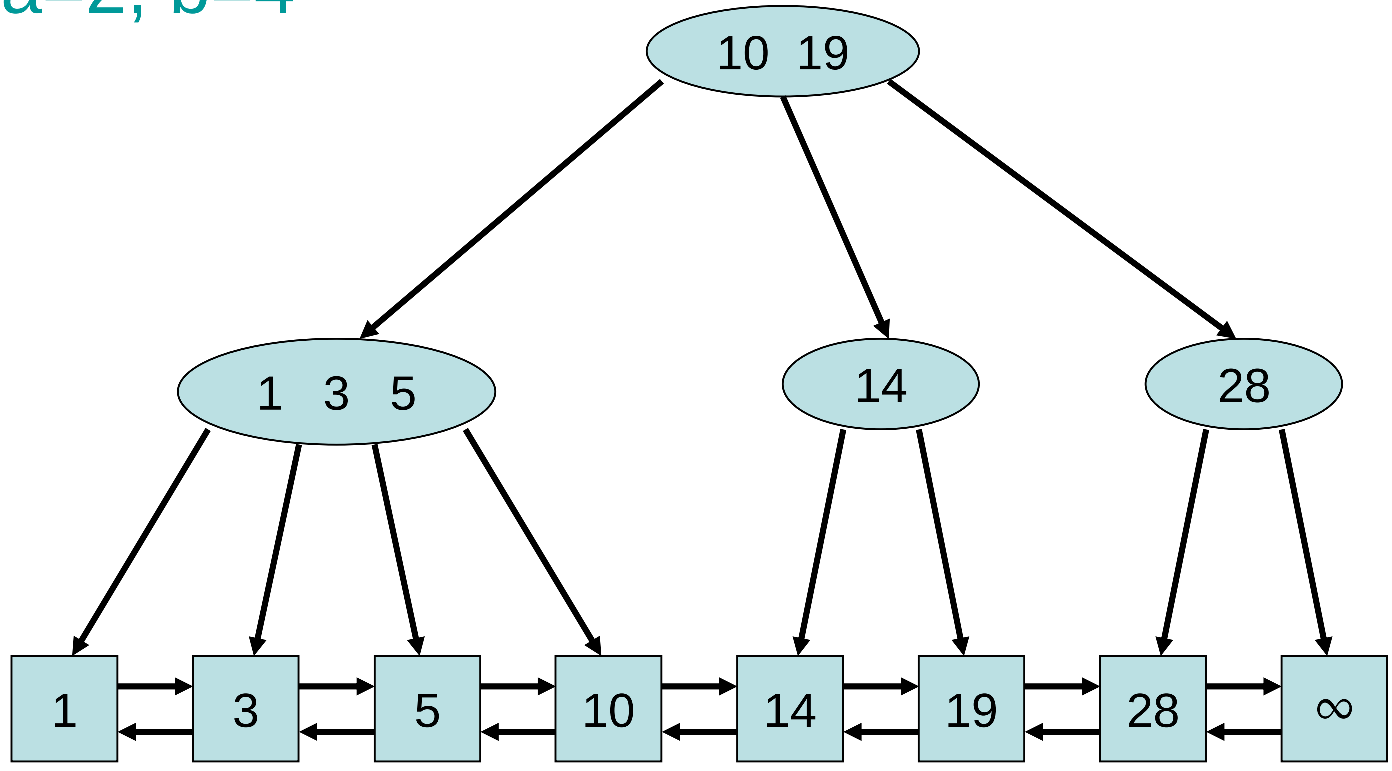
Insert(e) Operation

- If for the root v of T, d(v)>b, then cut v into two nodes and create a new root node.



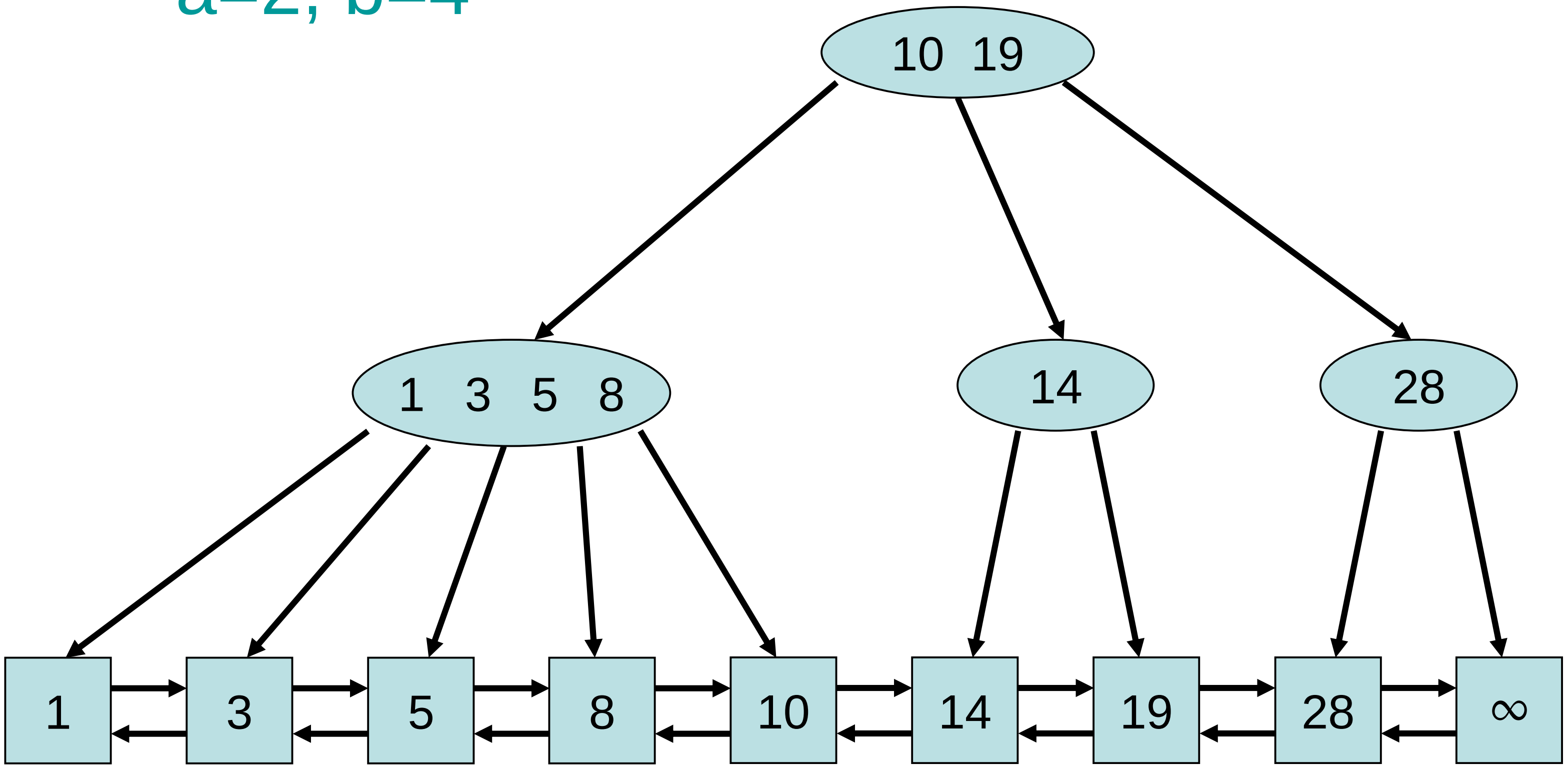
Insert(8)

a=2, b=4



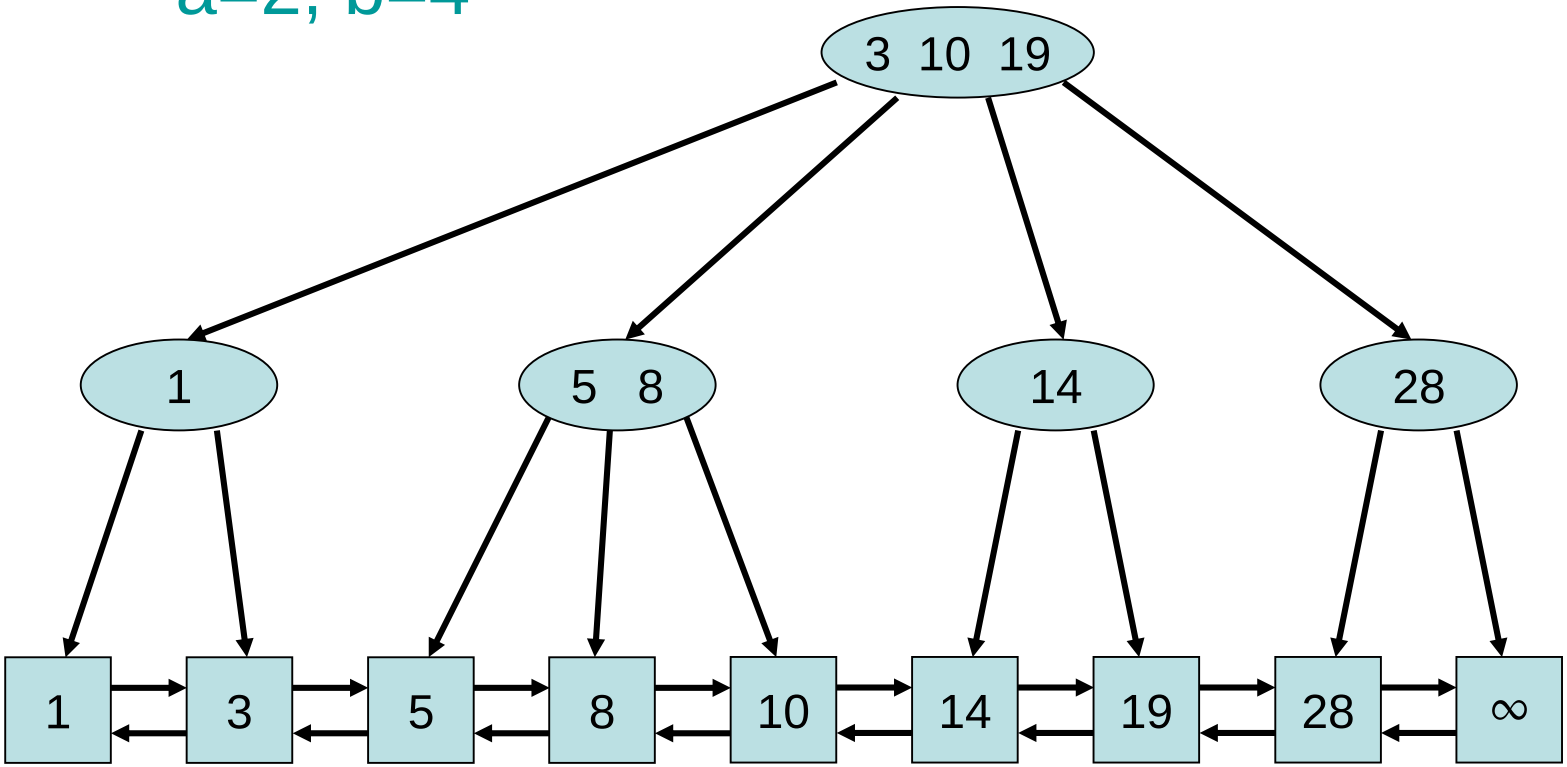
Insert(8)

a=2, b=4



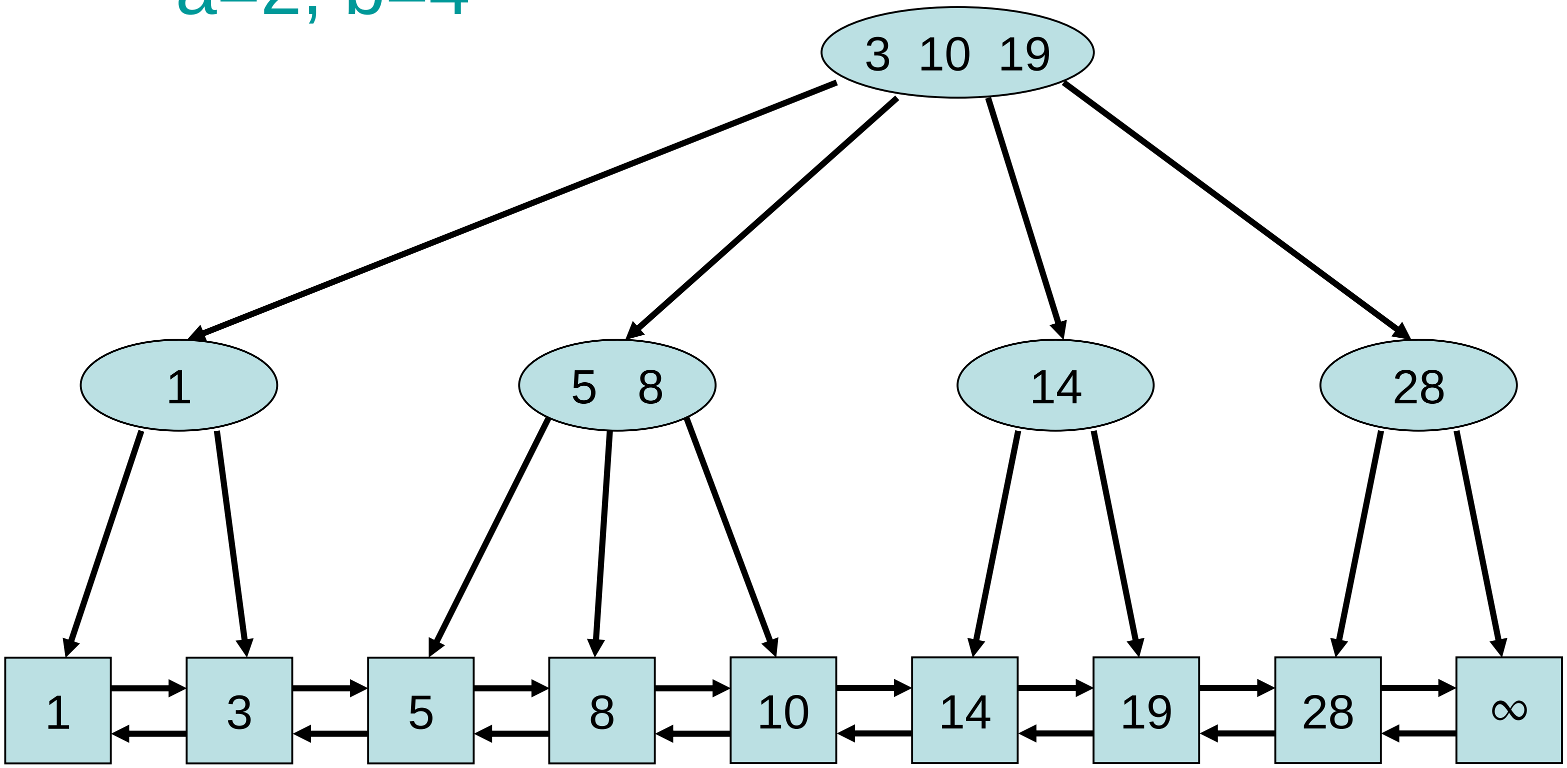
Insert(8)

a=2, b=4

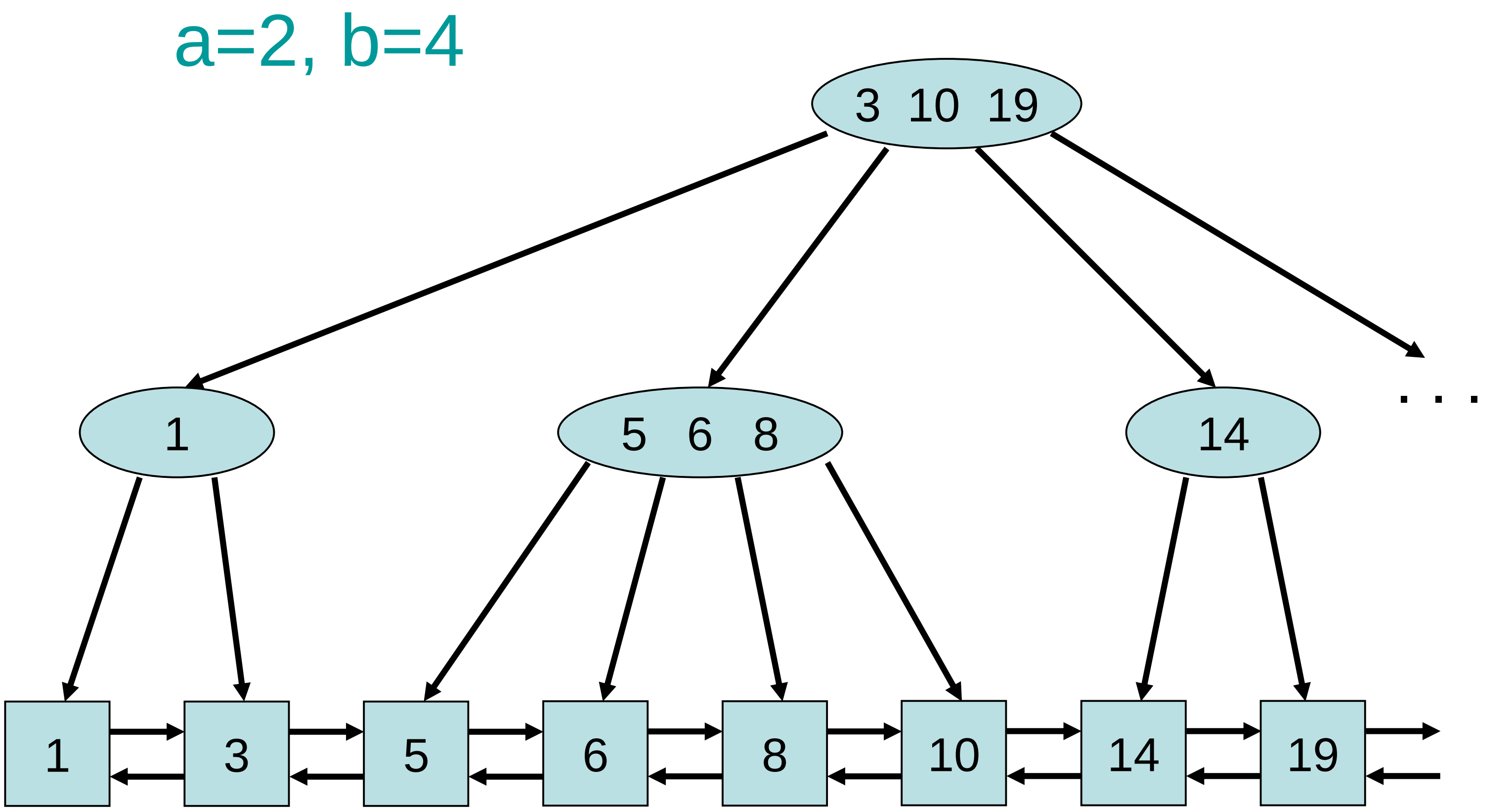


Insert(6)

a=2, b=4

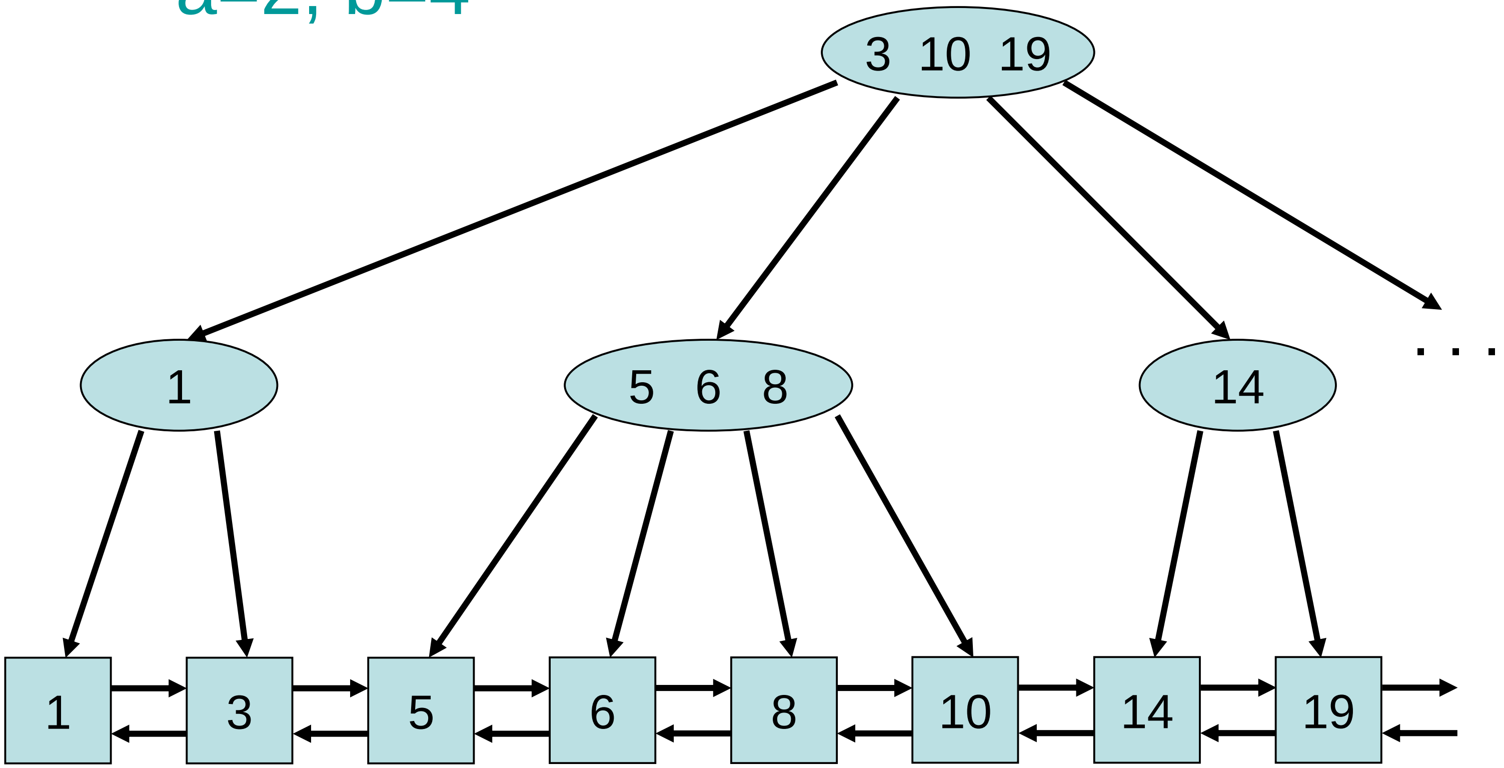


Insert(6)



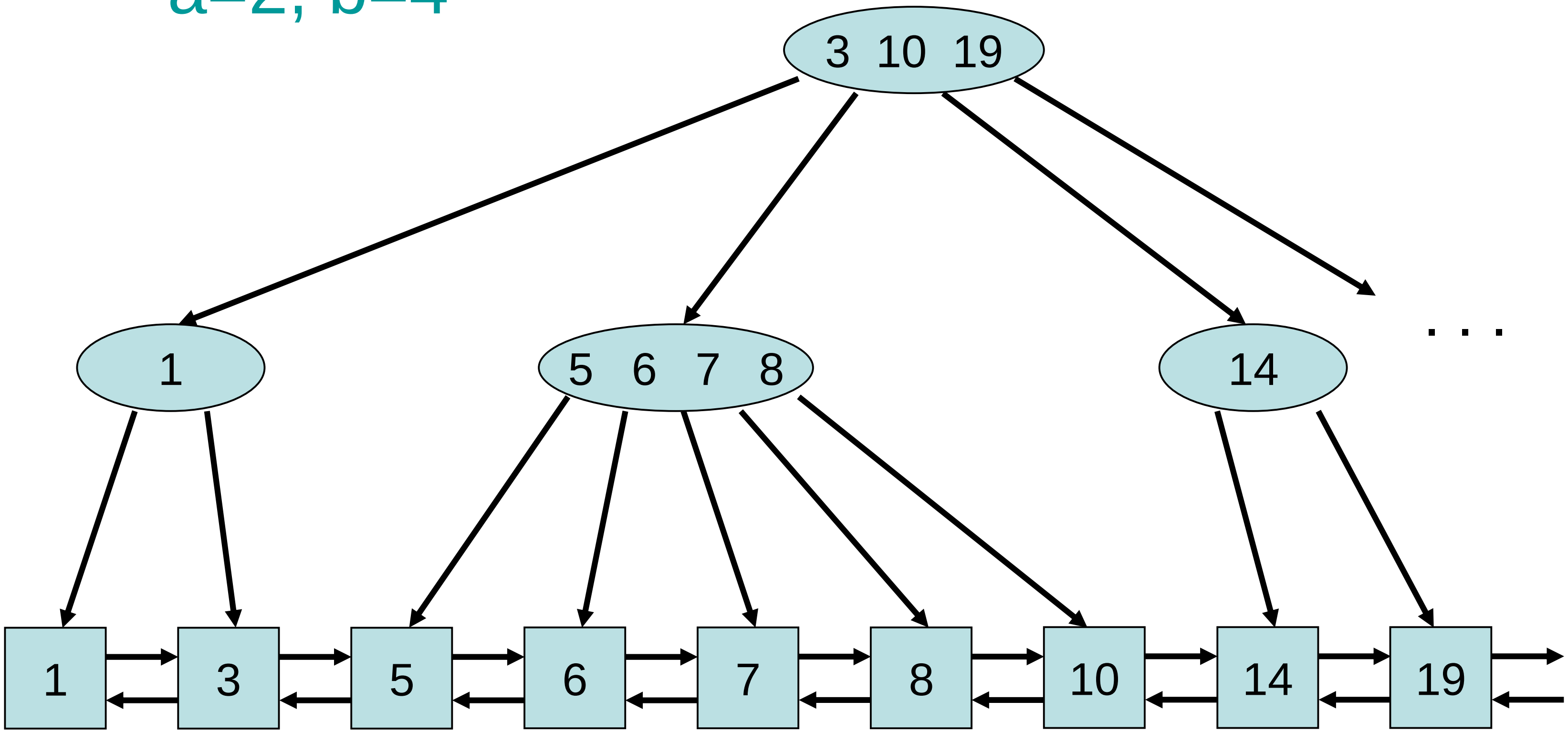
Insert(7)

a=2, b=4



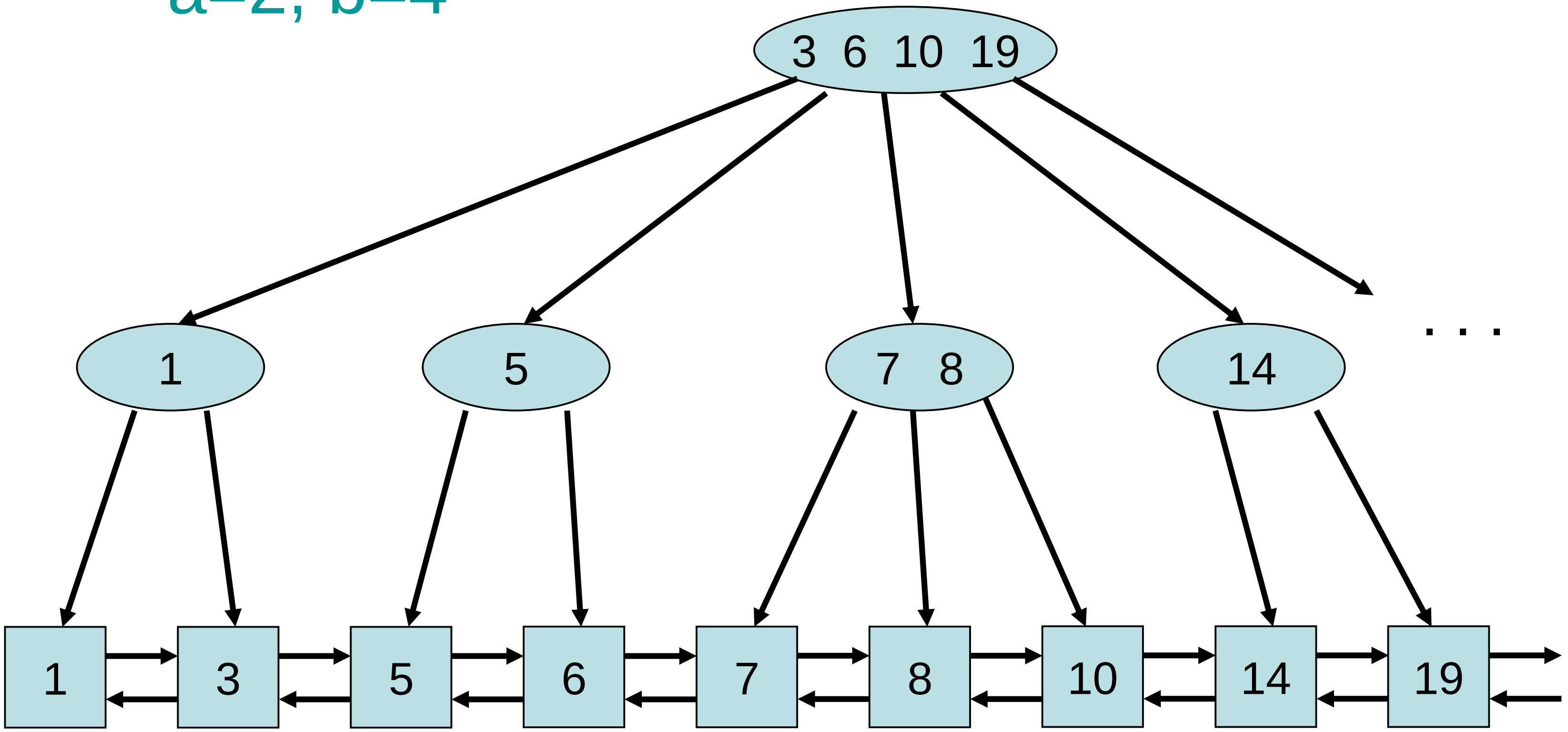
Insert(7)

a=2, b=4



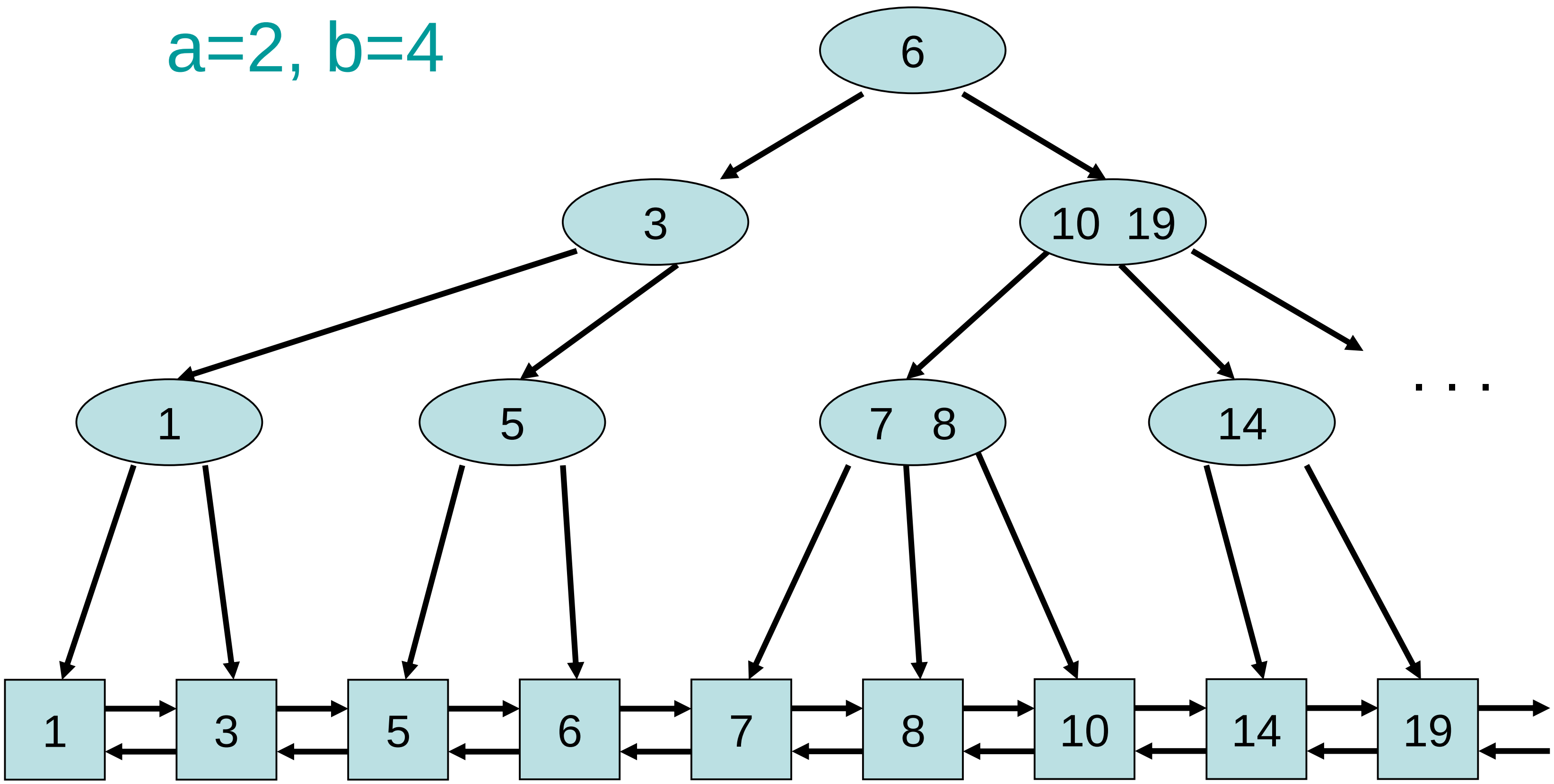
Insert(7)

a=2, b=4



Insert(7)

a=2, b=4



Insert Operation

- Form Invariant:
  For all leaves v,w: $t(v)=t(w)$
  Satisfied by Insert!

- Degree Invariant:
  For all inner nodes v except for the root:
  $d(v)\in[a,b]$, for root r: $d(r)\in[2,b]$

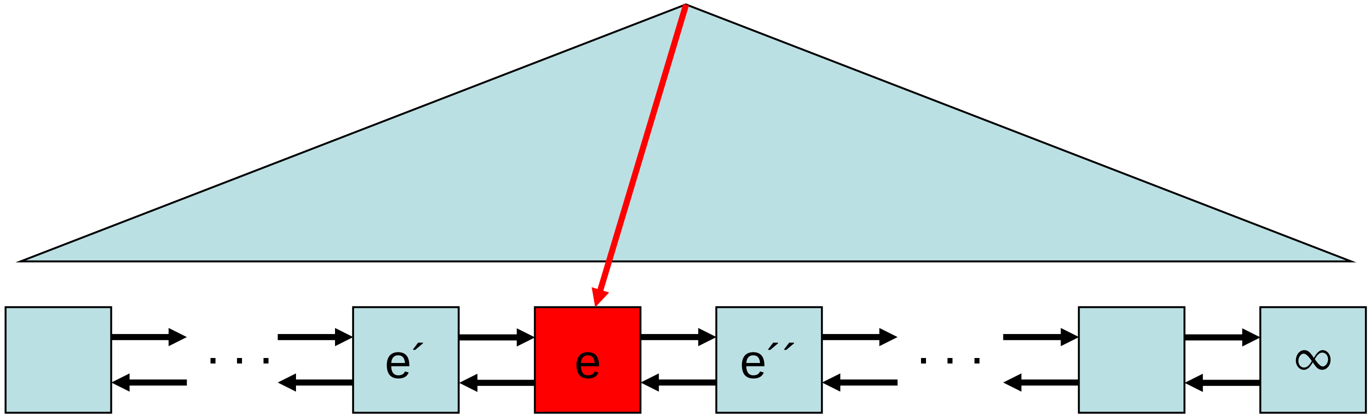
1) Insert splits nodes of degree $b+1$ into nodes of degree $\lfloor(b+1)/2\rfloor$ and $\lceil(b+1)/2\rceil$. If $b\geq 2a-1$, then both values are at least a.

2) If root has reached degree $b+1$, then a new root of degree 2 is created.

Delete(k) Operation

Strategy:

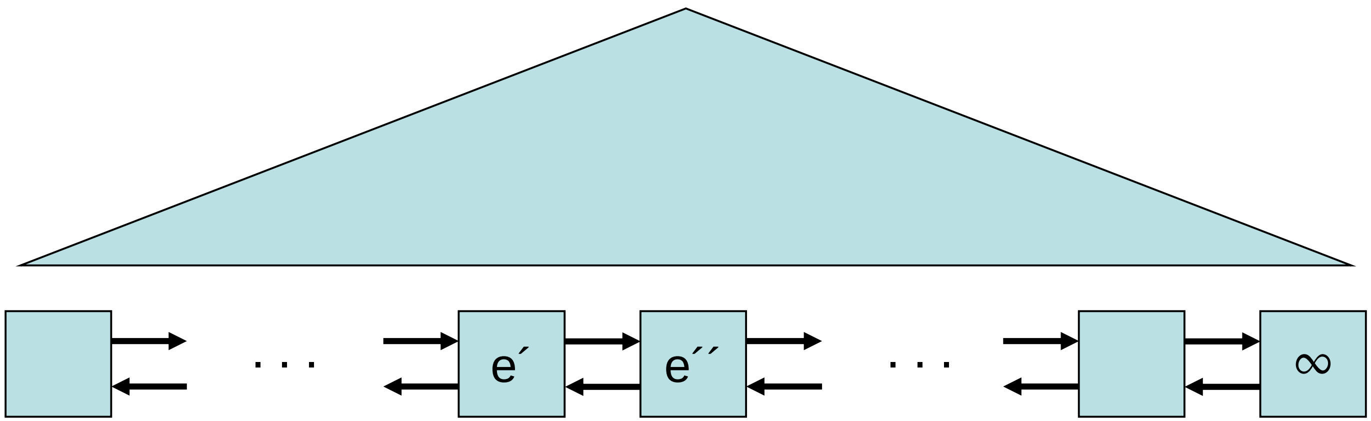
- First search(k) until some element e is reached in the list. If key(e)=k, remove e from the list, otherwise we are done.



Delete(k) Operation

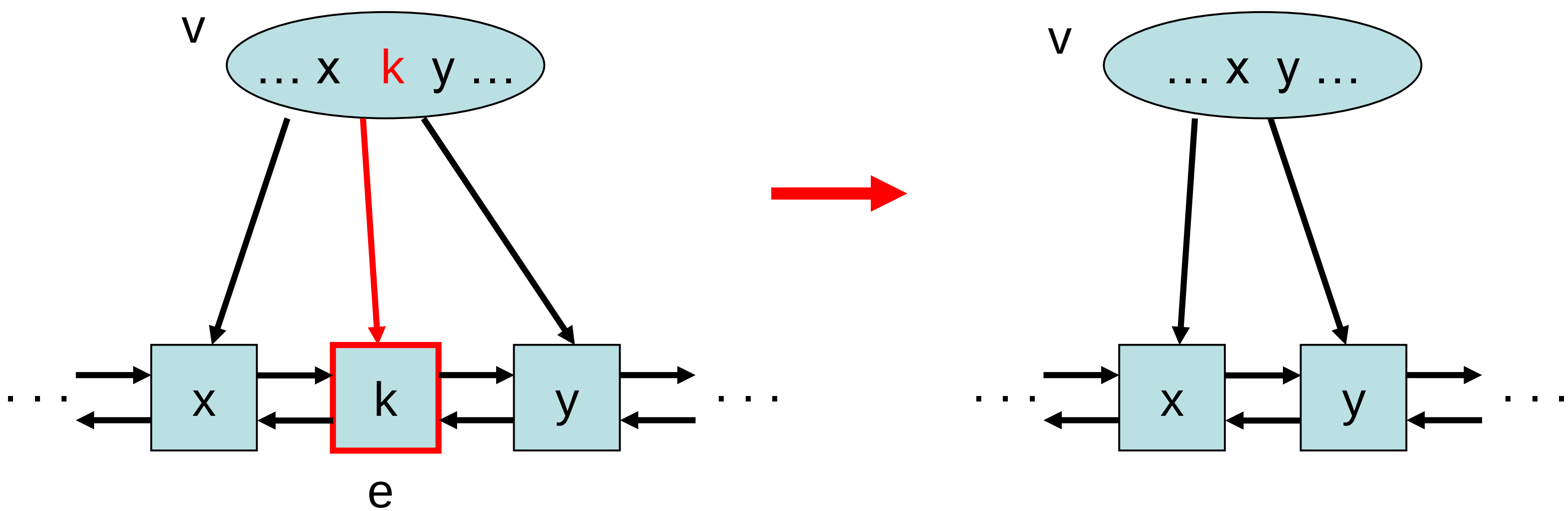
Strategy:

- First search(k) until some element e is reached in the list. If key(e)=k, remove e from the list, otherwise we are done.



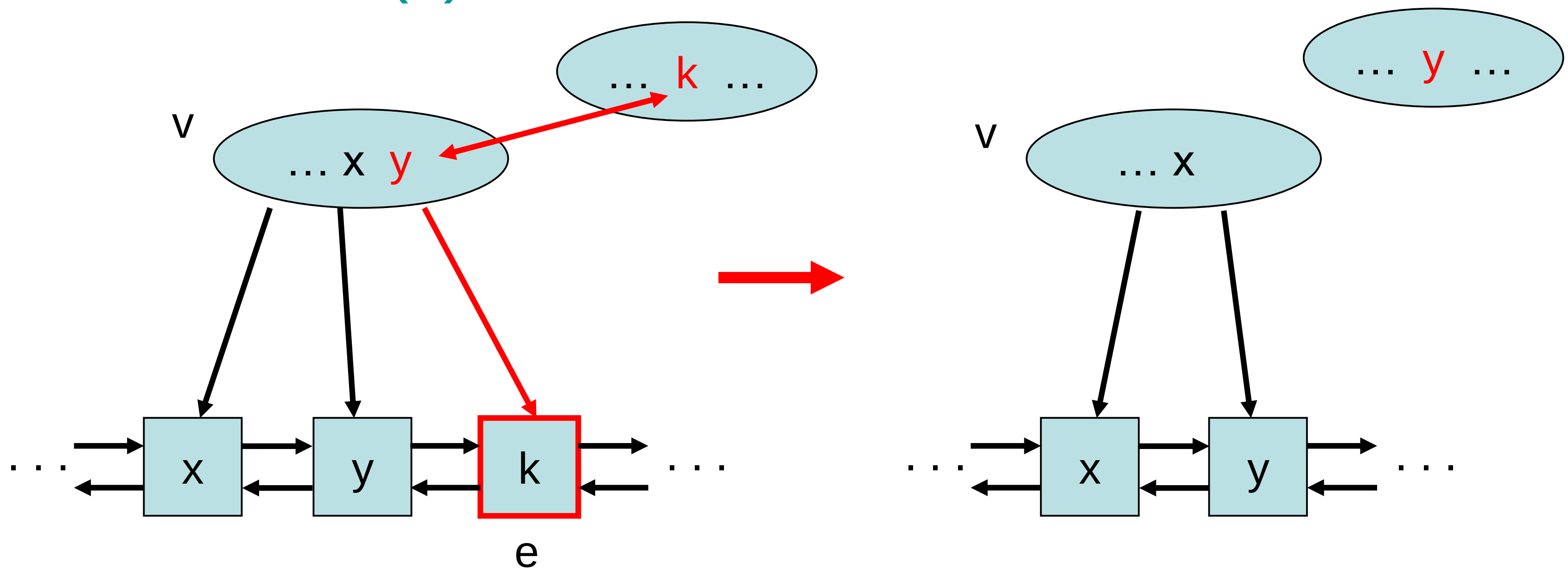
Delete(k) Operation

- Remove pointer to e and key k from the leaf node v above e. (e rightmost child: perform key exchange like in binary tree!) If afterwards we still have d(v)≥a, we are done.



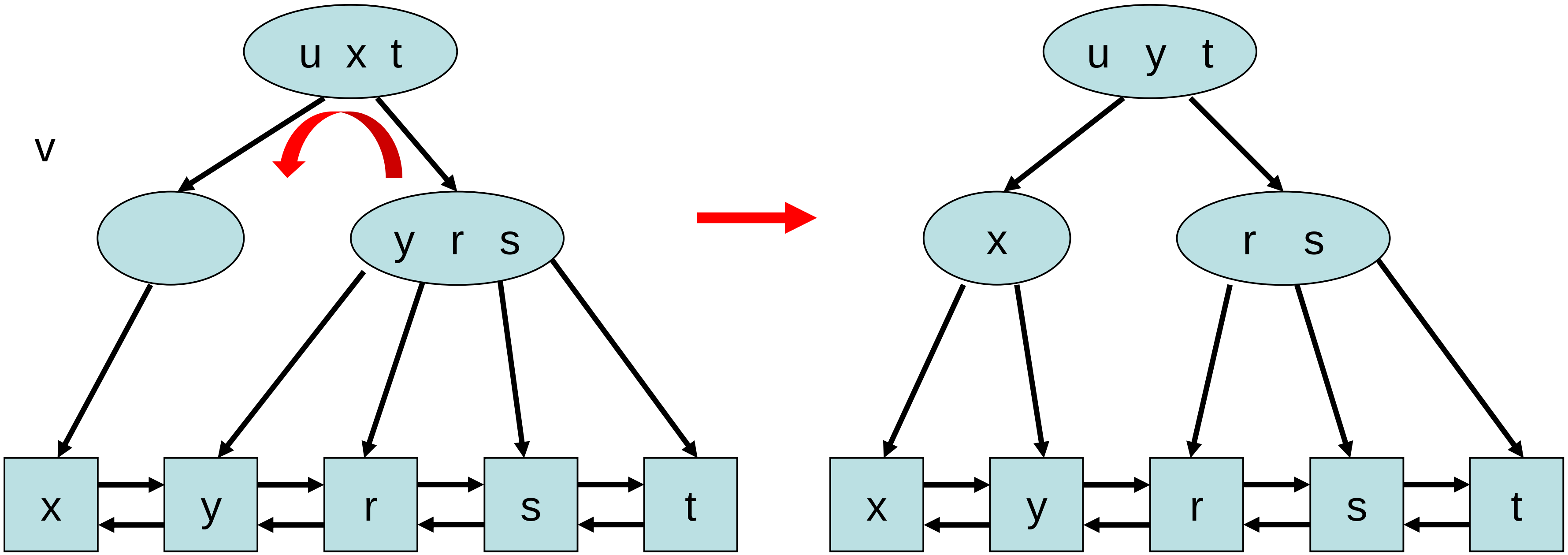
Delete(k) Operation

- Remove pointer to e and key k from the leaf node v above e. (e rightmost child: perform key exchange like in binary tree!) If afterwards we still have $d(v) \geq a$, we are done.



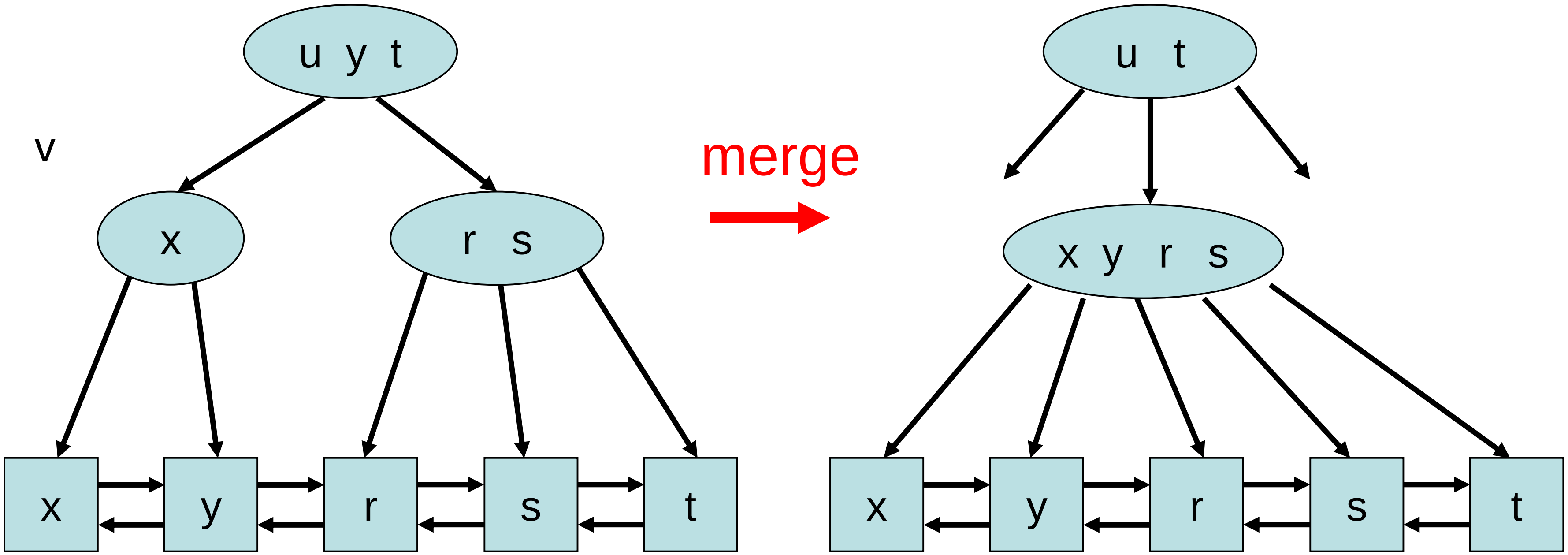
Delete(k) Operation

- If d(v)<a and the preceding or succeeding sibling of v has degree >a, steal an edge from that sibling. (Example: a=2, b=4)



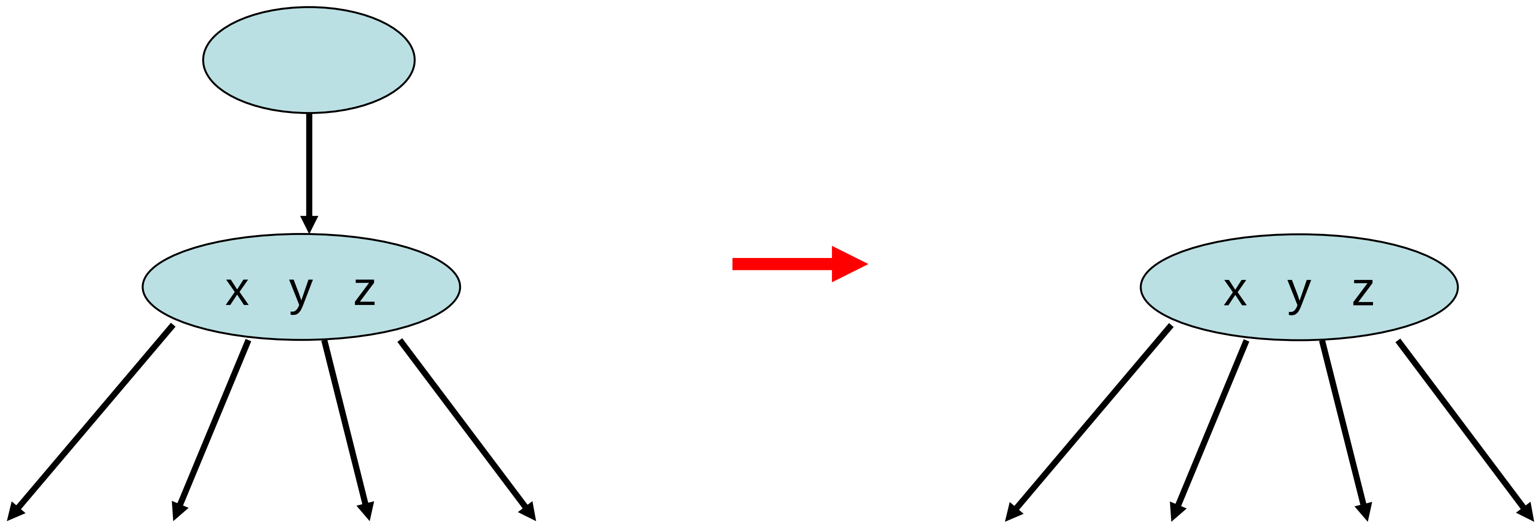
Delete(k) Operation

- If d(v)<a and the preceding and succeeding siblings of v have degree a, merge v with one of these. (Example: a=3, b=5)



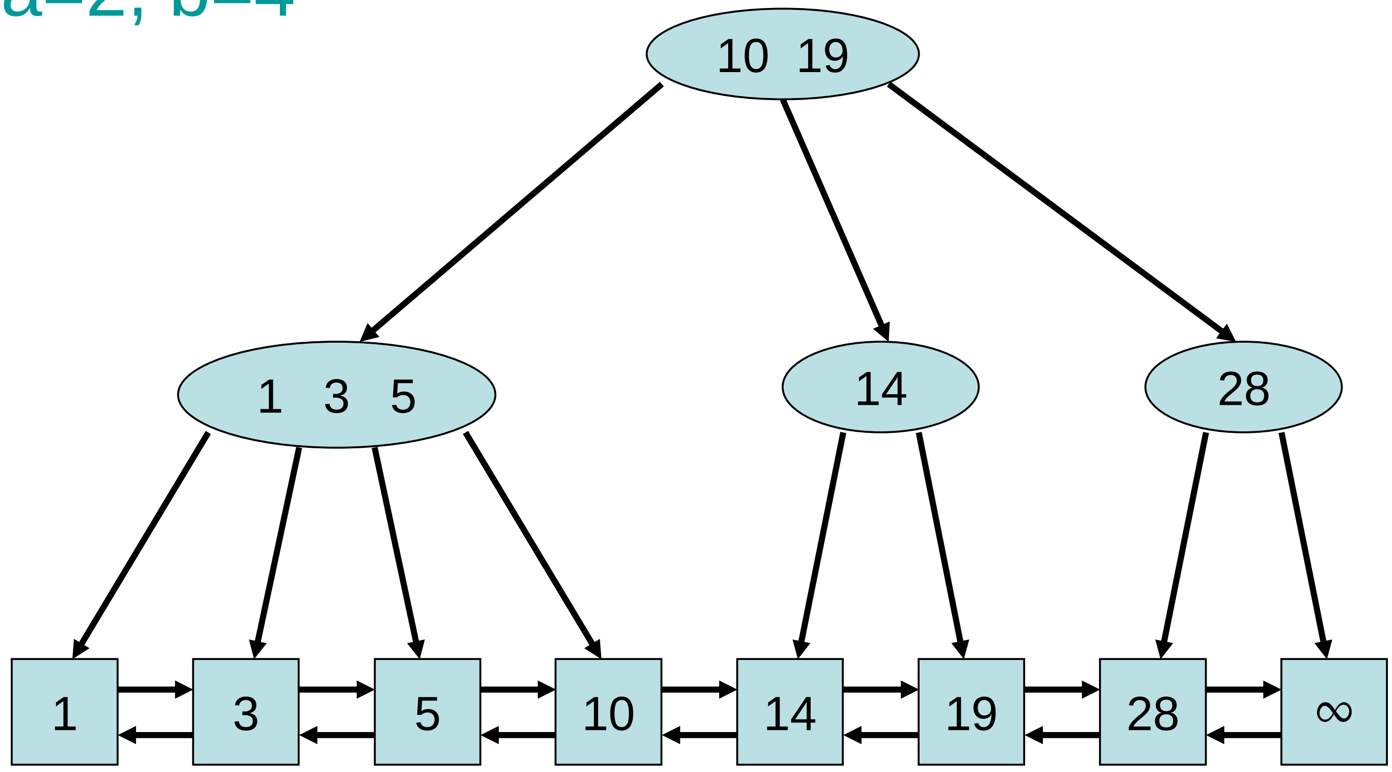
Delete(k) Operation

- Perform changes upwards until all inner nodes (except for the root) have degree $\geq a$. If root has degree <2: remove root.



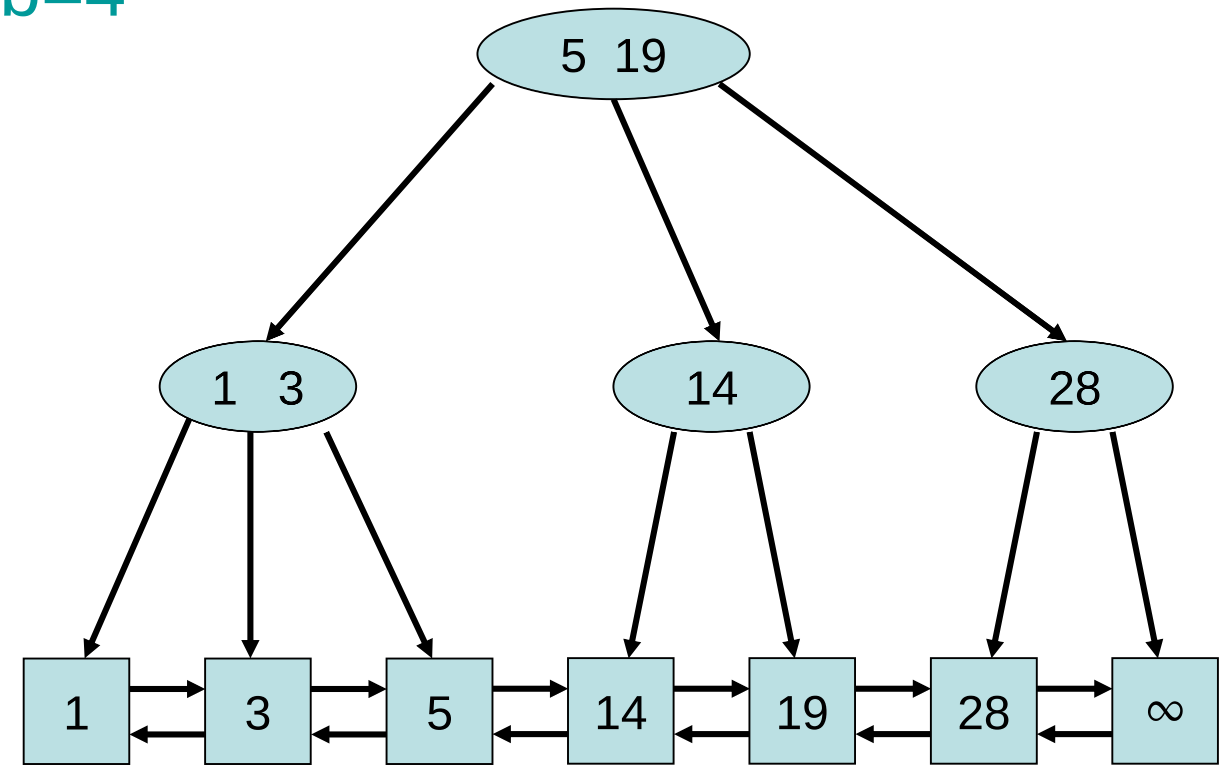
Delete(10)

a=2, b=4



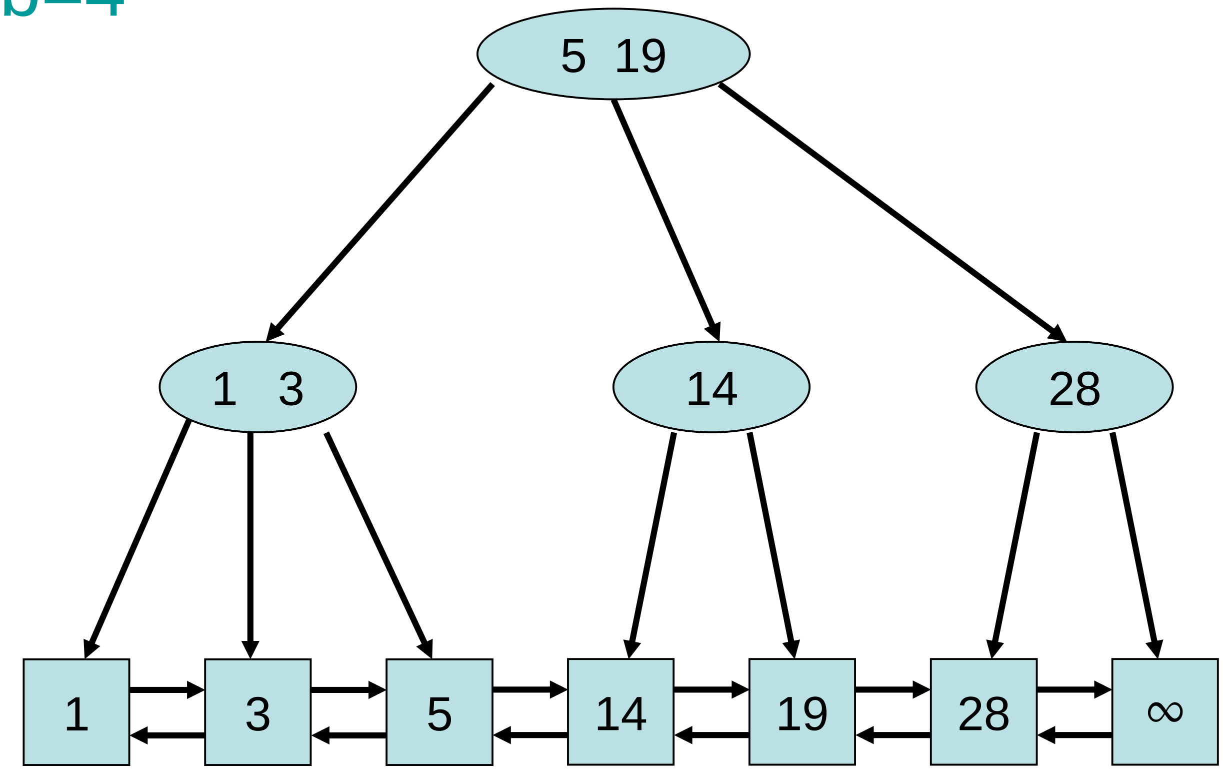
Delete(10)

a=2, b=4



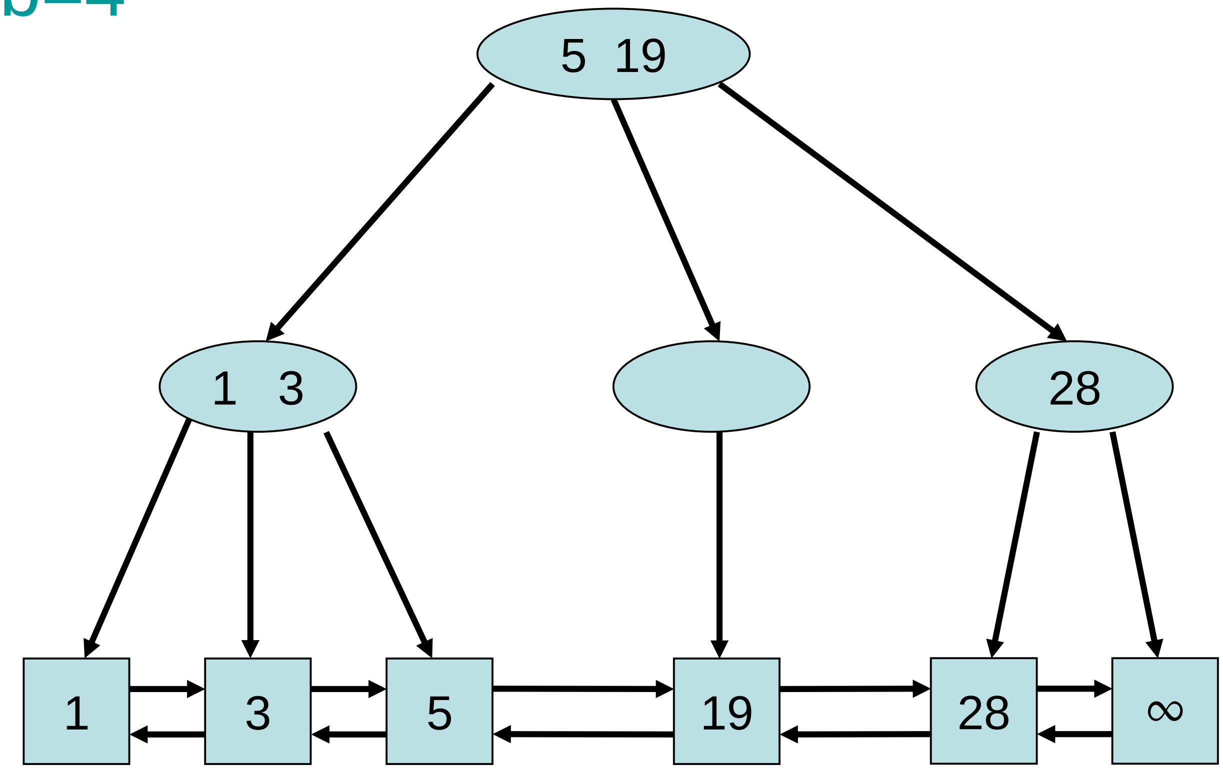
Delete(14)

a=2, b=4



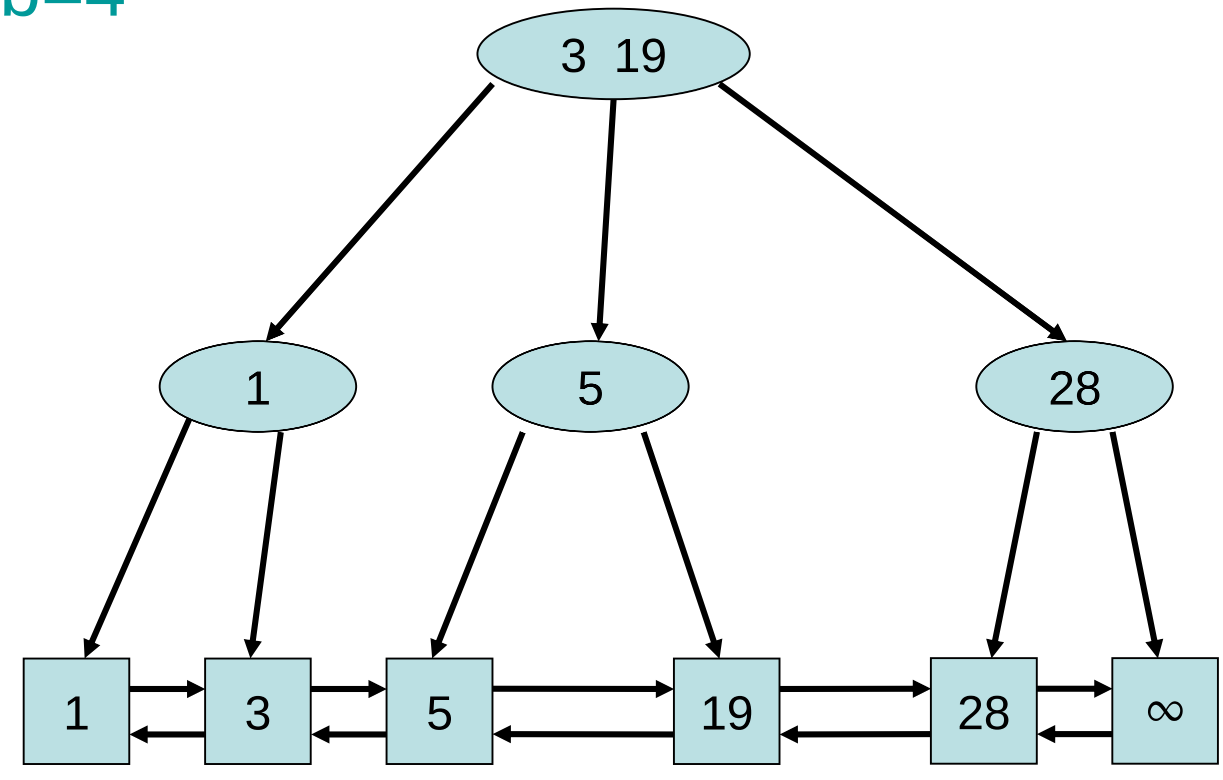
Delete(14)

a=2, b=4



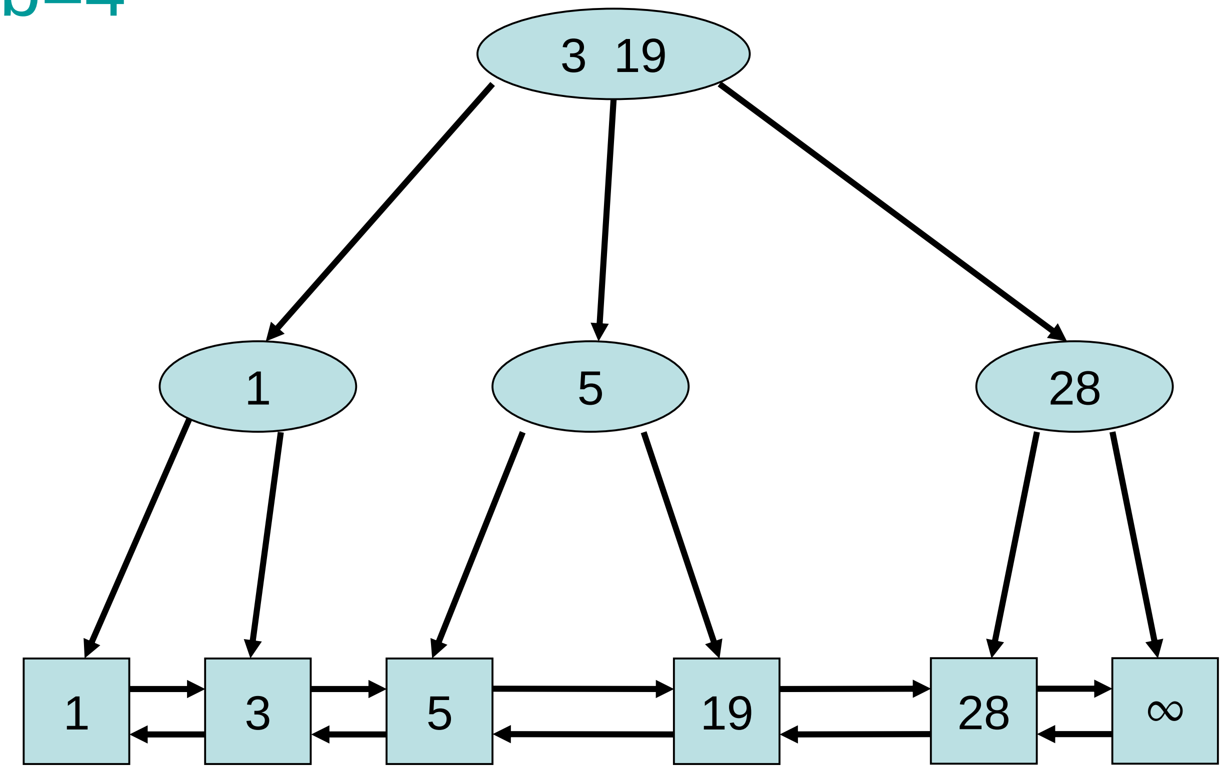
Delete(14)

a=2, b=4



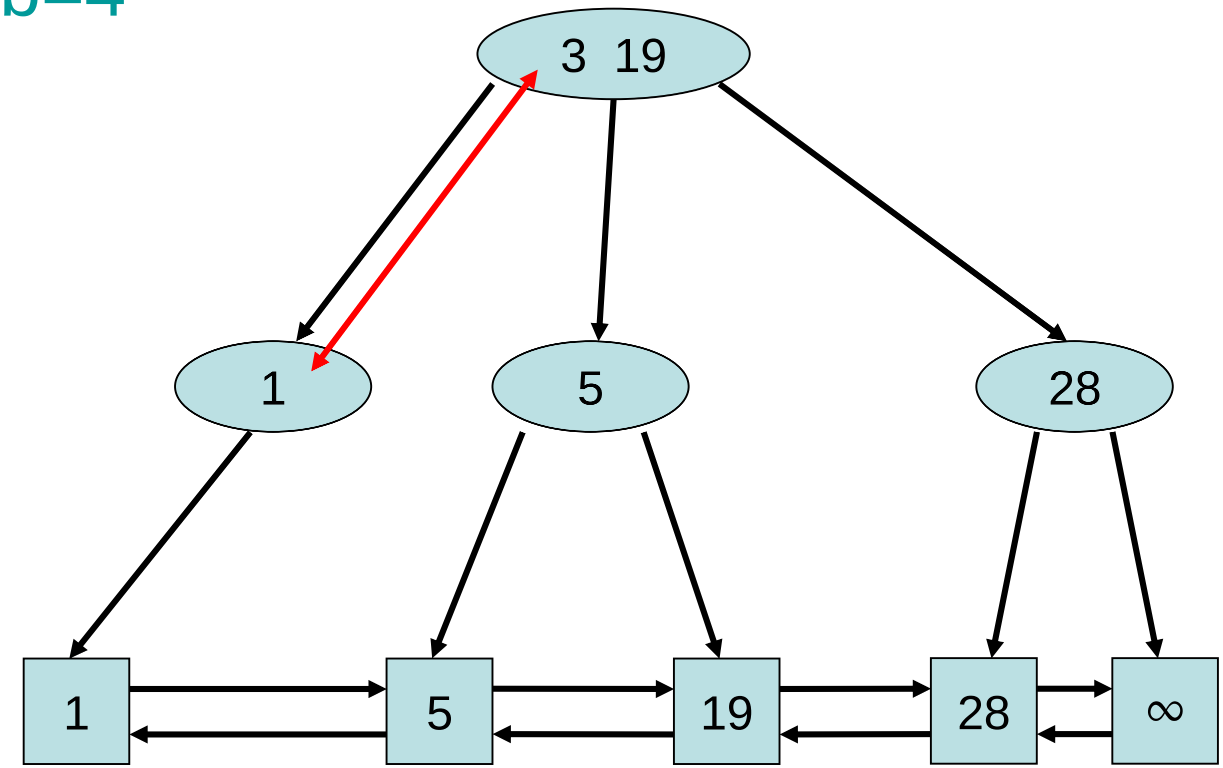
Delete(3)

a=2, b=4



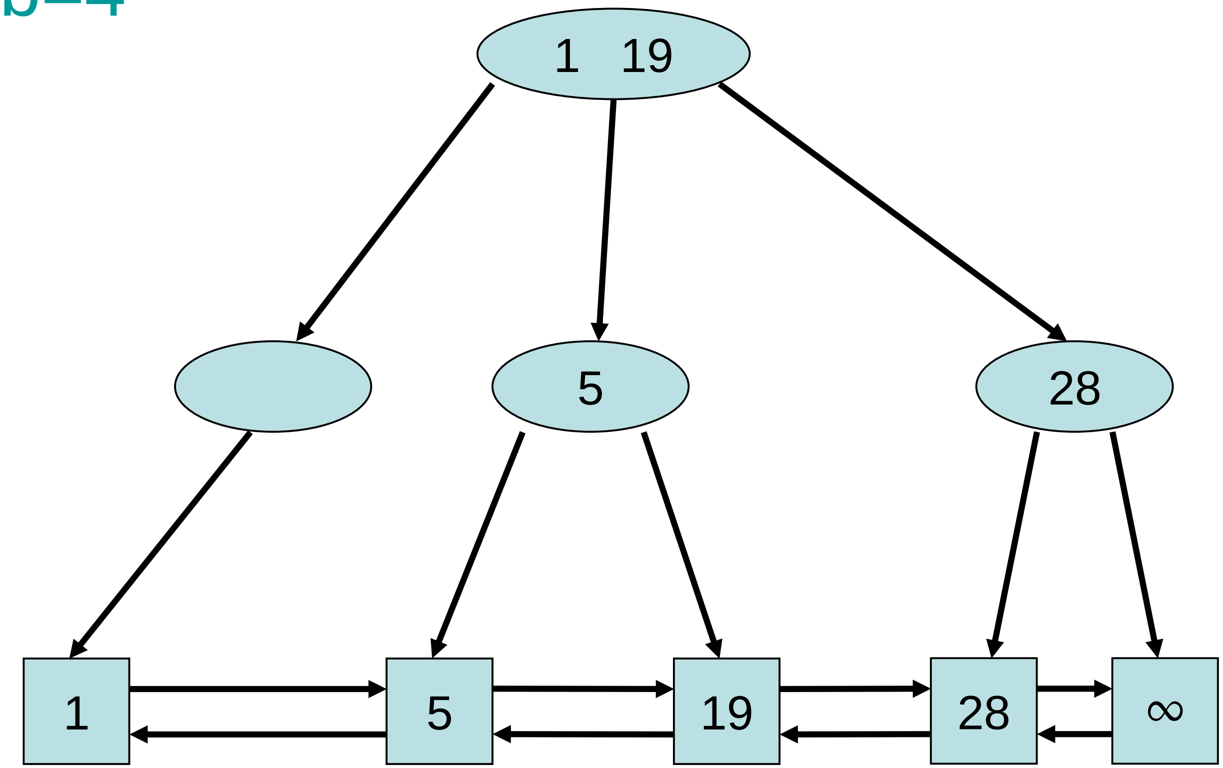
Delete(3)

a=2, b=4



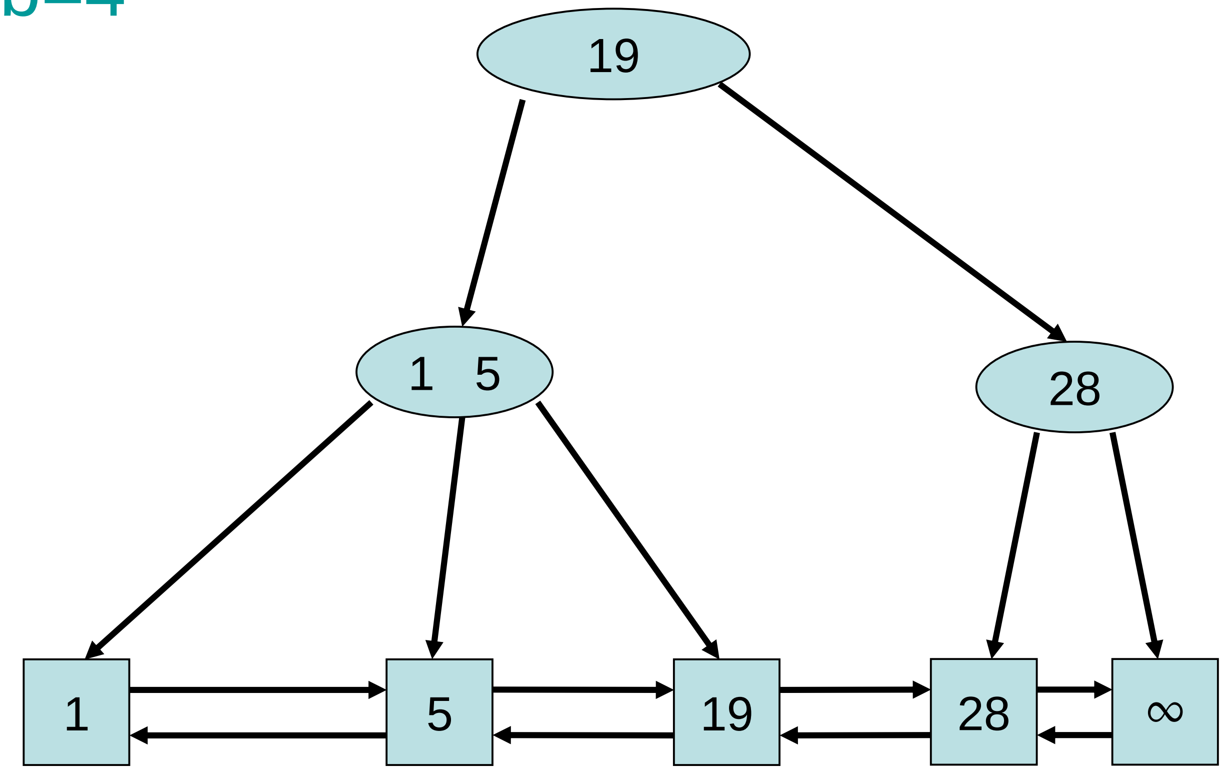
Delete(3)

a=2, b=4



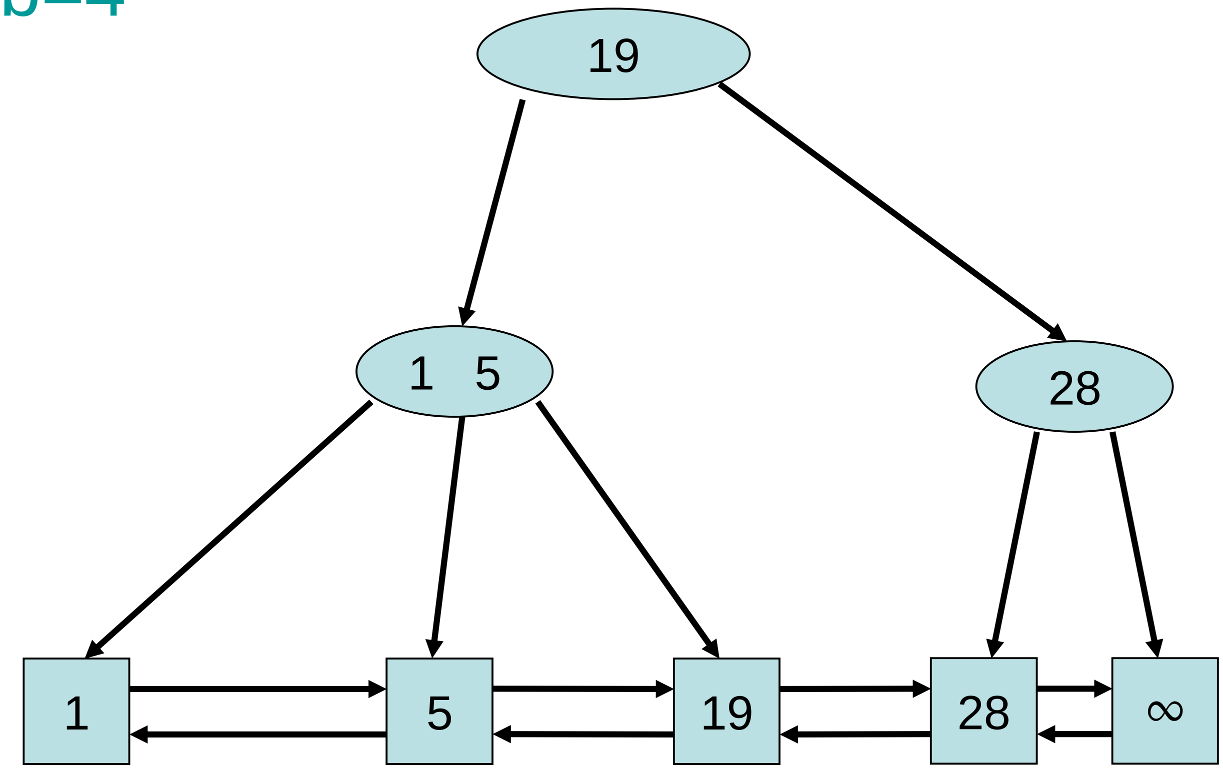
Delete(3)

a=2, b=4



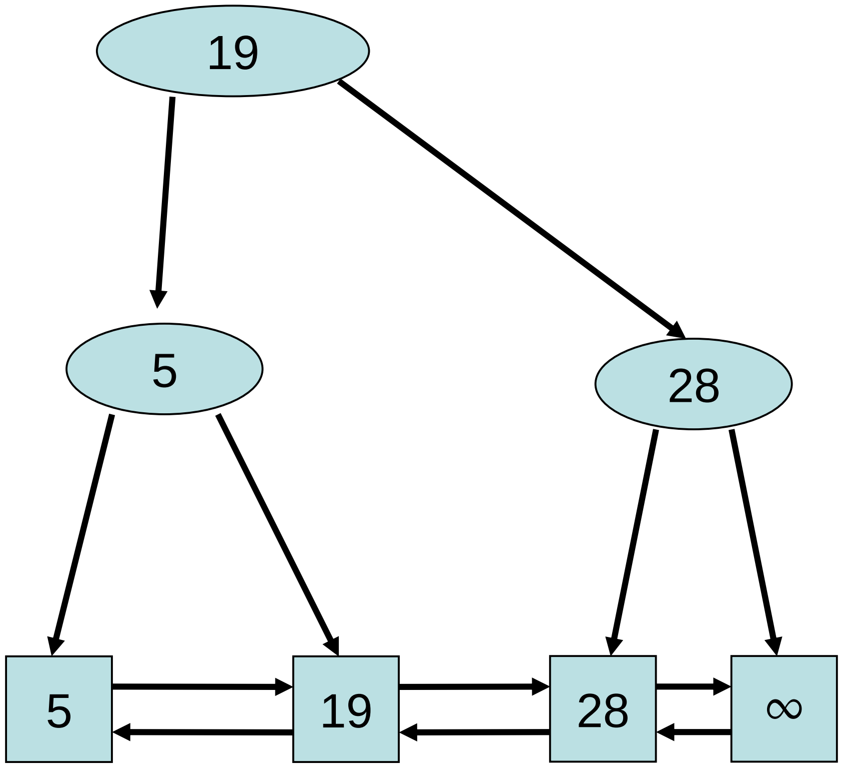
Delete(1)

a=2, b=4



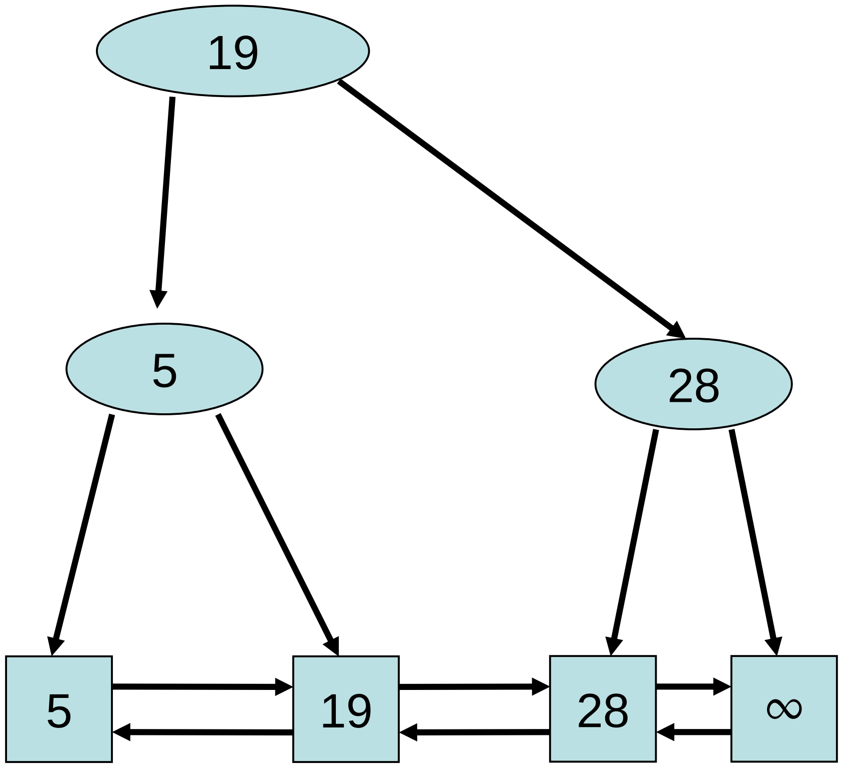
Delete(1)

a=2, b=4



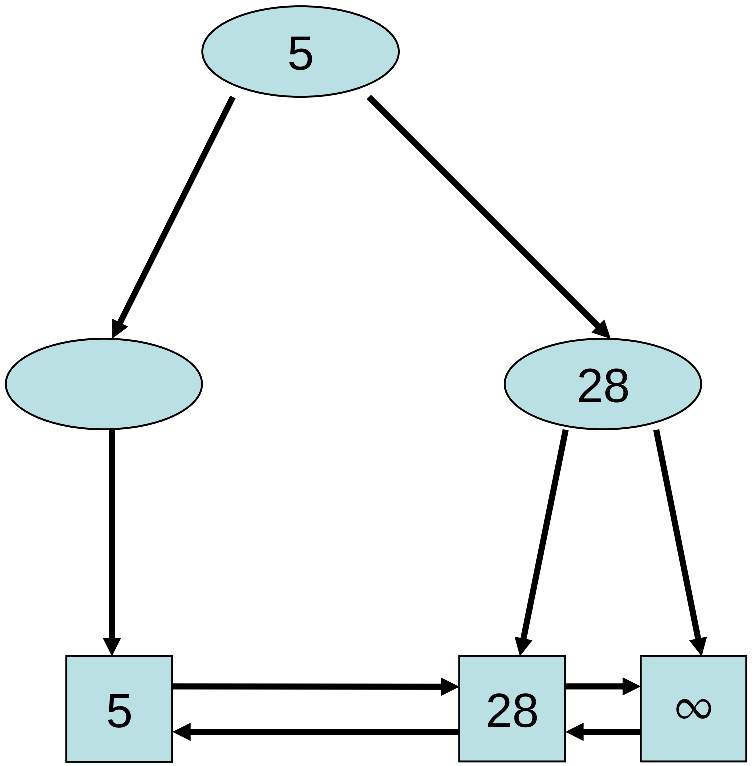
Delete(19)

a=2, b=4



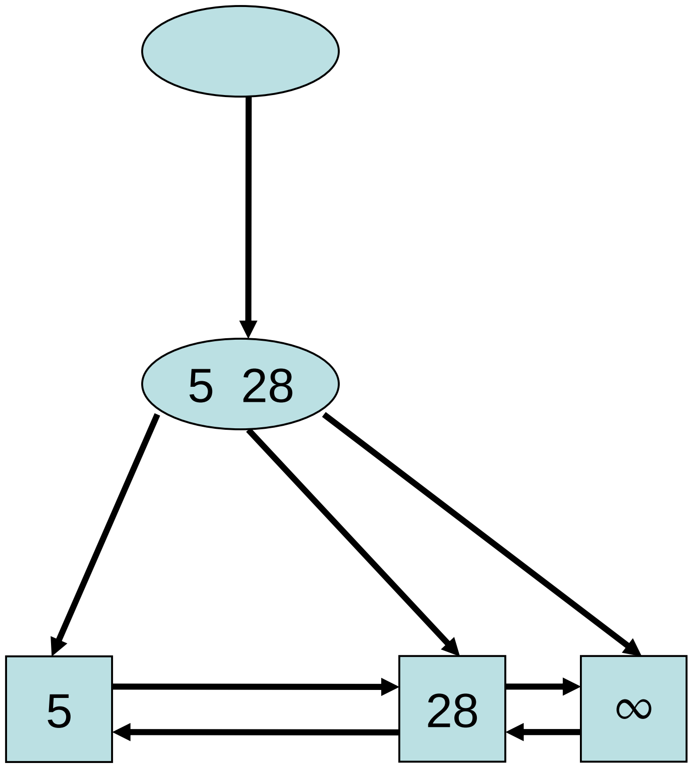
Delete(19)

a=2, b=4



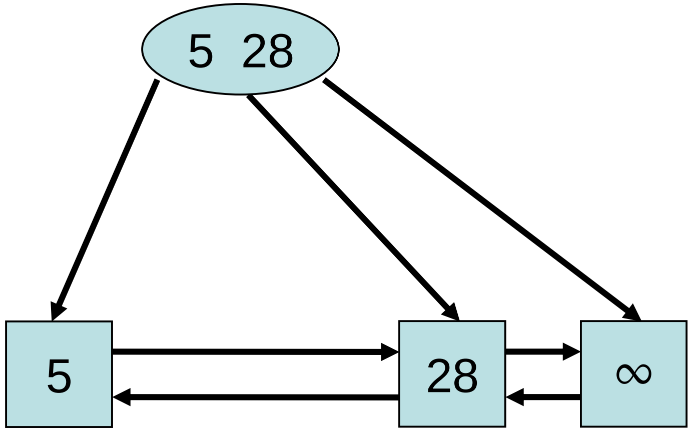
Delete(19)

a=2, b=4



Delete(19)

a=2, b=4



Delete Operation

- Form Invariant:
  For all leaves v,w: $t(v)=t(w)$
  Satisfied by Delete!

- Degree Invariant:
  For all inner nodes v except for the root: $d(v)\in[a,b]$, for root r: $d(r)\in[2,b]$

1) Delete merges node of degree $a-1$ with node of degree a. Since $b\geq 2a-1$, the resulting node has degree at most b.

2) Delete moves edge from a node of degree $>a$ to a node of degree $a-1$. Also OK.

3) Root deleted: children have been merged, degree of the remaining child is $\geq a$ (and also $\leq b$), so also OK.

More Operations

- min/max Operation:
  Pointers to both ends of list: time O(1).

- Range queries:
  To obtain all elements in the range [x,y], perform search(x) and go through the list till an element >y is found.
  Time O(log n + size of output).

n Update Operations

Theorem 3.11: There is a sequence of n insert and delete operations in a (2,3)-tree that require $\Omega(n \log n)$ many split and merge Operations.

Proof: Exercise

n Update Operations

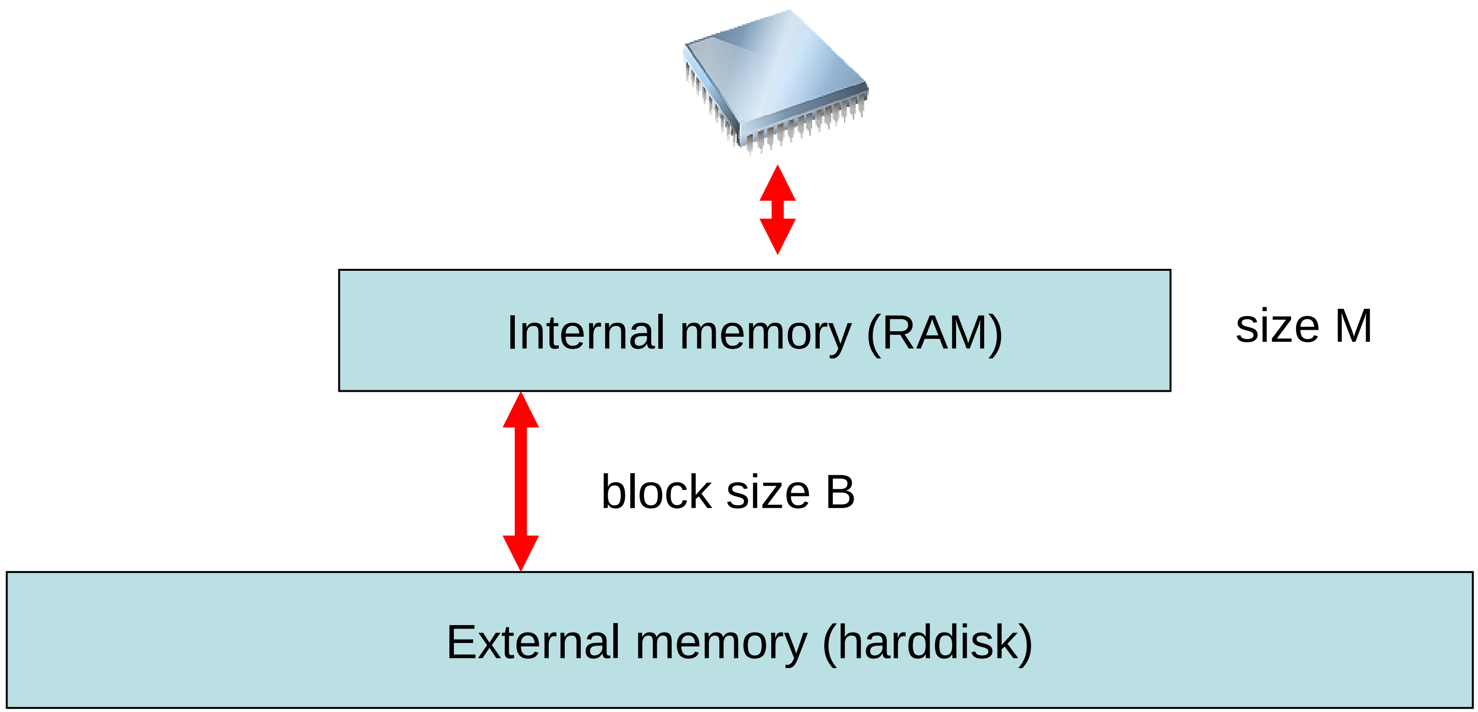
Theorem 3.12: Consider an (a,b)-tree with $b \geq 2a$ that is initially empty. For any sequence of n insert and delete operations, only O(n) split and merge operations are needed.

Proof:

Amortized analysis

External (a,b)-Tree

(a,b)-trees well suited for large amounts of data



External (a,b)-Tree

Problem: minimize number of block transfers between internal and external memory

Solution:

- use $b=B$ (block size) and $a=b/2$
- keep highest $(1/2)\cdot\log_a(M/b)$ levels of (a,b)-tree in internal memory (storage needed $\leq M$)
- Lemma 3.10: depth of (a,b)-tree $\leq 1+\lfloor\log_a (n/2)\rfloor$
- How many levels are not in internal memory?

  $\log_a[n/2] - (1/2)\cdot\log_a(M/b) \leq \log_a[n/(2\sqrt{M})] + O(1)$ (a, b are O(1))
- Cost for insert, delete and search operations: $O(\log_B(n/\sqrt{M}))$ block transfers

External (a,b)-Tree

Problem: minimize number of block transfers between internal and external memory

A better analysis can show (exercise):

- Cost for insert, delete and search operations: $\sim 2\log_{B/2}(n/M)+1$ block transfers (+1: list access)

Example:

- n = 100,000,000,000,000 keys
- M = 16 Gbyte (~4,000,000,000 keys)
- B = 256 Kbyte (~64,000 keys)
- $2\log_{B/2}(n/M)+1 \leq 3$

Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- hashed Patricia trie
  (loglog-search time)

Applications

Longest Prefix Search

- All keys are encoded as binary sequence $\{0,1\}^W$
- Prefix of a key $x \in \{0,1\}^W$: arbitrary subsequence of x that starts with the first bit of x (example: 101 is a prefix of 10110100)

Problem: given a key $x \in \{0,1\}^W$, find a key $y \in S$ with longest common prefix

Solution: Trie Hashing

Trie

A trie is a search tree over some alphabet Σ that has the following properties:

- Every edge is associated with a symbol $c \in \Sigma$
- Every key $x \in \Sigma^k$ that has been inserted into the trie can be reached from the root of the trie by following the unique path of length k whose edge labels result in x.

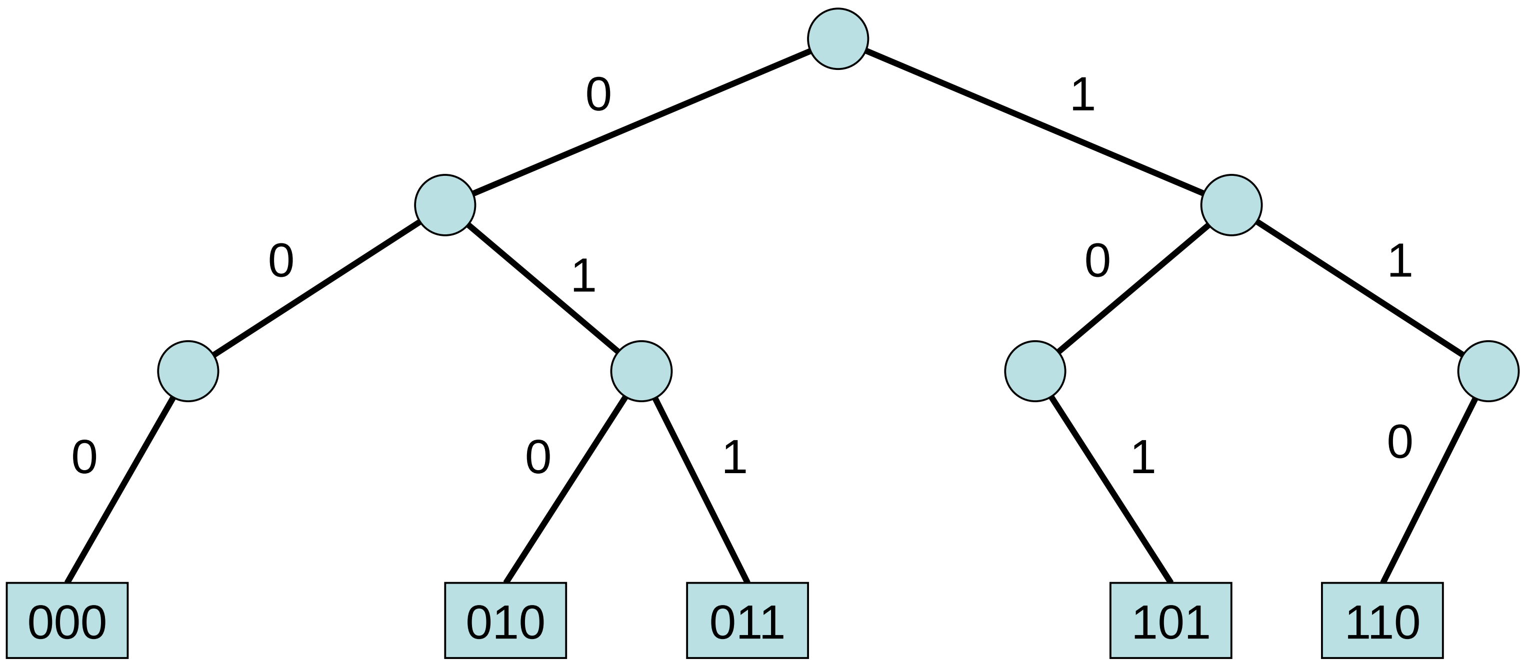
For simplicity: all keys from $\{0,1\}^W$ for some $W \in \mathbb{N}$.

Example:

(0,2,3,5,6) with W=3 results in (000,010,011,101,110)

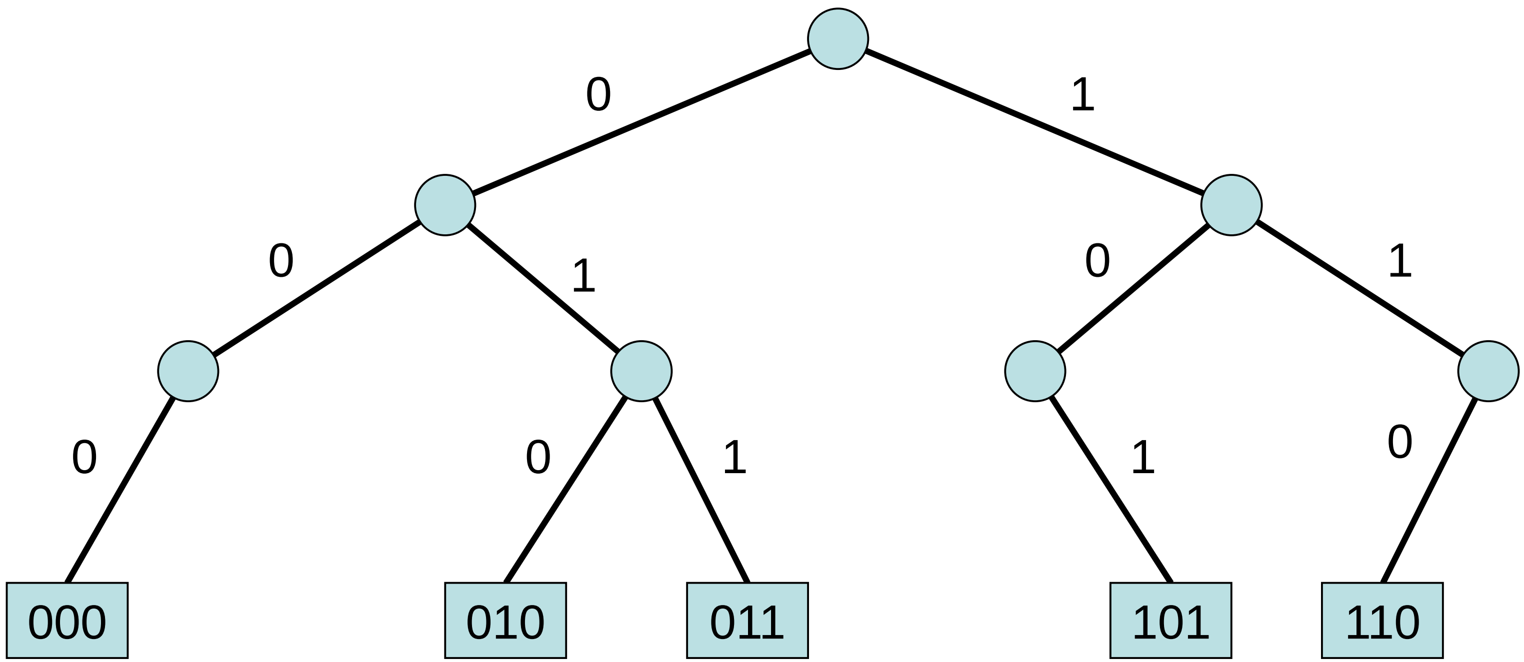
Trie

Example: (without list at bottom)



Trie

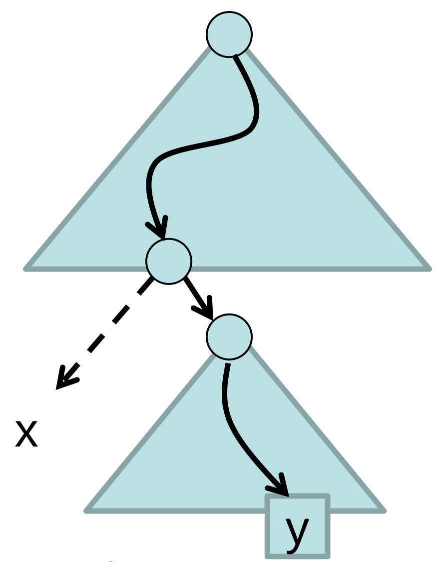
search(4) (4 corresponds to 100):



Output: 5 (longest common prefix)

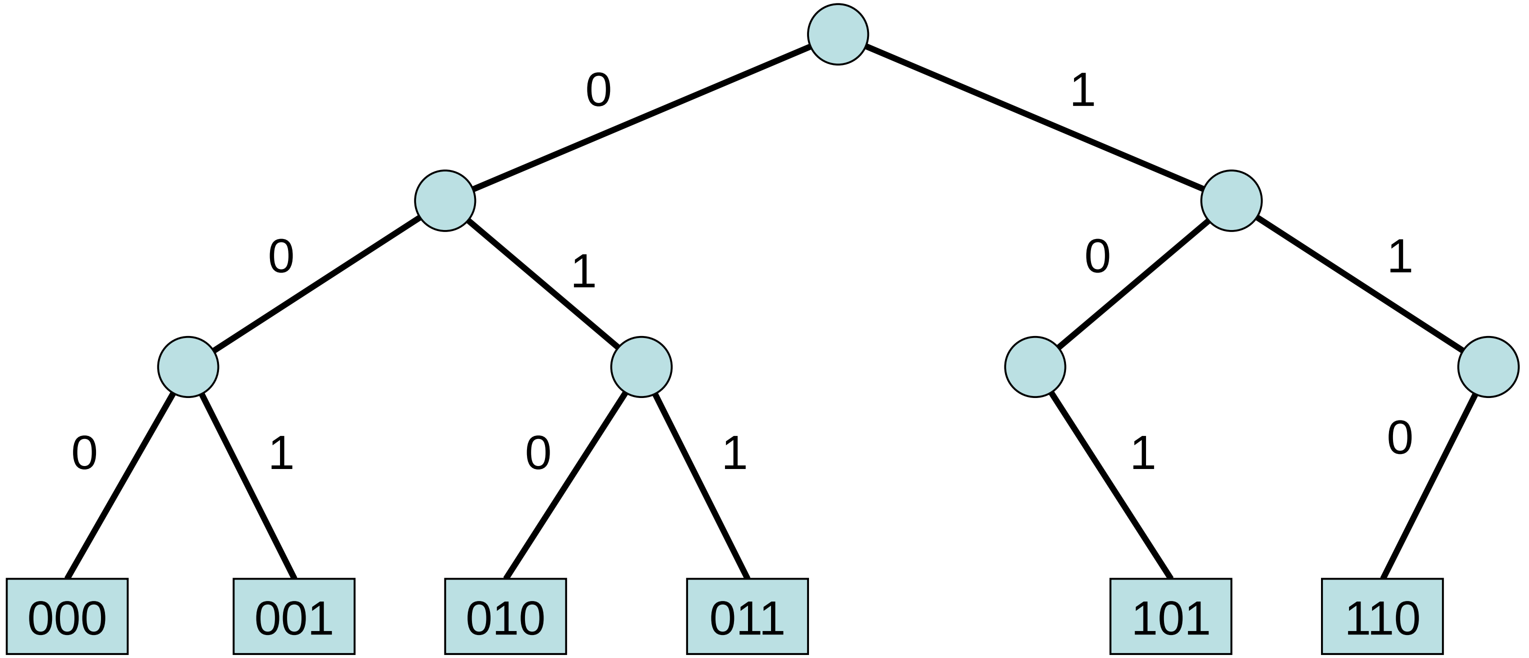
Trie

In general: a search(x) request follows the edges in the trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, the request may be forwarded to any leaf y in the subtrie below since all of them have the same longest prefix match with x.



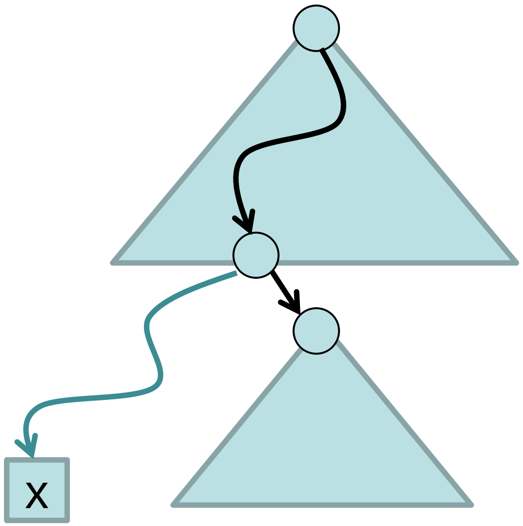
Trie

insert(1) (1 corresponds to 001):



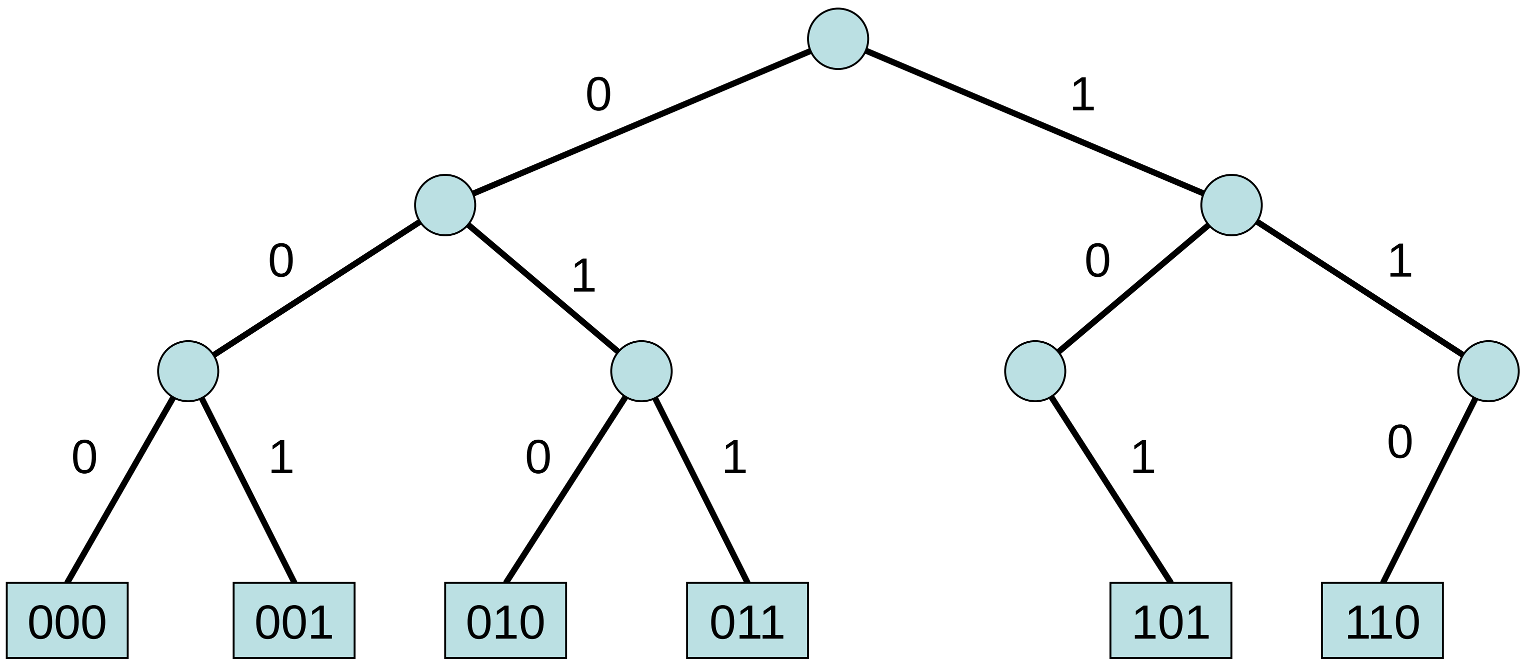
Trie

In general: an insert(x) request follows the edges in the trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, a new path (of length the remaining bits in x) is created that leads to the new leaf x.



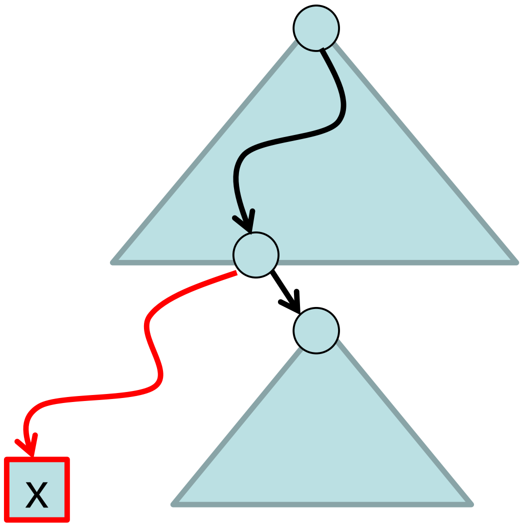
Trie

delete(5):



Trie

In general: a delete(x) request follows the edges in the trie down to the leaf x. If x does not exist, the delete operation terminates. Otherwise, x as well as the chain of nodes upwards till the first node with at least two children is deleted.



Patricia Trie

Problem:

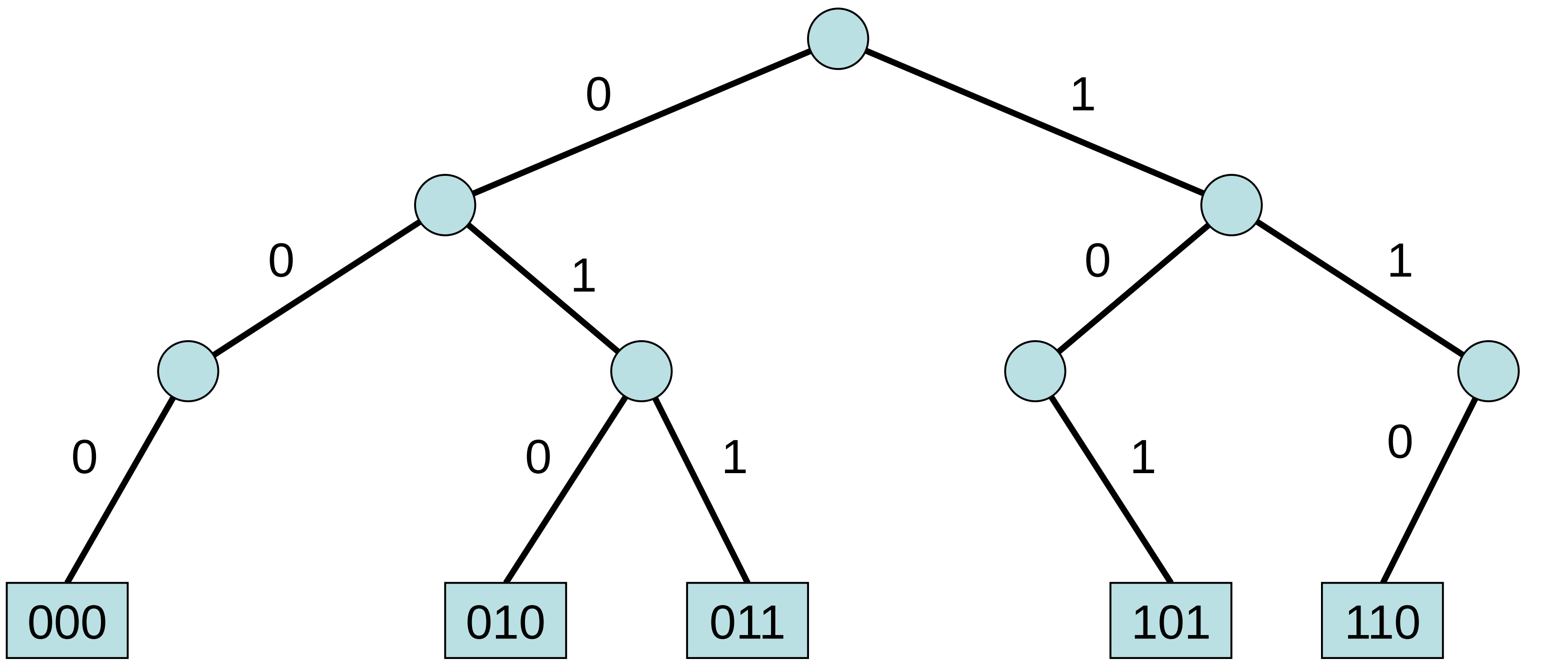
- Longest common prefix search for some $x \in \{0,1\}^W$ can take $\Theta(W)$ time.
- Insert and delete may require $\Theta(W)$ structural changes in the trie.

Improvement: use Patricia trie

A Patricia trie is a compressed trie in which all chains (i.e., maximal sequences of nodes of degree 1) are merged into a single edge whose label is equal to the concatenation of the labels of the merged trie edges.

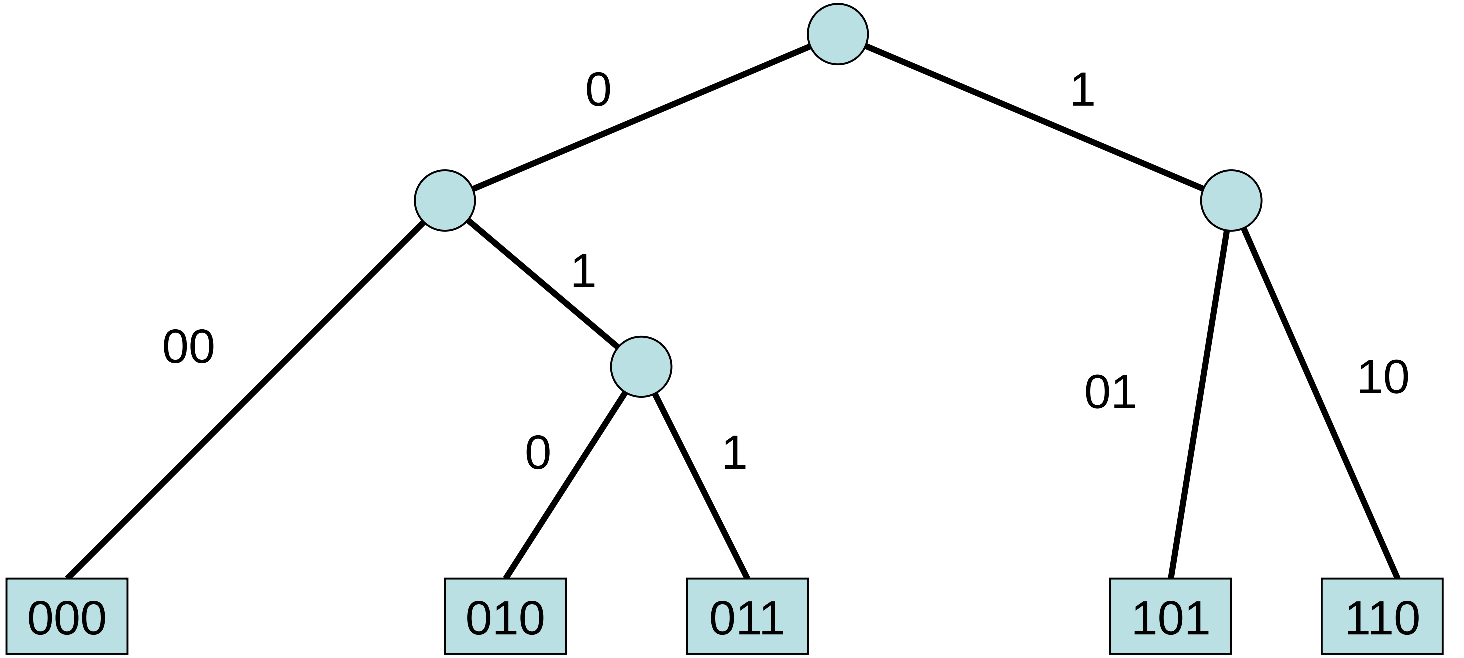
Trie

Example 1:



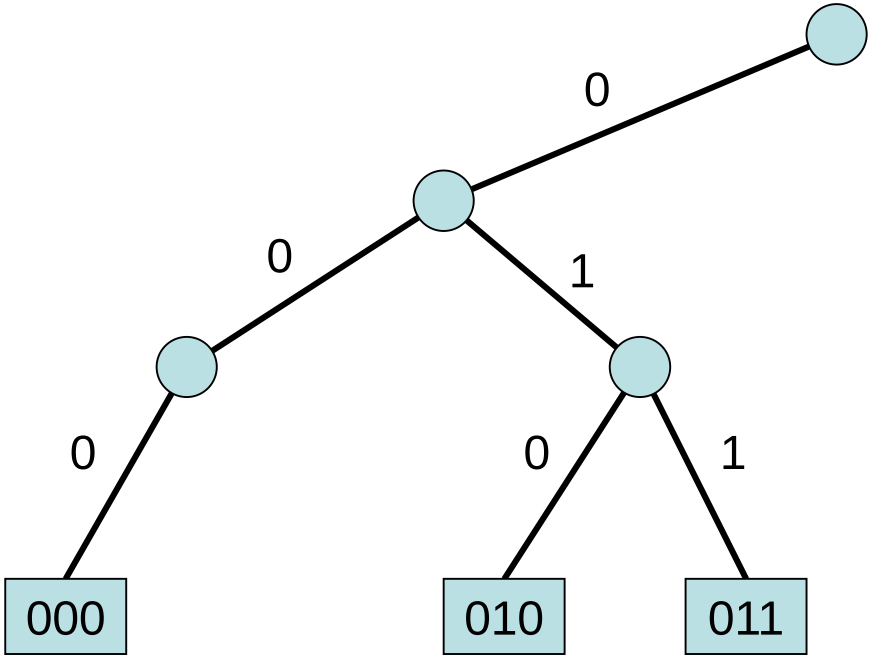
Patricia Trie

Example 1:



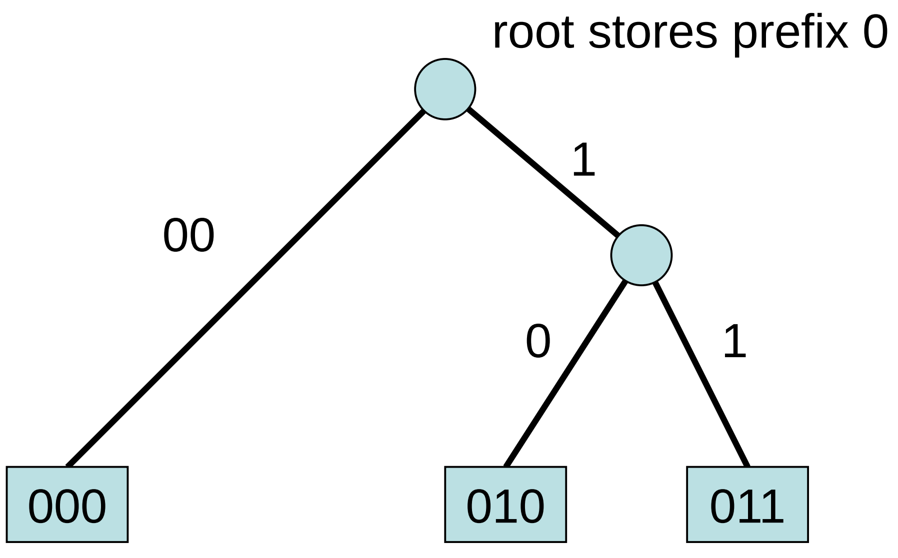
Trie

Example 2:



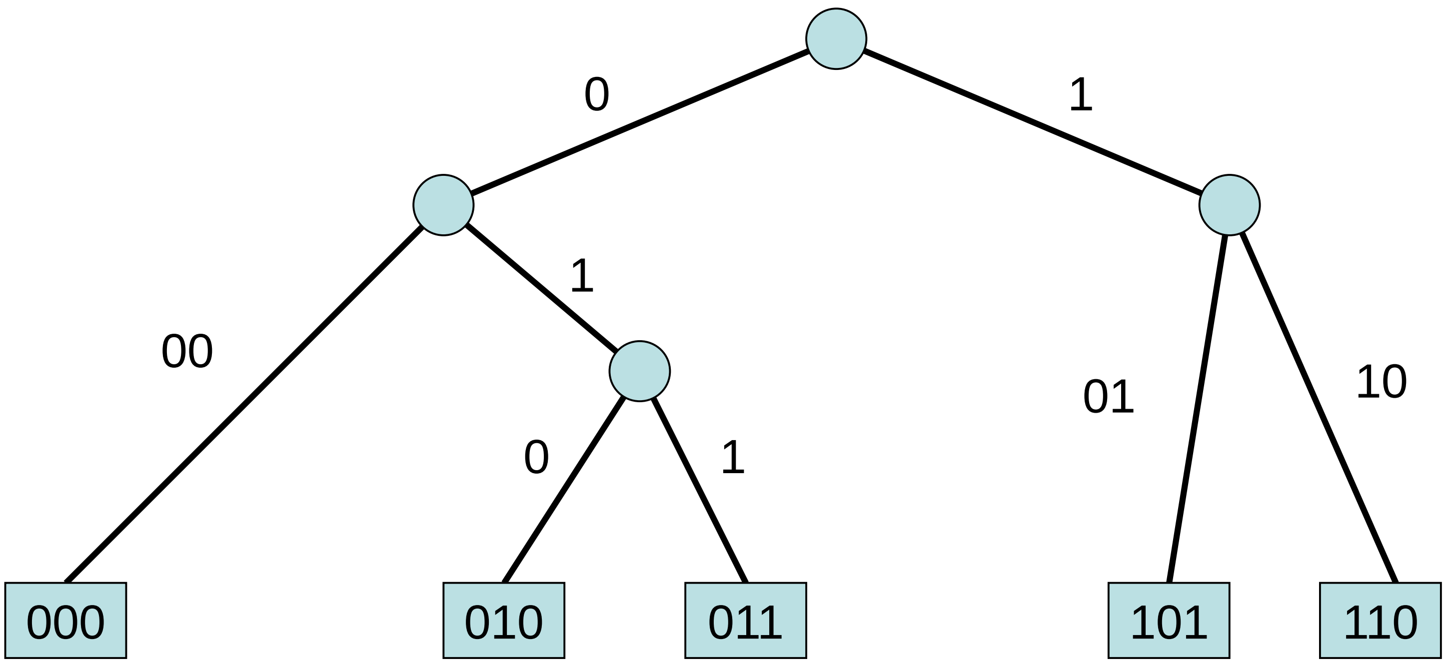
Patricia Trie

Example 2:



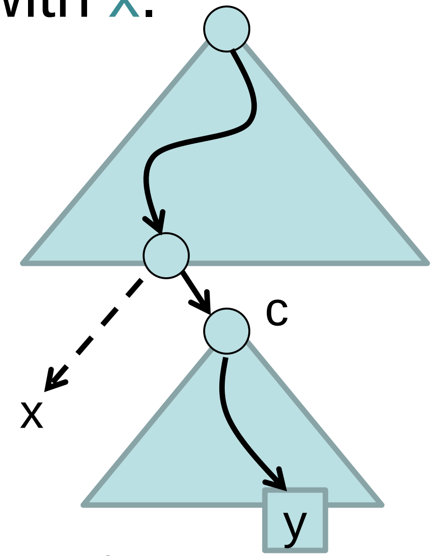
Patricia Trie

search(4):



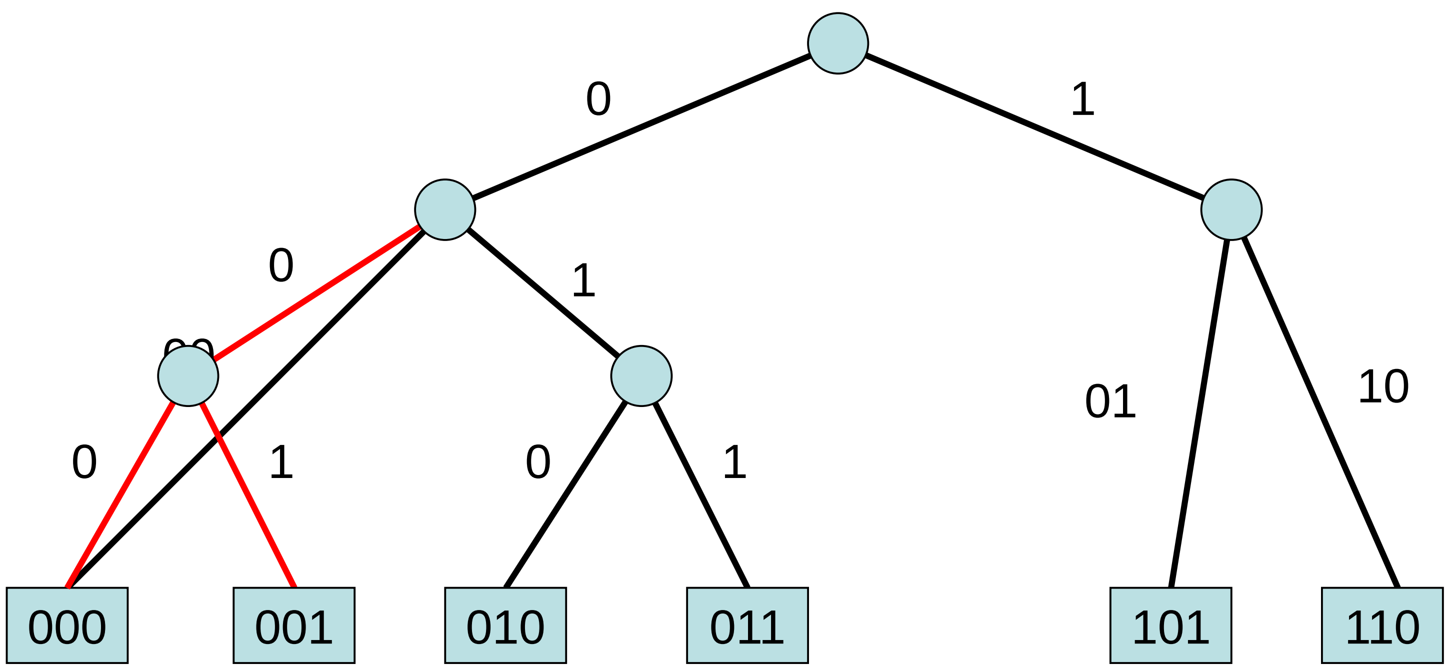
Patricia Trie

In general: a search(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, choose the current child c with longest common prefix. Then, the request may be forwarded to any leaf y in the subtrie rooted c at below since all of them have the same longest prefix match with x.



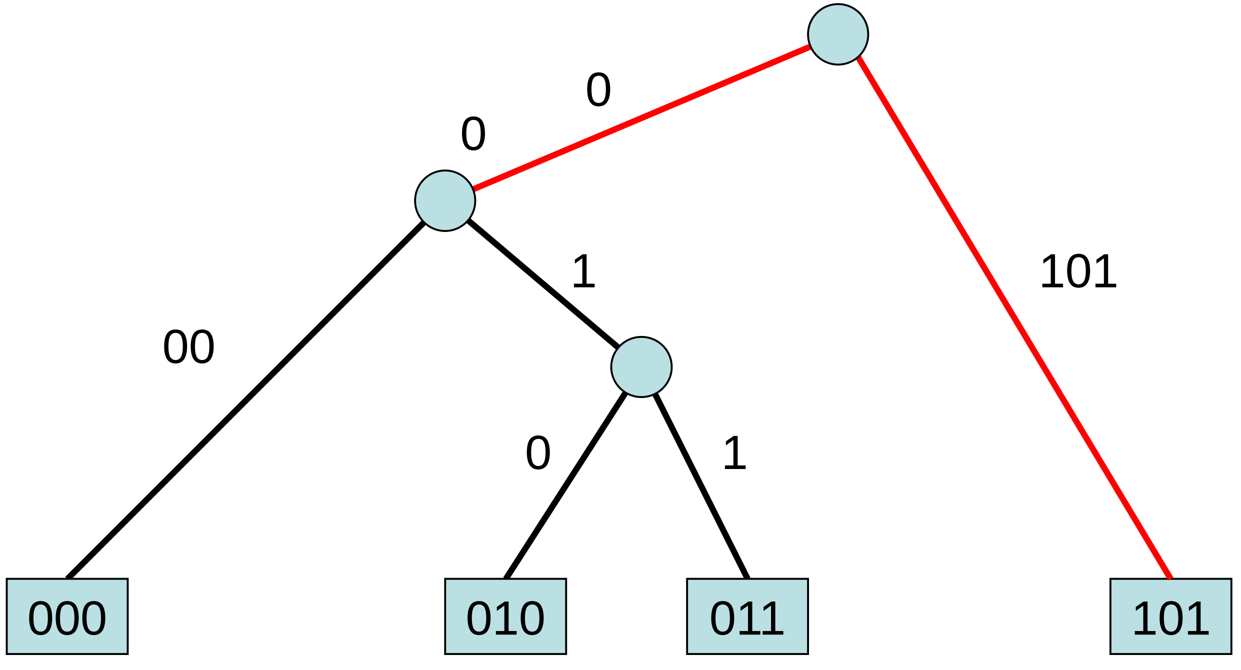
Patricia Trie

insert(1):



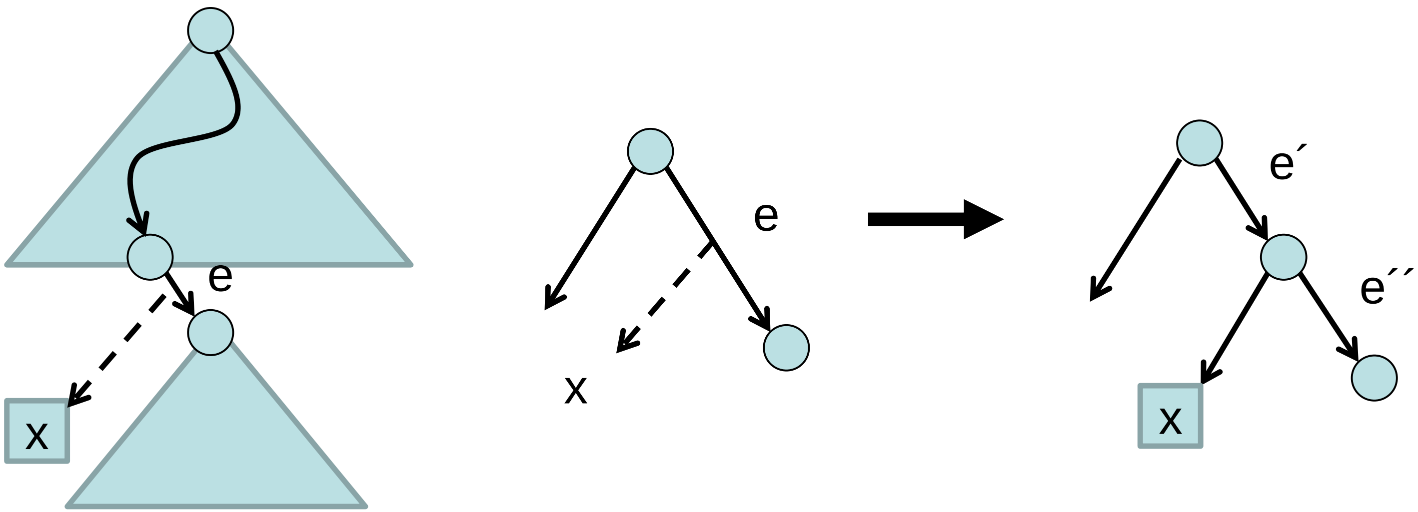
Patricia Trie

Insert(5):



Patricia Trie

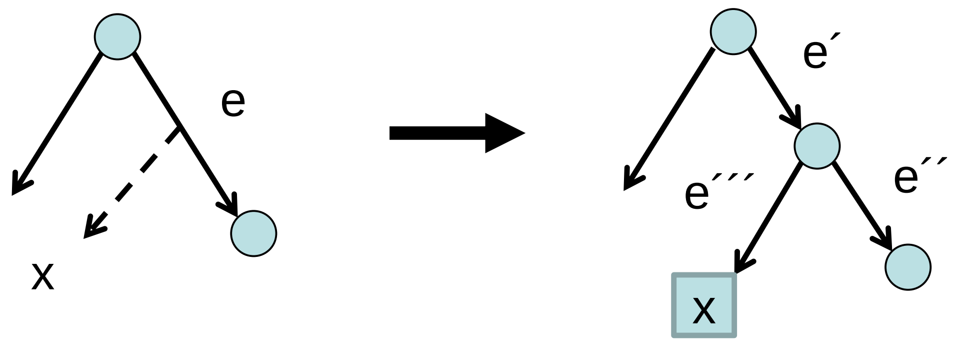
In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.



Patricia Trie

In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.

Example: l(e)=10010, x=…10110100

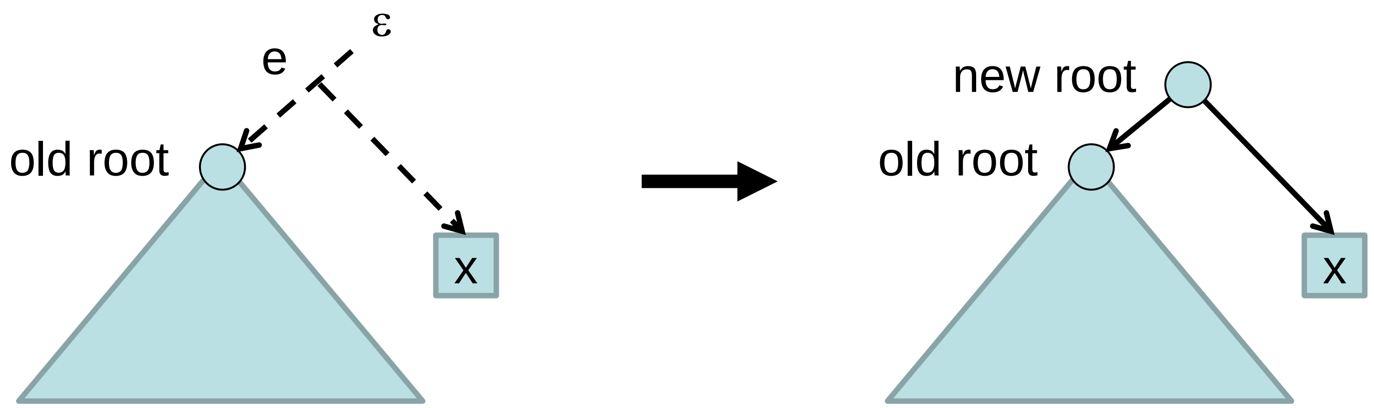


l(e´)=10
l(e´´)=010
l(e´´´)=110100

Patricia Trie

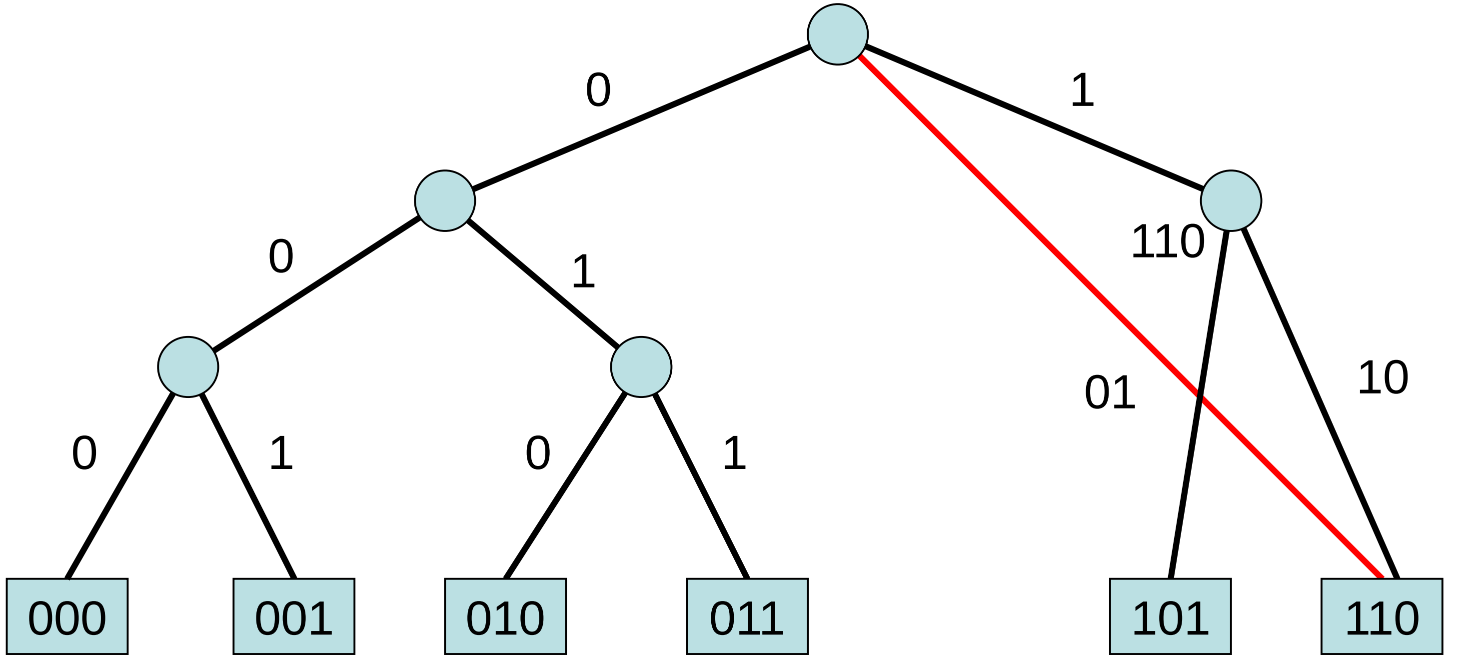
In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.

Special case:



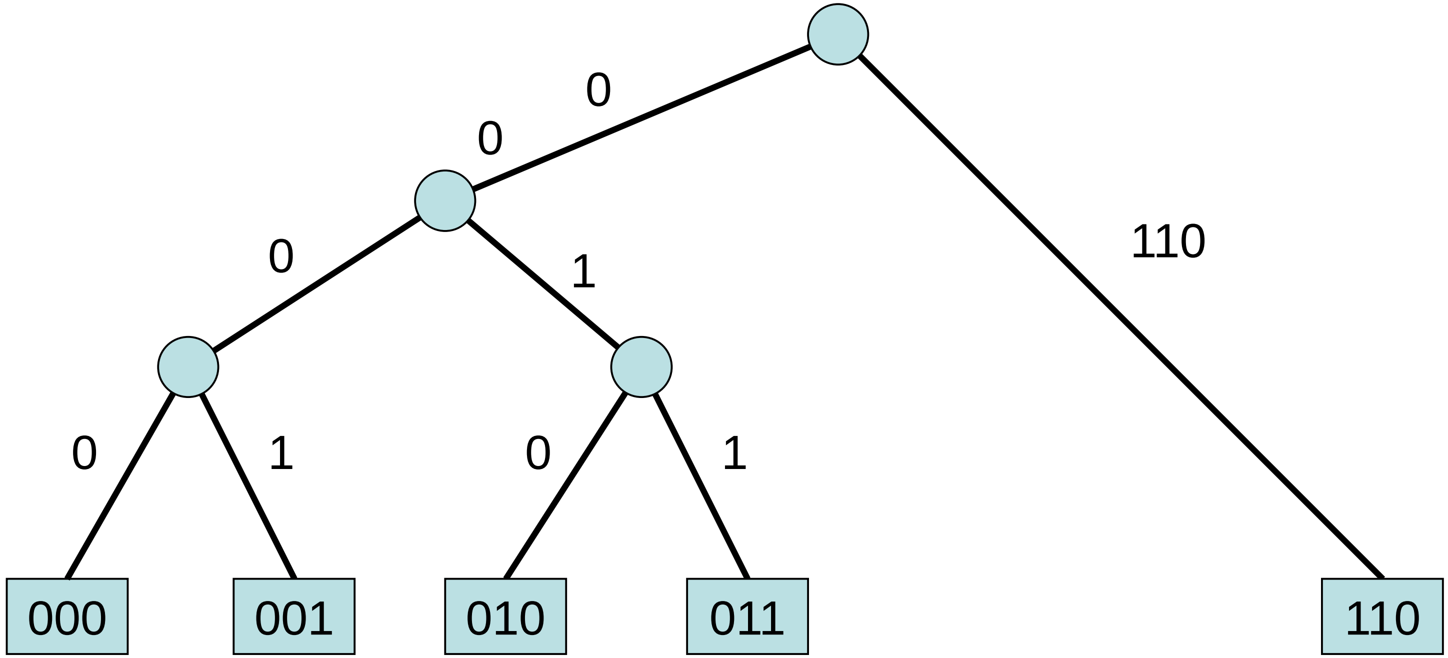
Patricia Trie

delete(5):



Patricia Trie

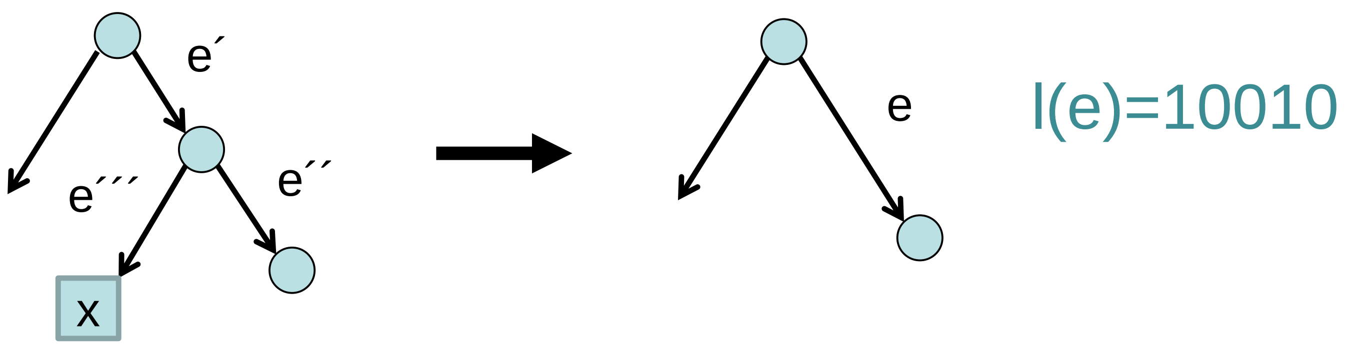
delete(6):



Patricia Trie

In general: a delete(x) request follows the edges in the Patricia trie down to the leaf x. If x does not exist, the delete operation terminates. Otherwise, x as well as its parent are deleted.

Example: l(e´)=10, l(e´´)=010, l(e´´´)=110100, x=…10110100



Patricia Trie

- Search, insert, and delete like in an ordinary binary tree, with the difference that we have labels at the edges.
- Search time still $O(W)$ in the worst case, but just $O(1)$ structural changes.

Patricia Trie

- History:
  - Invented independently by D. R. Morrison (1968) and G. Gwehenberger (1968).
  - Morrison called them „Patricia trees“, where PATRICIA stands for Practical Algorithm To Retrieve Information Coded in Alphanumeric.
  - Patricia trees are also referred to as *radix* trees (with radix 2).

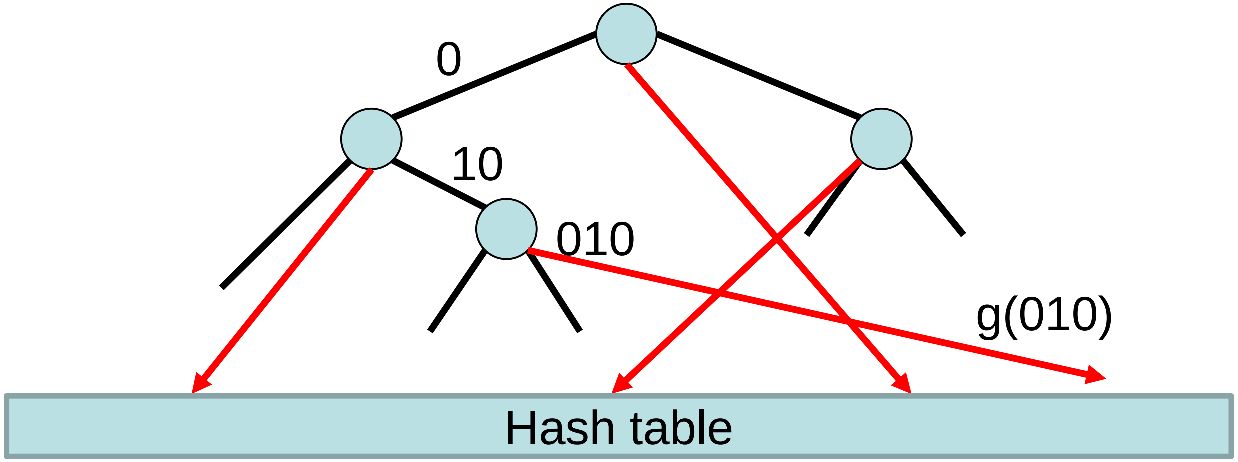
Idea (Kniesburges and Scheideler, 2011):

- To improve search time in Patricia trie, we hash the Patricia trie to some hash table.

Patricia Trie

Hashing to some hash table:

- Idea: Work over nodes rather than edges.
- Add labels to nodes: concatenation of edge labels from root
- Every node is hashed according to its node label.



- Then every Patricia node can directly be accessed via a HT-lookup if its label is known.

Patricia Trie

Observation:

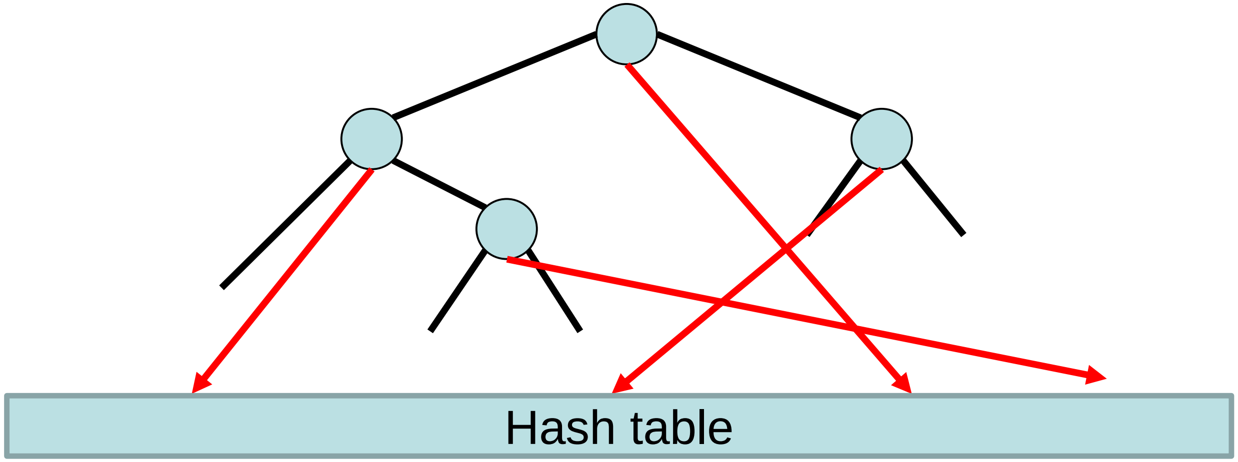
If one calls Search(x) when x is already in the tree (i.e. there exists a node with label x in tree), then a single lookup to the hashtable suffices to solve Search(x). Easy!

But what if x is *not* in the tree? Need to find a string in tree with largest matching prefix with x. We henceforth assume this case.

Patricia Trie

Hashing to some hash table:

- Idea: Work over nodes rather than edges.
- Add labels to nodes: concatenation of edge labels from root
- Every node is hashed according to its node label.



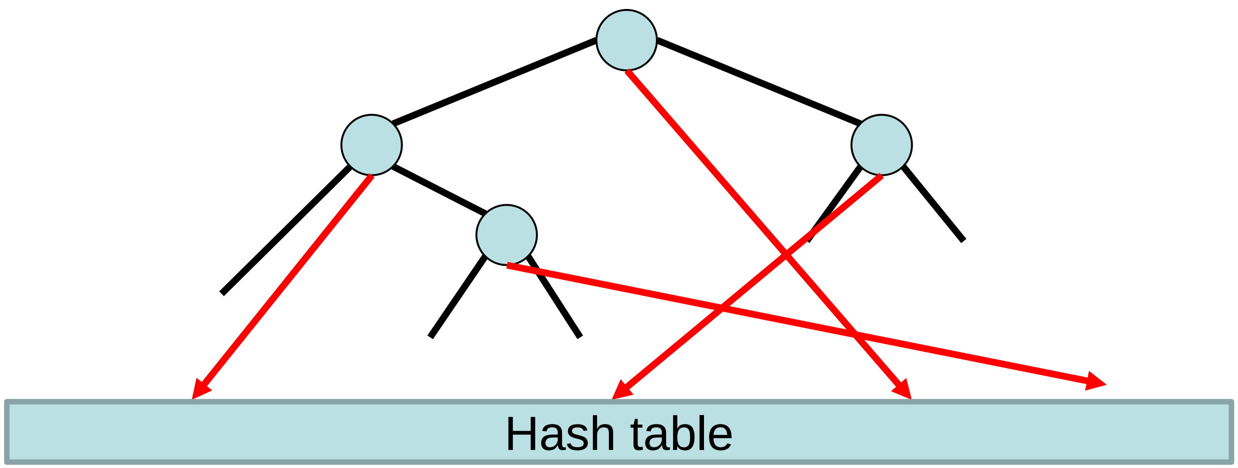
Next idea: Use binary search over node labels via HT-lookups to find the desired maximum prefix. This would run in time O(log W) instead of O(W)!

Problem: Max prefix is *not* necessarily attained at a node! (Why?)

Patricia Trie

Hashing to some hash table:

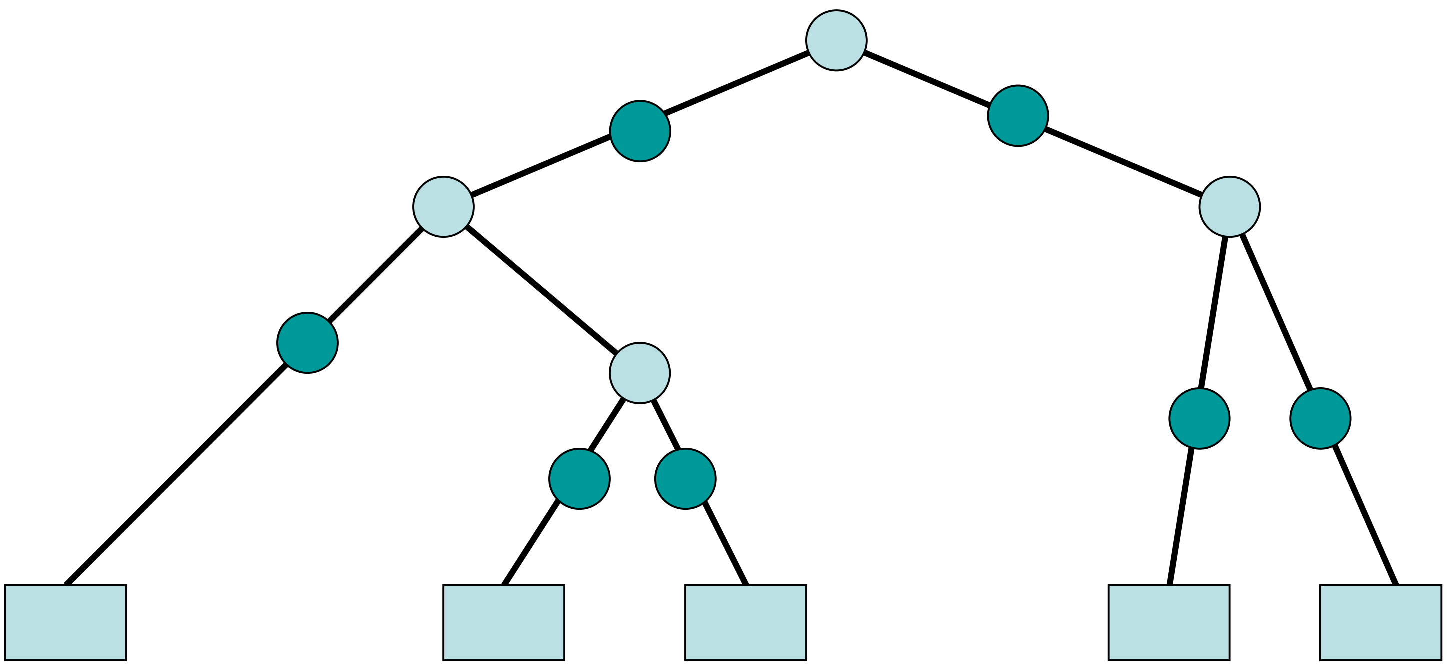
- Idea: Work over nodes rather than edges.
- Add labels to nodes: concatenation of edge labels from root
- Every node is hashed according to its node label.



Solution: add extra „intermediate“ nodes, called msd-nodes.

Patricia Trie Hashing

Solution: add msd-node ( ) for each edge.



Patricia Trie Hashing

- $|x|$: length of a bit sequence x.
- $b(v)$: label of node v.
- Recall: $msd(f,f')$ for two bit sequences f and f' is most significant bit (starting with position 0 from right) in which f and f' differ.
- Consider a bit sequence b with $(x_k,\ldots,x_0)$ being the binary representation of $|b|$. Let b' be a prefix of b. The msd-sequence $m(b',b)$ of b' and b is the prefix of b of length $l(|b|,j)=\sum_{i=j}^{k} x_i 2^i$ with $j=msd(|b|,|b'|)$.

  (Note: read the definition above carefully, noting the use of parameters b', b', $|b|$, and $|b'|$.)

Patricia Trie Hashing

- $|x|$: length of a bit sequence x.
- $b(v)$: label of node v.
- Recall: $msd(f,f')$ for two bit sequences f and f' is most significant bit (starting with position 0 from right) in which f and f' differ.
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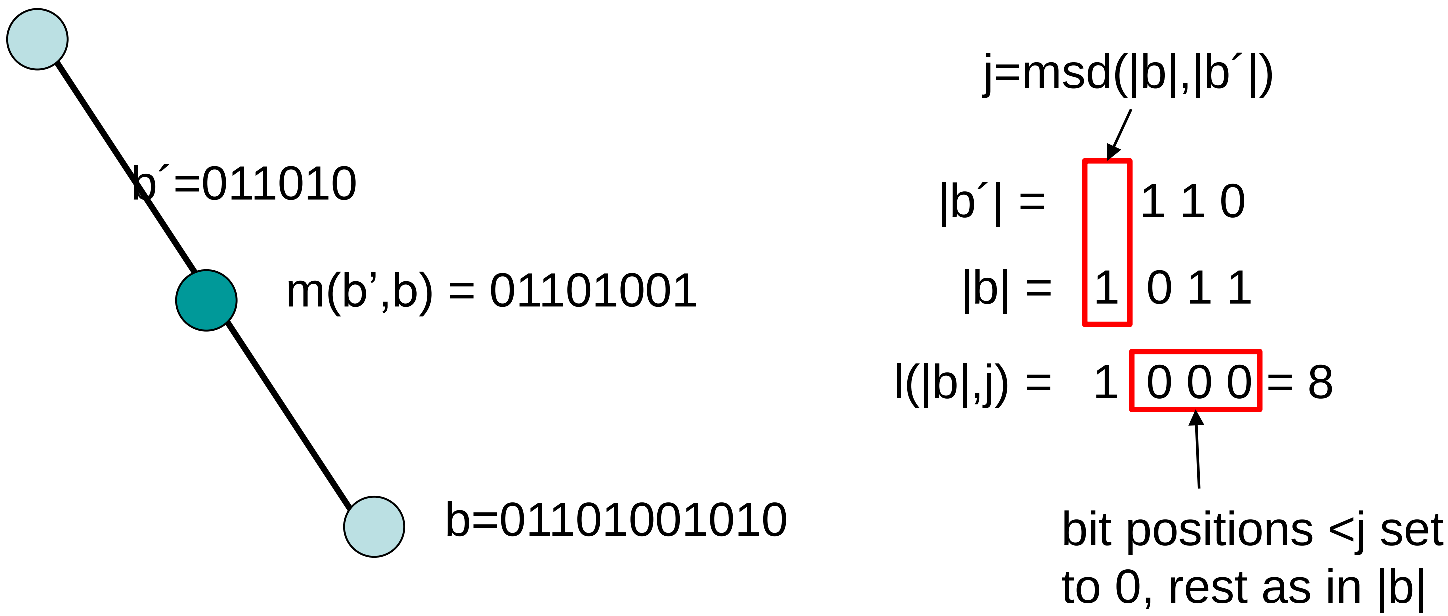
The msd-sequence $m(b',b)$ of b' and b is the prefix of b of length $l(|b|,j)=\sum_{i=j}^{k} x_i\, 2^i$ with $j=msd(|b|,|b'|)$.

Example: Consider $b=01101001010$ and $b'=011010$.
Then $|b|=1011_2$, and $|b'|=110_2$, i.e., $msd(|b|,|b'|)=3$. Hence, $l(|b|,j)=8$ and $m(b',b)= 01101001$.

Q: Why is msd used on |b| and |b'|, instead of b and b'?

Patricia Trie Hashing

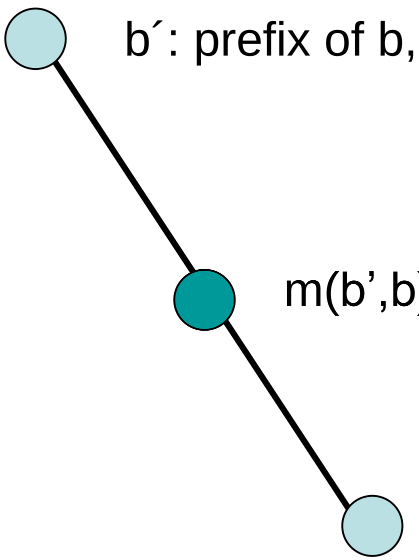
Example: Consider b=01101001010 and b’=011010.
Then $|b|=1011_2$, and $|b'|=110_2$, i.e., msd(|b|,|b’|)=3. Hence, l(|b|,j)=8 and m(b’,b)= 01101001.



Since we will binary search over label *lengths*, the new msd node is chosen to be of the „right length“ so as to help our binary search find it as we go down the tree.

Patricia Trie Hashing

Another example:



b´: prefix of b, |b´|=36

m(b’,b) is first 40 bits of b

b: string of length |b|=43

j=msd(|b|,|b´|)

|b´| = 1 0 0 1 0 0

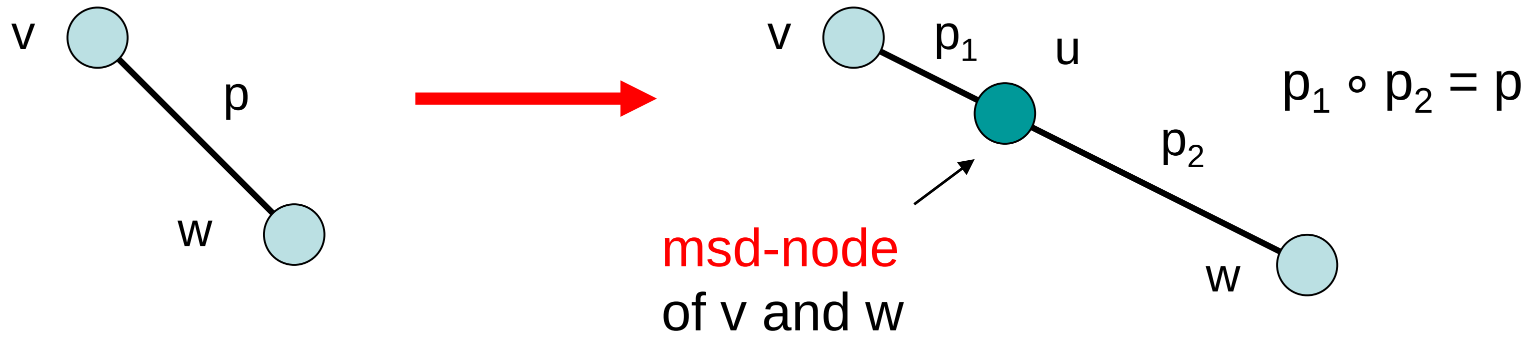
|b| = 1 0 1 0 1 1

l(|b|,j) = 1 0 1 0 0 0 = 40

bit positions <j set to 0, rest as in |b|

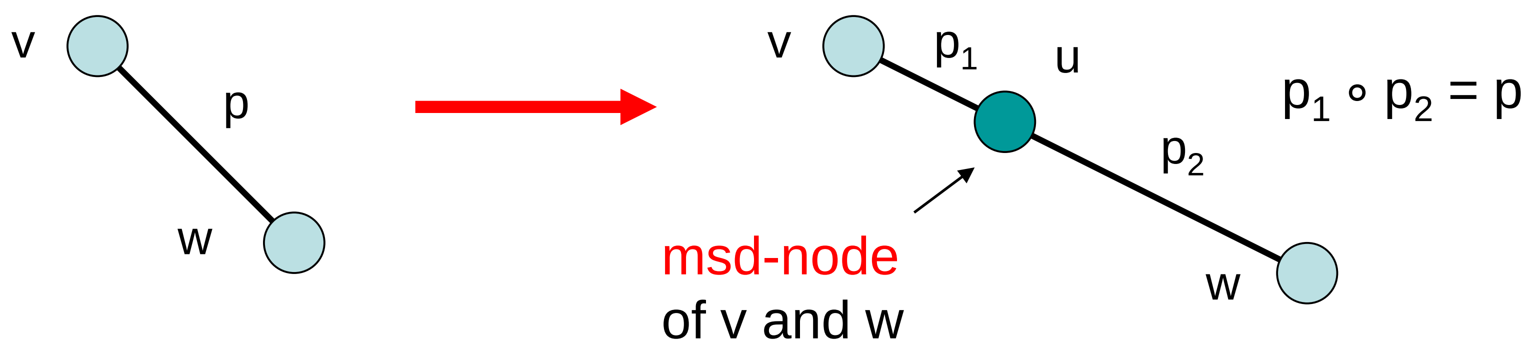
Patricia Trie Hashing

Approach: We replace every edge e={v,w} in the Patricia trie by two edges {v,u} and {u,w} with b(u)=m(b(v),b(w)) and hash the labels on each node to the hash table.

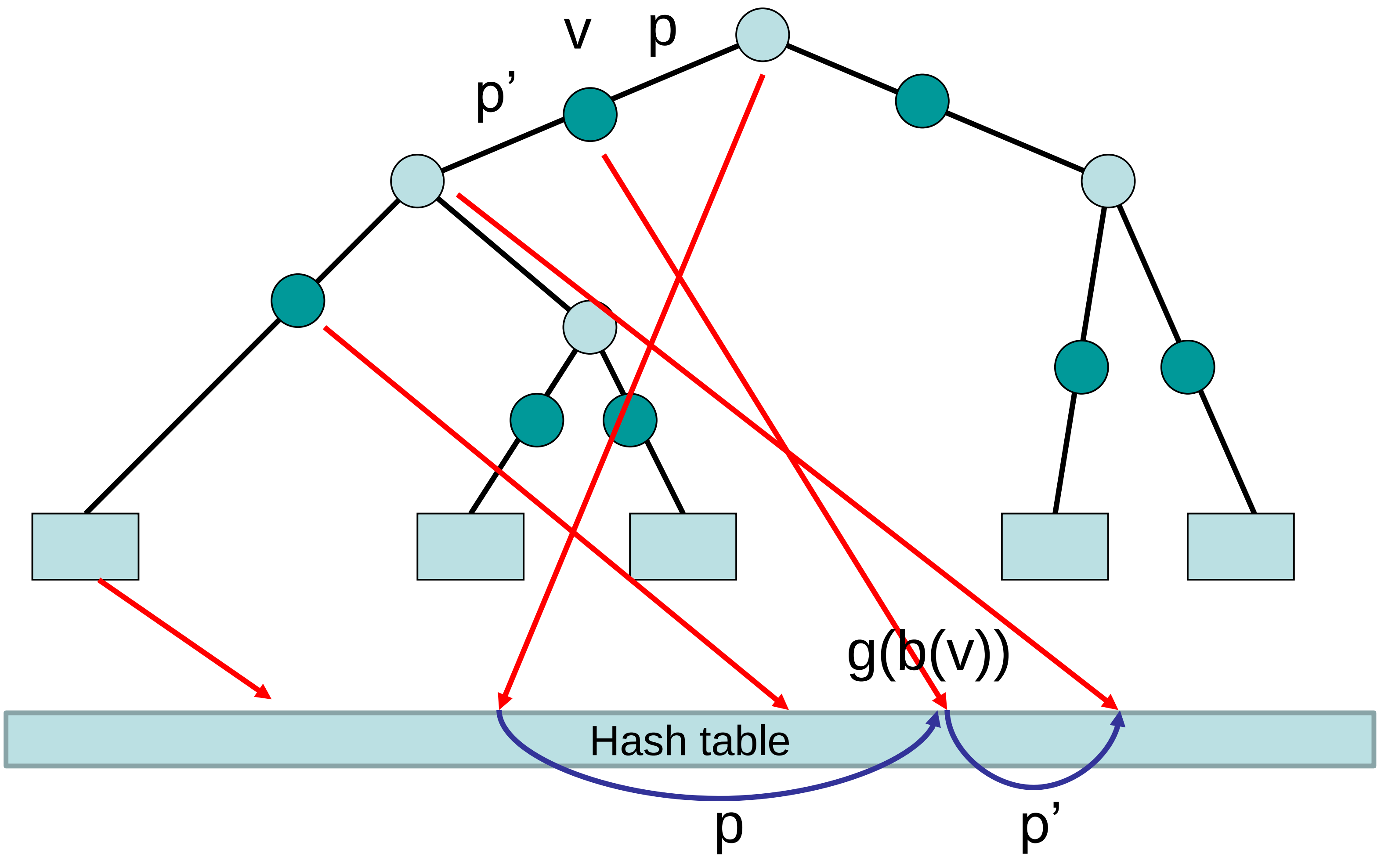


Patricia Trie Hashing

Motivation for inserting msd-nodes: msd-node placed at the position where binary search on the node label length will look for the first time for a node label of length between |b(v)| and |b(w)|.



Patricia Trie Hashing



Patricia Trie Hashing

Data structure for longest prefix search:

Every hash entry of a tree node v stores:

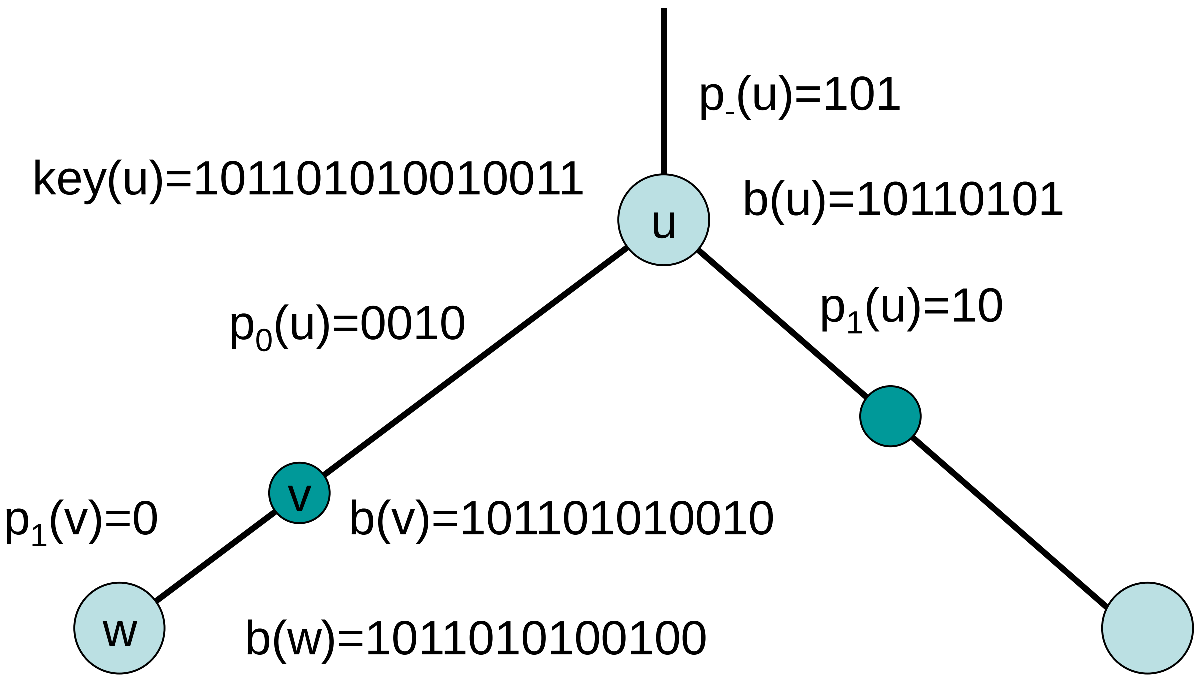
1. Label $b(v)$ of v (always ε for the root!)
2. Key $key(v)$ of an element e below the subtree of v, if v is an original Patricia trie node. (As in splay tree, allows us to directly jump to an element e in O(1) time.)
3. Labels $p_x(v)$ of edges to children, $x \in \{0,1\}$
4. Label $p_-(v)$ of edge to parent (root: $p_-(v)=$prefix to root)

Every hash entry of a list element e stores:

1. Key of e
2. Label $p_-(v)$ of edge to parent
3. Label of tree node storing $key(e)$

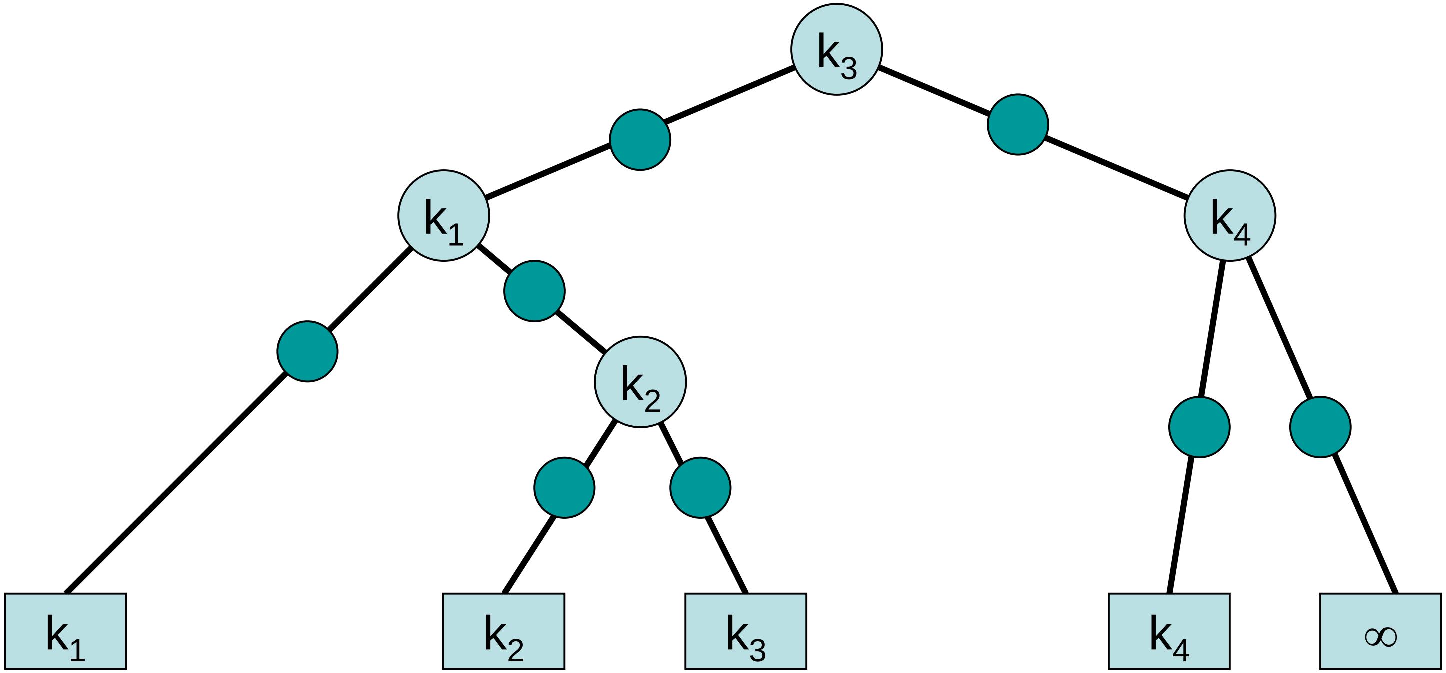
Patricia Trie Hashing

Example:



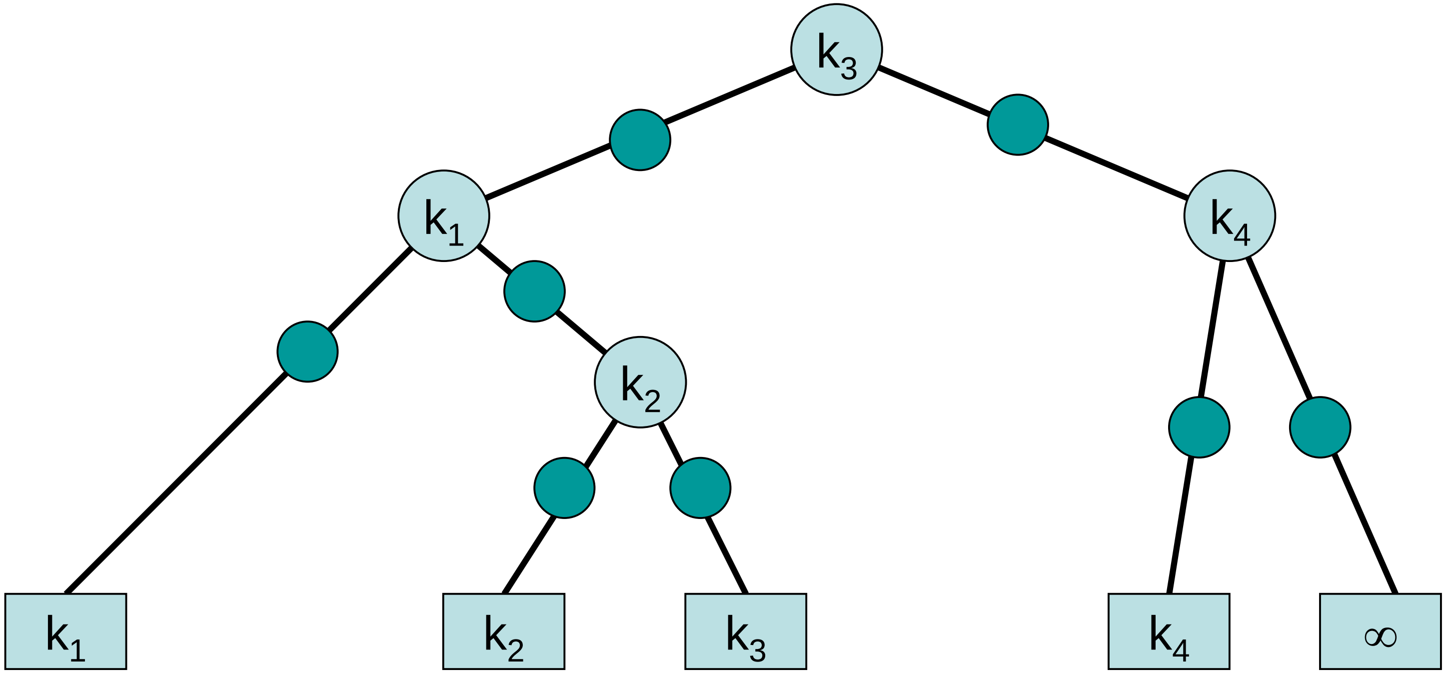
Patricia Trie Hashing

Requirement: every tree node stores key of exactly one element (possible with ∞).



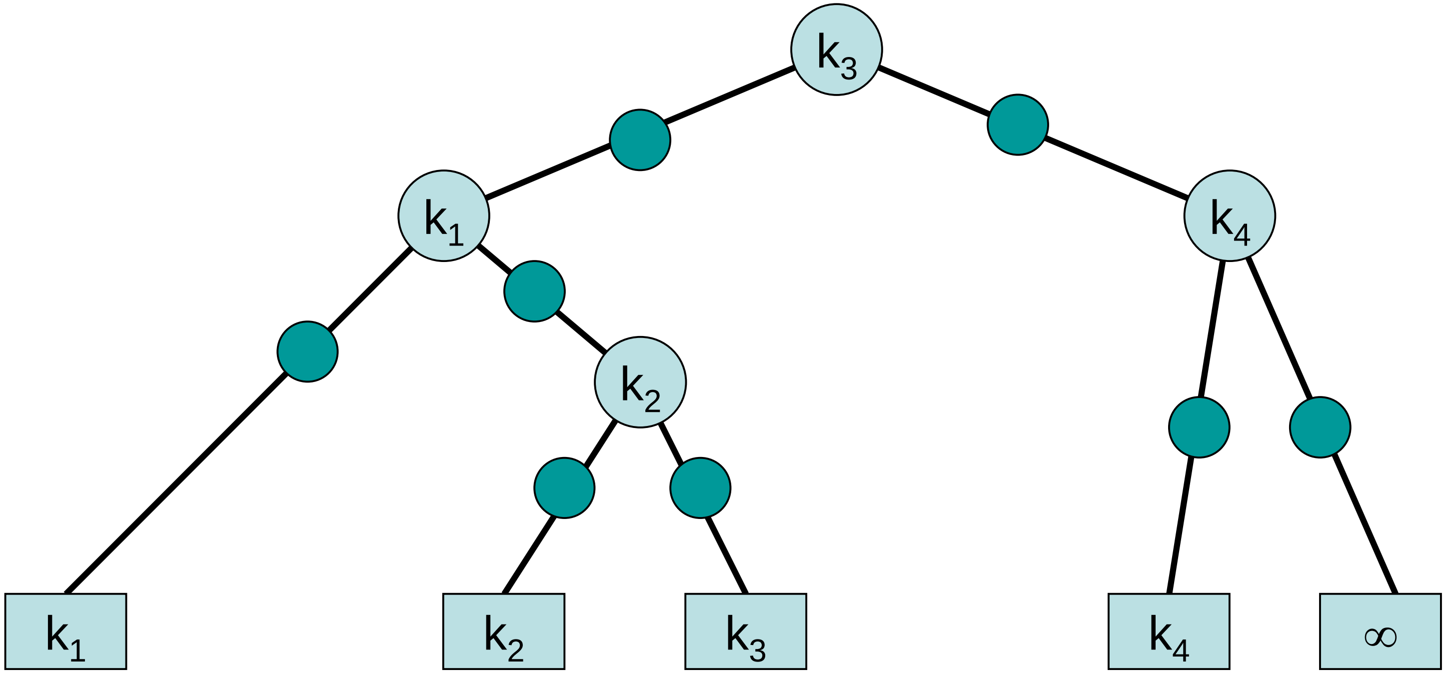
Patricia Trie Hashing

Invariant: the label of a tree node is a prefix of the key stored in it.



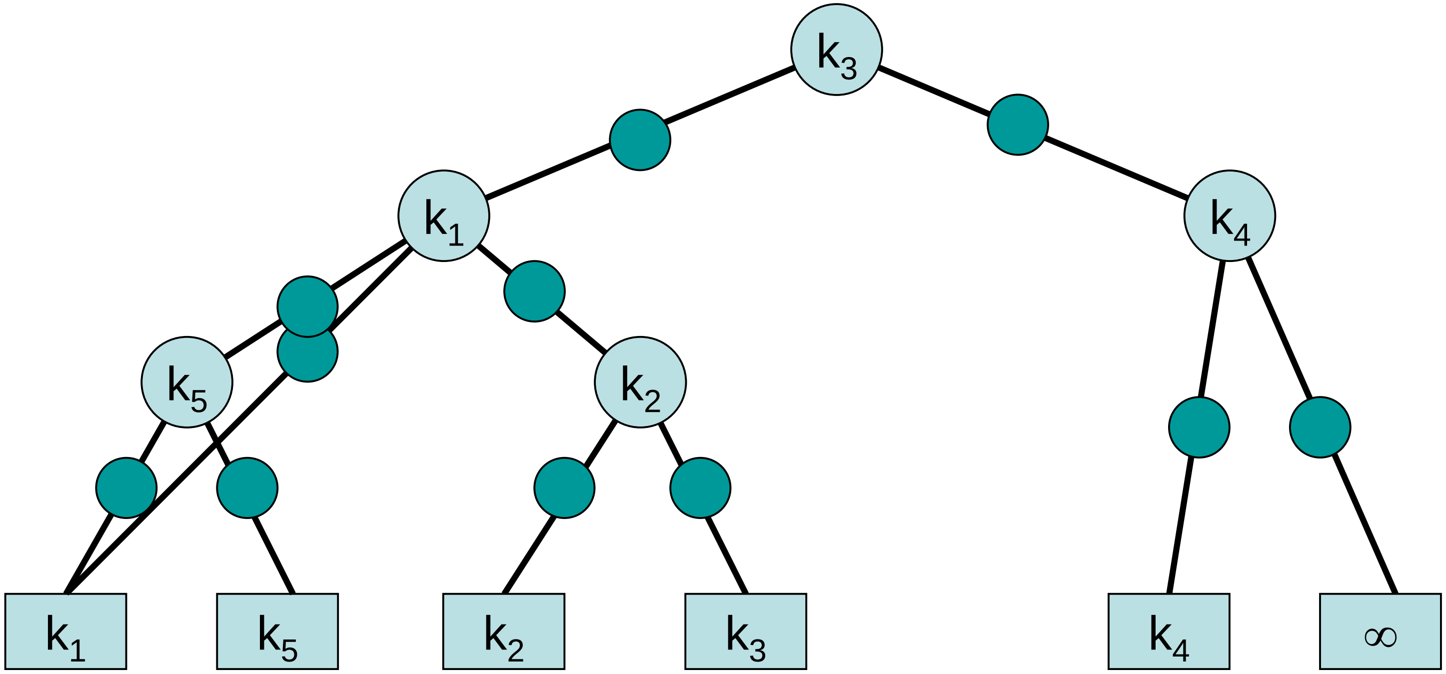
Patricia Trie Hashing

We first illustrate the structural changes for insert and delete.



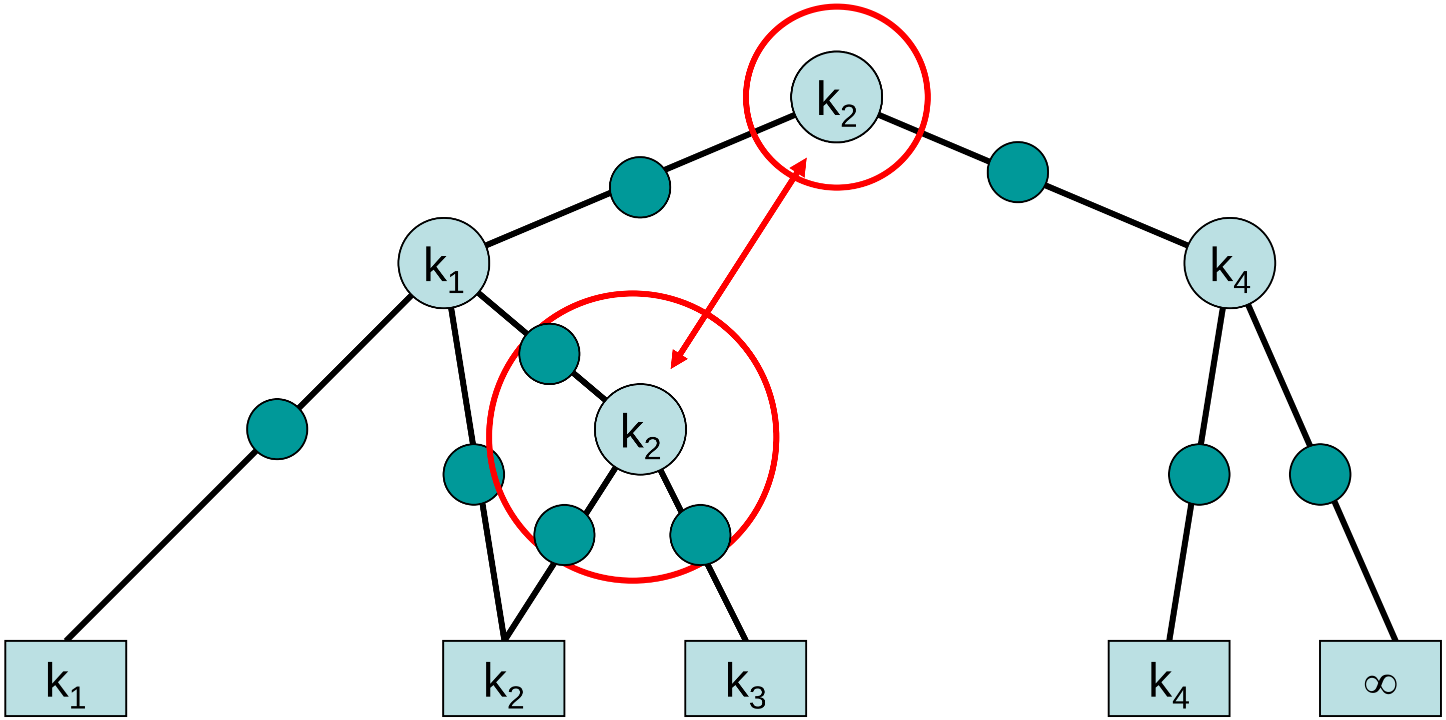
Patricia Trie Hashing

Insert(e), key(e)=k_5: like in binary search tree



Patricia Trie Hashing

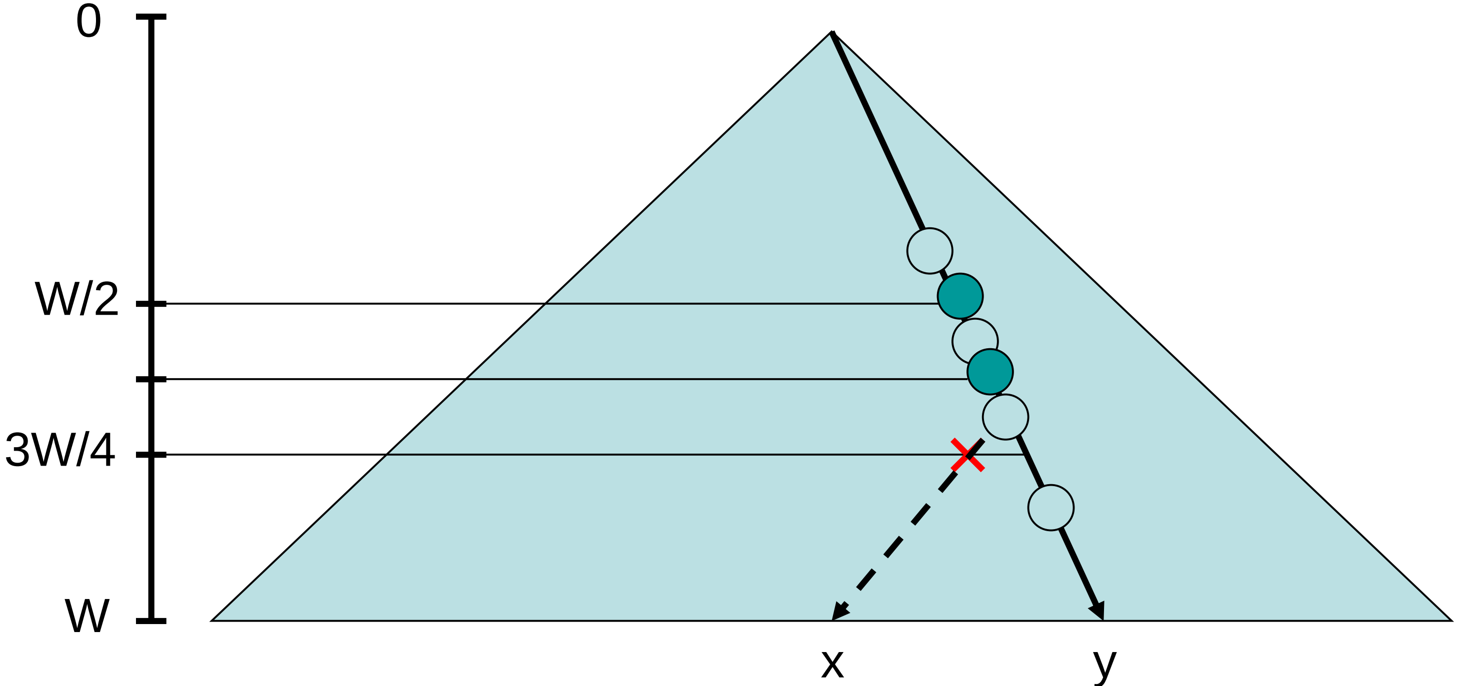
Delete(k_3): like in binary search tree



Patricia Trie Hashing

Search(x): (W: power of two)

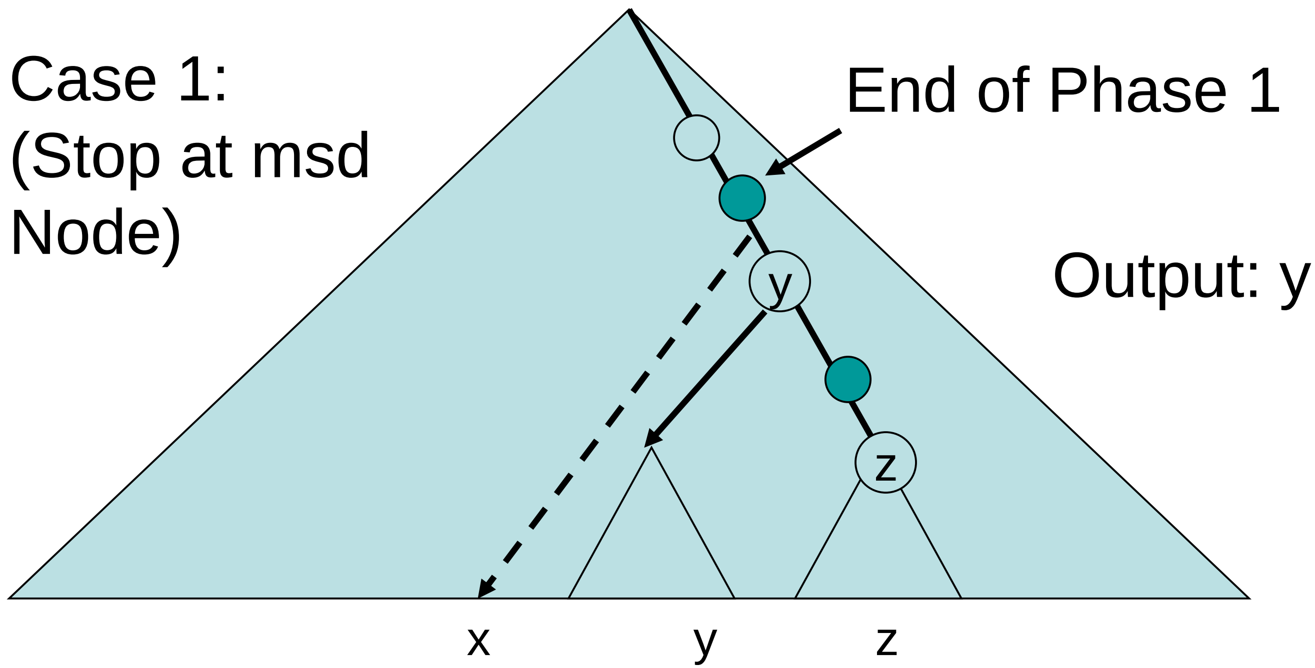
Phase 1: binary search on *length* of longest matching prefix of x via msd-nodes to find „good starting point" for a brute force search (Phase 2)



Patricia Trie Hashing

Search(x): (W: power of two)

Phase 2: Do brute force traversal downward

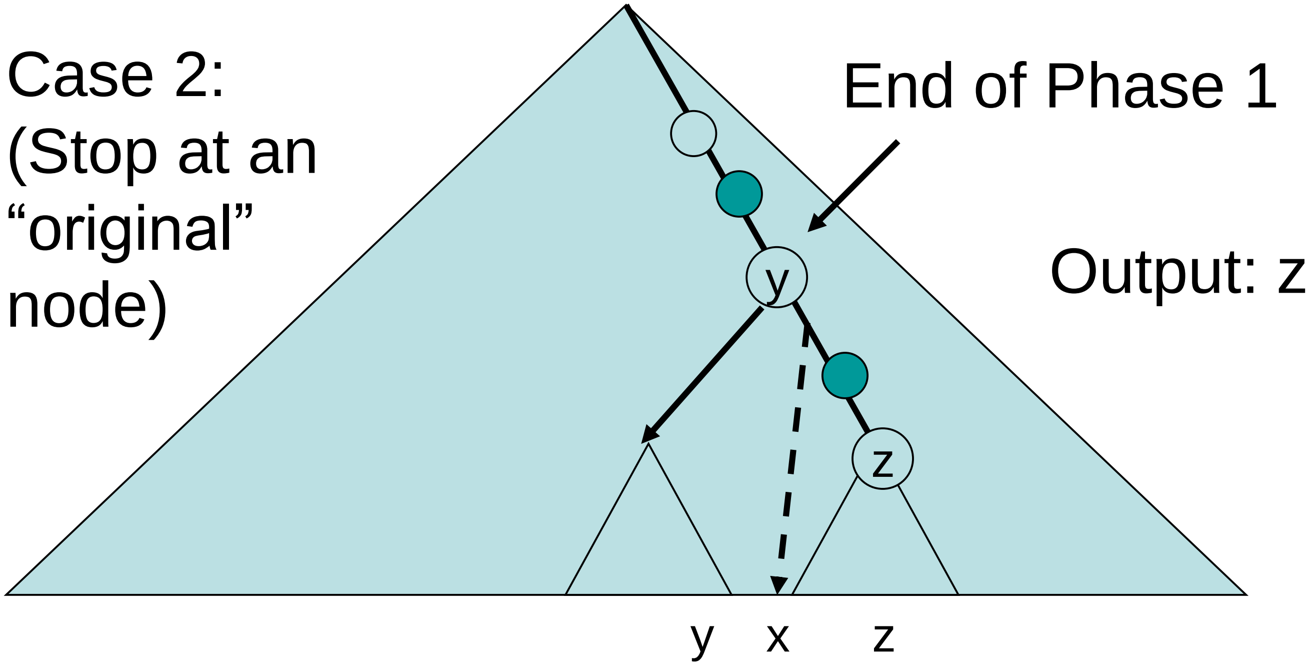


Patricia Trie Hashing

Search(x): (W: power of two)

Phase 2: Do brute force traversal downward

Case 2:
(Stop at an
“original”
node)



Patricia Trie Hashing

- Let $x \in \{0,1\}^W$ be represented by $(x_1,\dots,x_W)$
- Hash function: $h: U \rightarrow [0,1)$, Hash table: T

search(x):

    // Easy case: x is already in tree
    if $key(T[h(x)])=x$ then return $T[h(x)]$
    // Phase 1: binary search for x
    $s:=\lfloor \log W \rfloor$; $k:=0$; $v:=T[h(\varepsilon)]$; $p:=p_-(v) \circ p_{x_1}(v)$ // v: root of Patricia trie
    while $s >= 0$ do
        // is there node with label $(x_1,\dots,x_{k+2^s})$?
        if $(x_1,\dots,x_{k+2^s}) = b(T[h(x_1,\dots,x_{k+2^s})])$ // yes
            then $k:=k+2^s$; $v:=T[h(x_1,\dots,x_k)]$; $p:= (x_1,\dots,x_k) \circ p_{x_{k+1}}(v)$
        else if $(x_1,\dots,x_{k+2^s})$ is prefix of p
            // edge from v covers $(x_1,\dots,x_{k+2^s})$
            then $k:=k+2^s$
        $s:=s-1$
    // end while – end of Phase 1 – continues next slide

Patricia Trie Hashing

search(x): (continued from previous slide)

// Phase 1 stops at deepest *node* v with b(v) being a prefix of $(x_1,\ldots,x_W)$

// Phase 2: brute force to find correct max prefix

if $p_{x_{k+1}}(v)$ exists then

$v{:=}T[h(b(v) \circ p_{x_{k+1}}(v))]$

else

$v{:=}T[h(b(v) \circ p_{\bar{x}_{k+1}}(v))]$

if v is msd-node then //jump to next original node

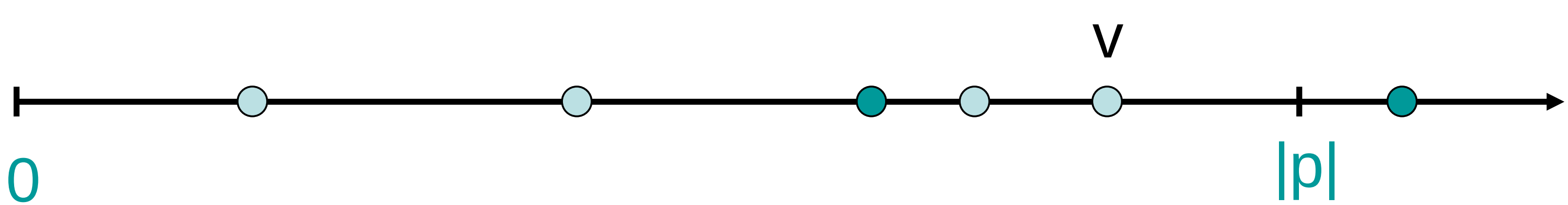
$v{:=}T[h(b(v) \circ p)]$ for bit sequence p out of v

return key(v)

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)_2$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node



Binary representation of $|b(v)|$ has ones at positions $i_1, i_2, \ldots$ (i_1: maximal position)

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)_2$.
- Patricia trie contains a route for prefix p
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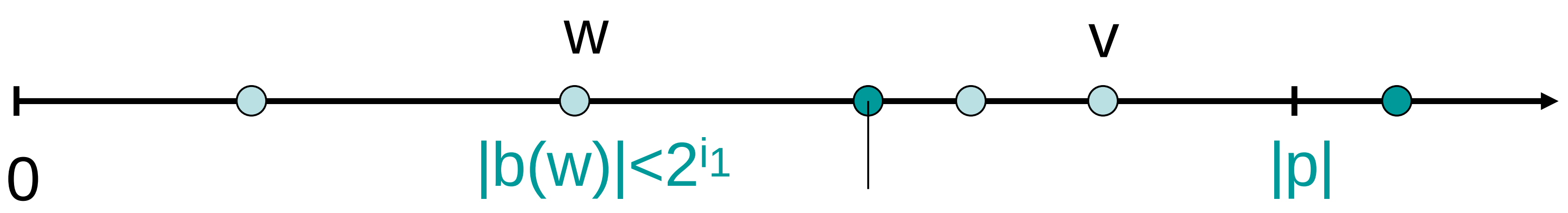
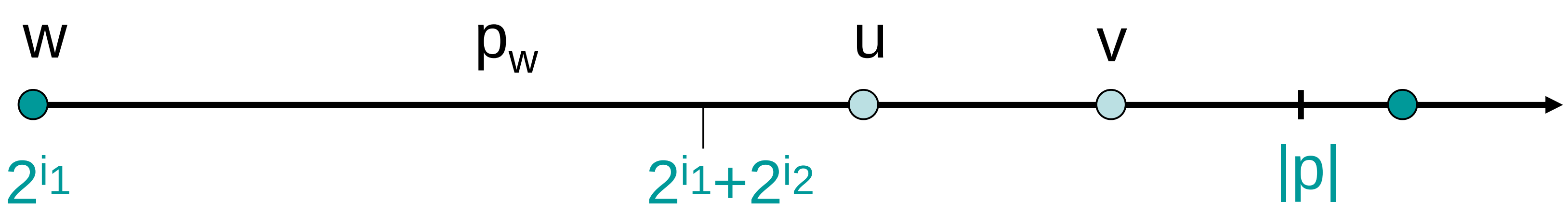


msd-node must exist at 2^{i_1},
will be found by binary search

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)_2$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node

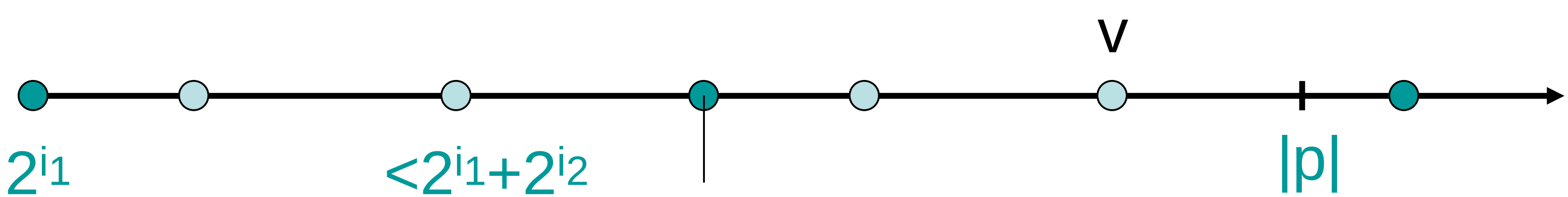


a) no msd-node at $2^{i_1}+2^{i_2}$: only if no Patricia node u with $2^{i_1}<|b(u)|\leq 2^{i_1}+2^{i_2}$, but this can be recognized via p_w

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)_2$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node

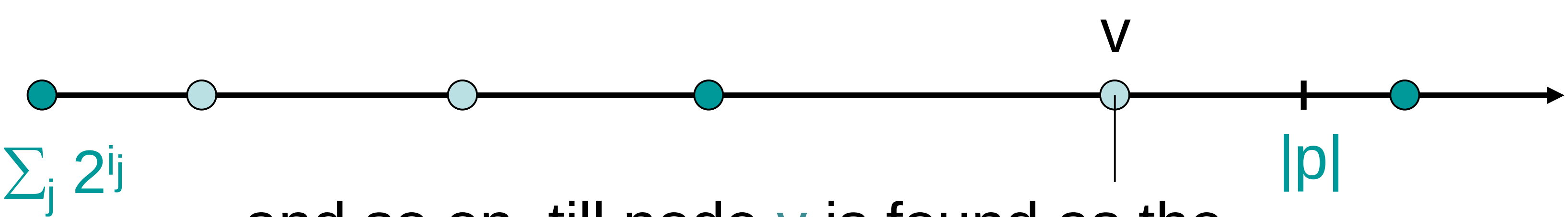


b) msd-node at $2^{i_1}+2^{i_2}$: is found by binary search

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)_2$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node

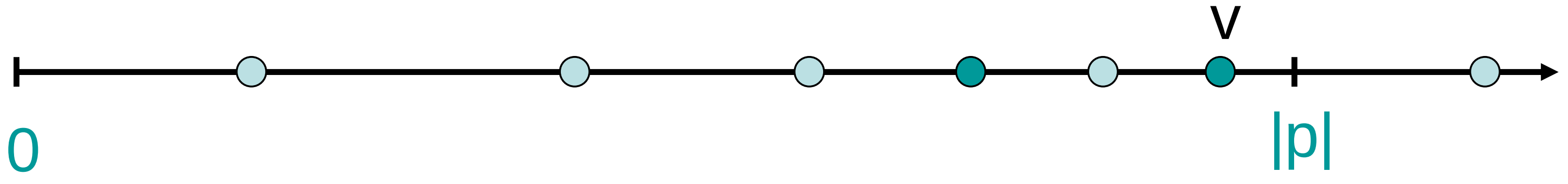


and so on, till node v is found as the last node of the binary search

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)_2$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 2: v is msd-node



v will also be the last node of binary search if it is an msd-node (argue like in case 1)

Patricia Trie Hashing

Number of HT accesses for longest prefix search:

- O(log W) HT-lookups, where W is key length

Number of HT accesses for insert:

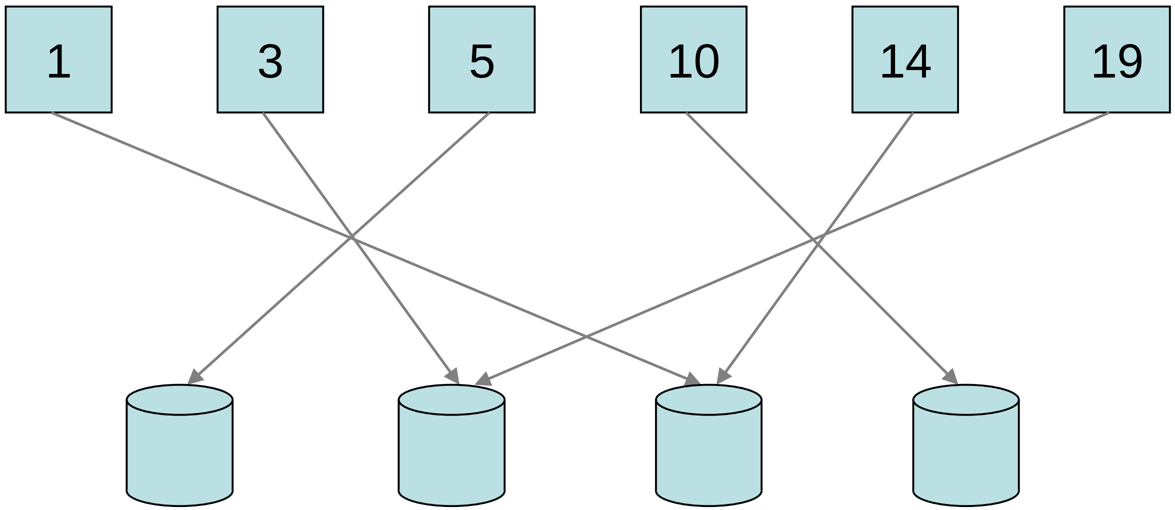
- O(log W) HT-lookups
- O(1) HT-updates

Number of HT accesses for delete:

- O(1) HT-lookups (why not O(log W)?)
- O(1) HT-updates

Patricia Trie Hashing

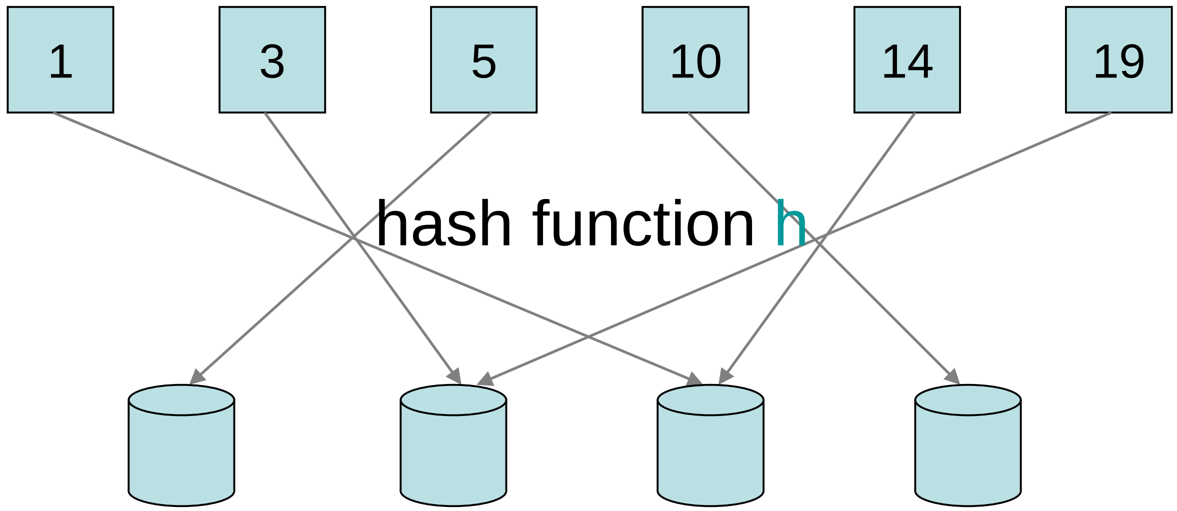
Application: distributed storage system



Goal: minimize number of accesses to servers for longest prefix match

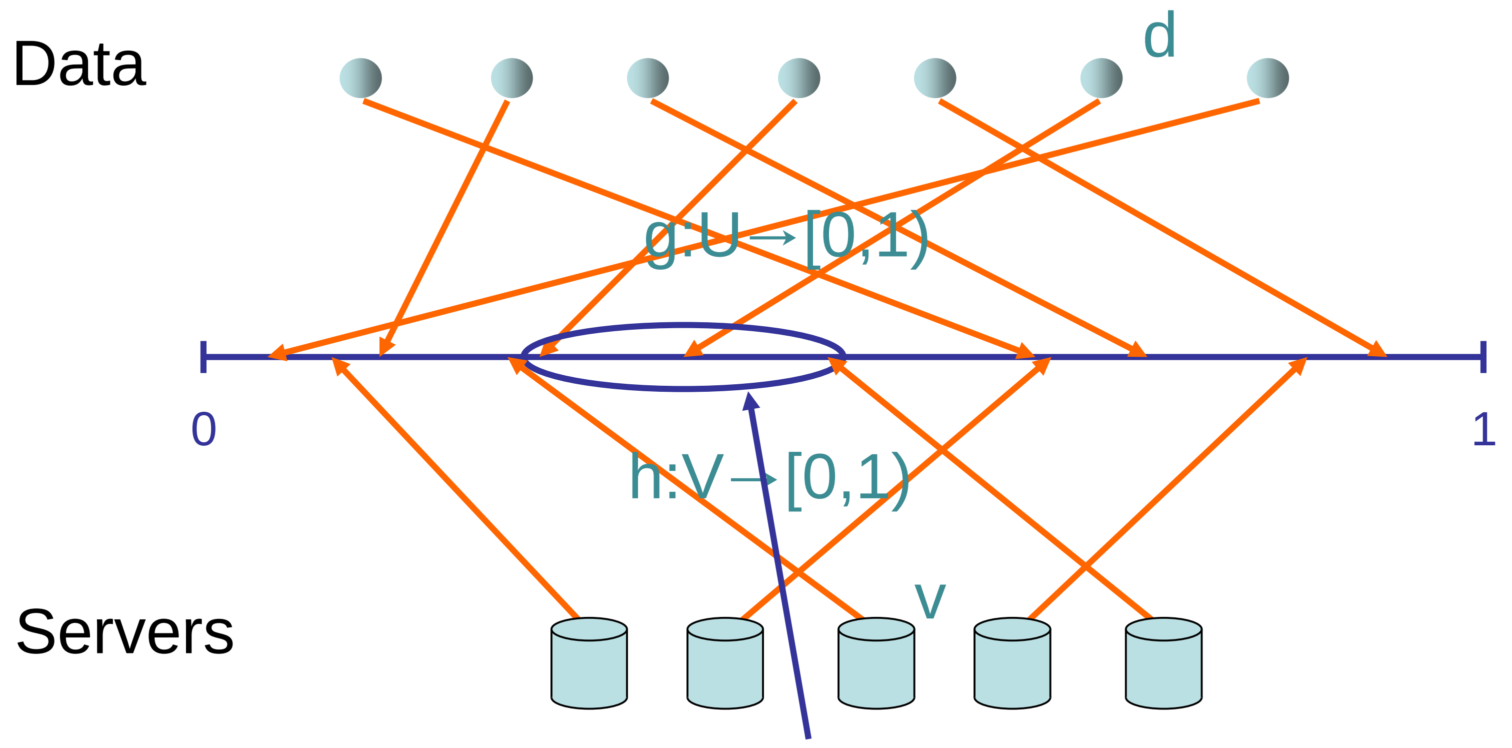
Distributed Storage System

Standard approach for exact search:
distributed hash table (DHT)



Consistent Hashing

Choose two random hash functions h, g



Region that server v is responsible for

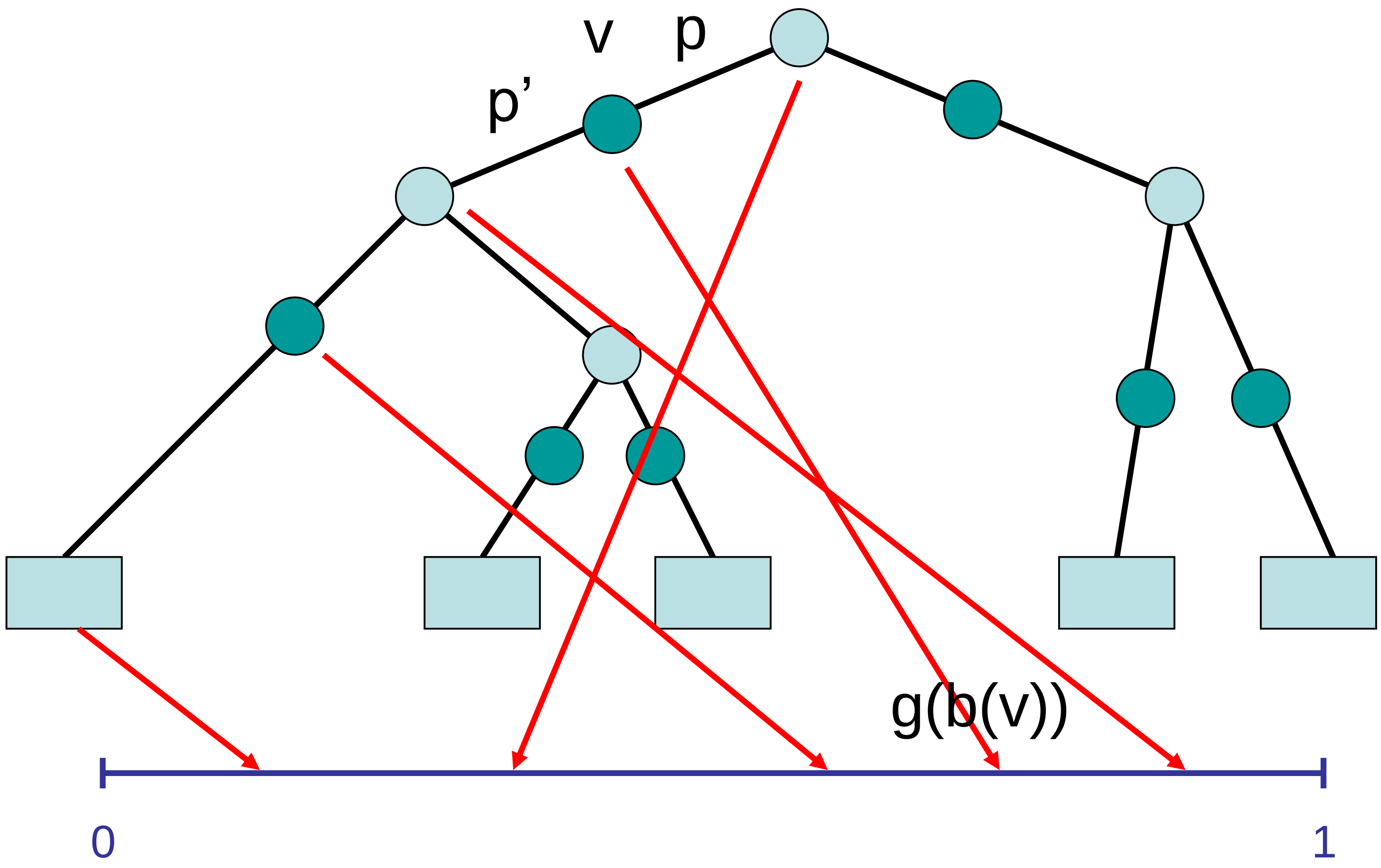
Consistent Hashing

- V: current set of servers
- succ(v): closest successor of v in V w.r.t. hash function h (where [0,1) is viewed as a cycle)
- pred(v): closest predecessor of v in V w.r.t. h

Assignment rules:

- One copy per data item: server v stores all items d with $g(d) \in I(v)$, where $I(v)=[h(v), h(succ(v)))$.
- $k>1$ copies per data item: d is stored in the above server v and its $k-1$ closest successors w.r.t. h

Distributed Patricia Trie Hashing



Distributed Patricia Trie Hashing

Number of DHT accesses for longest prefix search:

- O(log W), where W is key length

Number of DHT accesses for insert:

- O(log W) for lookups
- O(1) for updates

Number of DHT accesses for delete:

- O(1) for lookups
- O(1) for updates