Generalised Kripke semantics for various substructural logics

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Acknowledgements

During my master in mathematics, I became interested in linguistics and I took two courses that provided a bridge between these areas: *Meaning, reference and modality* and *Sémantique computationnelle*. Fortunately, the mathematical content of these courses fitted properly within the specialisation that I had already chosen for my master: Algebra and Logic. Mai Gehrke came up with a research subject for my master thesis that fulfilled my interests in both mathematics and linguistics, namely to find relational semantics for the Lambek-Grishin calculus, which is a logic that has applications in the field of linguistics. While working on this subject, we decided, together with Dion Coumans, to try to find relational semantics for linear logic as well.

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Introduction

A logic consists of a set of axioms and reasoning rules that define a derivability relation. A class of mathematical structures provides semantics for a logic if the structures model the behaviour of the logic and characterise it completely. Knowledge about this class of structures may be used to obtain insight in the logic.

The algebraic semantics for a logic is obtained by associating a class of ordered algebras to a logic, such that the properties of the logic are related to the algebraic properties of the associated algebras. The derivability relation is identified with the order on the algebra, and the connectives from the logic are identified with the operations of the algebra.

Besides algebraic semantics, some logics are naturally associated with a class of relational structures. For example, relational semantics for modal logic is given by (traditional) Kripke frames that consist of a set of worlds and an accessibility relation. The algebraic semantics for modal logic is given by modal algebras. The Kripke semantics arises from the representation theory for modal algebras in the following way. Every modal algebra embeds into a modal powerset algebra, and from this a Kripke frame can be built. For a Kripke frame, its corresponding algebraic structure is a modal powerset algebra based on the set of worlds. Since modal powerset algebras are special modal algebras, but are general enough that every modal algebra can be embedded in one by taking its canonical extension, the completeness of the algebraic semantics implies the completeness of the semantics based on Kripke frames. Kripke semantics often provides modular semantics allowing a clear treatment of various modal logics related by the addition of axioms or connectives.

Kripke semantics have also been applied for intuitionistic logic, for which the algebraic semantics is given by Heyting algebras. In the setting of intuitionistic logic, the sets of worlds of the Kripke frames are enriched with an order. The corresponding algebraic structures are up-set lattices. Similar to the case of the powerset modal algebras described above, the up-set lattices are special Heyting algebras, but form a class rich enough to allow the embedding of arbitrary Heyting algebras.

A logic is called substructural if it lacks one of the so-called structural rules of commutativity, contraction or weakening. In the algebraic semantics of substructural logics, there may be no lattice operations or very weak ones that are not necessarily distributive. Based on the representation theory of such posets and lattices, the papers [DGP05, Geh06] suggest a natural generalisation of Kripke semantics to an analogous kind of semantics (generalised Kripke semantics) for a broader setting including that of substructural logics. This generalisation allows us to build models on a more general algebraic structure, viz. on a kind of lattice generalising powersets and up-set lattices.

The generalised Kripke semantics of [Geh06] provide the possibility of a modular treatment of various substructural logics obtained by axiomatic extension or enrichment of the connective type. This treatment is based on canonicity and correspondence as in the classical modal logic setting. A modular treatment for the basic hierarchy of substructural logics based just on the implication-fusion fragment was given in the paper [DGP05], where the generalised Kripke semantics were first introduced.

Since generalised Kripke semantics are the ones corresponding to the topological representation theory for lattices, they are closely related to the semantics in the book by J. Michael Dunn and Gary Hardegree [DH01] and in [BD09]. However, the approach we take is a closer parallel to the classical canonicity and correspondence treatment as in [BdRV02].

In this thesis we will consider, in Part II, a kind of substructural logic (the so-called Lambek-Grishin calculus) that was first mentioned in [Gri83] and discussed in more detail in [Moo09]. This calculus extends the basic implication-fusion fragment (the non-associative Lambek calculus) in the following way: together with the fusion residuated family of connectives, there is a second residuated family. In [Gri83], Grishin identified four groups of additional axioms, of which two groups stipulate interactions between the two families of connectives. Traditional Kripke semantics for the Lambek-Grishin calculus was discussed in [Che07] and [KM10]. However, as soon as additional axioms or additional connectives are present, one must start over to obtain semantics for such richer logics. For example, in order to obtain semantics for the Lambek-Grishin calculus with interaction axioms in the presence of lattice operations, semantics based on phase spaces are announced in [Bas10]. This is hard to compare to the semantics given in [KM10].

Here, we consider semantics based on the generalised Kripke frames naturally associated with the algebraic semantics of the logics in question via their representation theory. The main advantage of this approach is that each axiom and each connective modularly slots in as an additional first order property or additional relational component. This allows a clear comparison of various logics and a fully modular family of completeness results. Accordingly, in addition to the basic Lambek-Grishin operations and axioms, we consider all four groups of additional axioms identified by Grishin, as well as associativity, commutativity, weakening and contraction, and additional connectives such as lattice operations and a negation. The presence of this negation tying together the two families of operations in the Lambek-Grishin calculus yields that all the possible interaction axioms are equivalent. In particular, the interaction axioms considered in [Moo09] are in this setting equivalent to associativity of both the product and the plus connective. A quick overview of our results on the family of Lambek-Grishin calculi may be found in Section 6.4.

Our approach to completeness for these various logics is parallel to the approach taken in [DGP05]. Thus, we prove canonicity and Sahlqvist-style correspondence, thereby providing complete relational semantics. Several of the equivalent versions of the interaction axioms in each group may be seen to be Sahlqvist in an appropriate generalised sense, and thus our derivations produce first-order conditions on the corresponding frames. Remarkably, each such frame condition turns out to be equivalent to a version of the initial axiom itself that is restricted to just the completely join resp. meet irreducible elements of the complex algebra.

Next to the Lambek-Grishin calculus and its extensions, we discuss in Part III another well-known substructural logic, called linear logic. Linear logic was introduced by Jean-Yves Girard in [Gir87] and has been identified as the most natural weakening of classical logic. An interpretation of this logic is that formulas stand for resources that may be used only once. This interpretation is made possible by the fact that the structural rules contraction and weakening do not hold in general. They are allowed only on formulas that are in the image of a special connective, called the exponential. Interpretation-wise, the exponential expresses the fact that a resource can be used endlessly.

To obtain complete relational semantics for linear logic, we use the method of canonicity and correspondence described above. We first derive in an algorithmic fashion complete relational semantics for the fragment of linear logic without the exponential (called MALL for multiplicative-additive linear logic). The modular set-up allows us to enrich the connective type with the exponential, and to augment the results for MALL into complete relational semantics for full linear logic. These results are given in Chapter 9. The frames that arise as the relational semantics for full linear logic are more complicated than the frames for the Lambek-Grishin calculus. The presence of the exponential requires some more work, as its axioms translate into second-order properties instead of first-order properties.

Traditionally, phase spaces are used as semantics for linear logic. These have the drawback that they do not allow a modular treatment of additional operations and axioms. The paper [CGvR11b] that is in preparation investigates the relation between our models and the traditional phase space models.

This thesis is divided into three parts. Part I describes the preliminaries needed to understand Parts II and III and is structured as follows: Chapter 1 deals with some basic algebraic notions. Those familiar with lattice theory can easily skip this chapter. Chapter 2 introduces the concepts of substructural logic and semantics. In Chapter 3, we describe the generalised Kripke semantics, and give in general the approach to completeness results via canonical extensions.

Part II reflects the results of a joint project with Anna Chernilovskaya and Mai Gehrke on developing relational semantics for the Lambek-Grishin calculus. Most of the results from Part II can also be found in the paper [CGvR11a] that materialised from this project. In Chapter 4, we introduce the Lambek-Grishin calculus with all of its relevant axiomatic extensions, and consider the influence of enriching the connective type with a linear logic-type negation. Also, we briefly look at the linguistic applications of this calculus. The theory of generalised Kripke semantics is applied in Chapter 5 to capture the basic Lambek-Grishin calculus. The approach to completeness results via canonical extensions is exploited in Chapter 6, where we give our results on complete relational semantics for the Lambek-Grishin calculus extended with different additional axioms that can be handled in a modular way.

Part III considers linear logic. This part contains the results of a joint project with Dion Coumans and Mai Gehrke. This project resulted in the paper [CGvR11b]. Chapter 7 introduces linear logic as a sequent calculus. In Chapter 8, we apply the theory of generalised Kripke semantics to obtain relational semantics for the multiplicative- additive fragment of linear logic. The content of this chapter originates from the paper [Cou11] that is written by Dion Coumans. In Chapter 9, the connective type of the multiplicative-additive fragment is enriched with the exponential. Complete relational semantics are obtained for full linear logic using the method of canonicity and correspondence. Finally, in Chapter 10, we look at the exponentials from the algebraic perspective.

Part I

Preliminaries

Chapter 1

Algebraic introduction

In this chapter we will briefly repeat the basic algebraic notions that we will use in the following chapters. Most importantly, we give the definition of a canonical extension in Section 1.1. A more detailed description of the notions introduced in this chapter can be found in [DP02] and [DGP05].

Definition 1. *Let P be a set. Then $(P, \leqslant)$ is a* ***partially ordered set*** *or* ***poset*** *if $\leqslant$ is a reflexive, transitive and anti-symmetric binary relation on P, i.e. for all $a, b, c \in P$,*

1. $a \leqslant a$;
2. $a \leqslant b$ and $b \leqslant c \;\Rightarrow\; a \leqslant c$;
3. $a \leqslant b$ and $b \leqslant a \;\Rightarrow\; a = b$.

We draw attention to a special notion, called a Galois connection, which we will use when defining the generalised Kripke semantics.

Definition 2. *Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be posets. A pair of maps (f, g), such that $f : P \to Q, g : Q \to P$, is called a* ***Galois connection*** *between P and Q if, for all $p \in P, q \in Q$:*

$$p \leqslant_P g(q) \;\Leftrightarrow\; q \leqslant_Q f(p)$$

Every poset gives rise to another poset, which is called its dual.

Definition 3. *Let $(P, \leqslant)$ be a poset. The* ***dual*** *of P, denoted by P^{dual}, is a poset that has P as underlying set, for which the order is the reversed order from $(P, \leqslant)$, i.e. for all $a, b \in P$,*

$$a \leqslant b \text{ in } P^{\text{dual}} \;\Leftrightarrow\; b \leqslant a \text{ in } P.$$

Every statement about a poset $(P, \leqslant)$ corresponds to a dual statement that is obtained by replacing every occurrence of the order of P by the order of P^{dual}. We will see an example of this in the following definition.

Definition 4. *Let $(P, \leqslant)$ be a poset and $S \subseteq P$. Then $a \in P$ is the* ***least upper bound*** *(also called* ***join*** *or* ***supremum****) of S, provided*

1. *for all* $s \in S$, $s \leqslant a$*;*
2. *if* $b \in P$ *is such that for all* $s \in S$, $s \leqslant b$*, then* $a \leqslant b$*.*

The element a *is denoted by* $\bigvee S$*. The* ***greatest lower bound*** *(or* ***meet*** *or* ***infimum****) of* S *in* P*, denoted by* $\bigwedge S$*, is defined dually as the least upper bound of* S *in* P^{dual}*.*

Posets that have the property that the join and meet of every finite subset exist are called **lattices**. If the join and meet of every subset exist, the lattice is called **complete**.

Example 5. *Let* X *be a set. The powerset* $\mathcal{P}(X)$ *consists of all subsets of* X *and is ordered by set inclusion. This is a poset, for which the join-operation is union and the meet-operation is intersection. Clearly,* $(\mathcal{P}(X), \subseteq)$ *is a complete lattice.*

1.1 Canonical extensions

The algebraic theory provides a means to embed an arbitrary poset into a complete lattice, called the canonical extension, which is unique with respect to a few simple abstract properties.

In order to be able to give the abstract definition of the canonical extension of a poset, we first make the following definitions:

Definition 6. *Let* P *be a poset.*

1. *A* ***filter*** *of* P *is a non-empty subset* F *of* P *satisfying:*
   (a) F *is an up-set, that is, if* $x \in F$, $y \in P$*, and* $x \leqslant y$*, then* $y \in F$*;*
   (b) F *is down-directed, that is, if* $x, y \in F$*, then there exists* $z \in F$ *with* $z \leqslant x$ *and* $z \leqslant y$*.*

   An ideal of P *is defined dually. That is,* I *is an* ***ideal*** *of* P *provided* I *is a non-empty up-directed down-set of* P*.*
2. *An* ***extension*** *of* P *is an order embedding* $e : P \to Q$*, i.e., for every* $x, y \in P$*,* $x \leqslant y$ *if and only if* $e(x) \leqslant e(y)$*. For ease of notation we will suppress* e *and call* Q *an extension of* P *and assume that* P *is a subposet of* Q*.*
3. *Given an extension* Q *of* P*, an element of* Q *is called a* ***filter element*** *provided it is the infimum in* Q *of some filter* F *of* P*. We denote the set of all filter elements of* Q *by* $F(Q)$*. Dually, an element of* Q *is called an* ***ideal element*** *provided it is the supremum in* Q *of some ideal* I *of* P*. We denote the set of all ideal elements of* Q *by* $I(Q)$*. Note that filter, respectively ideal, elements are also known under the names closed, respectively open, elements.*
4. *An extension* Q *of* P *is said to be* ***dense*** *provided each element of* Q *is both the supremum of all the filter elements below it and the infimum of all the ideal elements above it.*
5. *An extension* Q *of* P *is said to be* ***compact*** *provided* $\bigwedge F \leqslant \bigvee I$ *in* Q *implies* $F \cap I \neq \emptyset$ *for* F *a filter of* P *and* I *an ideal of* P*.*

6. *A **completion** of a poset P is an extension Q of P which also happens to be a complete lattice.*

We are now ready to give the abstract definition of the canonical extension of a poset.

Definition 7. *Let P be a poset. A **canonical extension** of P is a dense and compact completion of P.*

In [DGP05], it is shown that every poset P has a canonical extension that is unique up to an isomorphism fixing P. We will denote this extension by P^δ. Earlier, the notation P^σ was used instead of P^δ, but we will use the new notation as introduced in [GP08].

A canonical extension is a special kind of lattice - not only is it complete, but it is also generated by a certain subset of its elements. To have all the information about the canonical extension, it thus suffices to remember this subset of elements. To be able to state this property more formally, we first define the following notions:

Definition 8. *An element a of a complete lattice is called **completely join irreducible** if $a = \bigvee S$ implies that $a \in S$, for every subset S of the lattice. For a **completely meet irreducible** element the order dual statement holds: $a = \bigwedge S$ implies that $a \in S$, for every subset S of the lattice.*

Using the axiom of choice one can prove that the canonical extension P^δ of an arbitrary poset P is a **perfect lattice**, that is, it is a complete lattice such that for all $x \in P^\delta$:

$$x = \bigvee \Big\{ j \in \mathcal{J}^\infty(P^\delta) : j \leqslant x \Big\} \text{ and } x = \bigwedge \Big\{ m \in \mathcal{M}^\infty(P^\delta) : x \leqslant m \Big\},$$

where $\mathcal{J}^\infty(P^\delta)$, respectively $\mathcal{M}^\infty(P^\delta)$, denotes the set of completely join, respectively meet, irreducible elements of P^δ.

Example 9. *In the complete lattice $(\mathcal{P}(X), \subseteq)$, the completely join irreducible elements are the singletons $\{a\}$, for $a \in X$. The completely meet irreducible elements are the co-singletons $X - \{a\}$, for $a \in X$. Every subset of X is both the union of all the singletons that it contains and the intersection of all co-singletons that it is contained in, hence $(\mathcal{P}(X), \subseteq)$ is a perfect lattice.*

When a poset comes with additional operations, these have to be extended to the canonical extension as well. First, we look at posets that come with an additional order-preserving unary map f. There are two different ways of extending f to the canonical extension, namely:

Definition 10. *Let P and Q be posets, and $f : P \to Q$ an order-preserving map. Define maps $f^\sigma, f^\pi : P^\delta \to Q^\delta$ by setting:*

$$\begin{aligned} f^\sigma(u) &= \bigvee \Big\{ \bigwedge \{ f(p) : x \leqslant p \in P \} : u \geqslant x \in F(P^\delta) \Big\}, \\ f^\pi(u) &= \bigwedge \Big\{ \bigvee \{ f(p) : y \geqslant p \in P \} : u \leqslant y \in I(P^\delta) \Big\}. \end{aligned}$$

It is clear that for $u \in P^\delta$, $x \in F(P^\delta)$, and $y \in I(P^\delta)$, this gives

$$\begin{aligned} f^\sigma(u) &= \bigvee\{f^\sigma(x) : u \geqslant x \in F(P^\delta)\}, \\ f^\sigma(x) &= \bigwedge\{f(p) : x \leqslant p \in P\}, \\ f^\pi(u) &= \bigwedge\{f^\pi(y) : u \leqslant y \in I(P^\delta)\}, \\ f^\pi(y) &= \bigvee\{f(p) : y \geqslant p \in P\}. \end{aligned}$$

Furthermore, it can be shown that f^σ and f^π are order-preserving extensions of f, that send filter elements to filter elements and ideal elements to ideal elements.

Canonical extension commutes with taking the order dual version of a poset and with taking a finite product of posets, that is $(P^{\mathrm{dual}})^\delta = (P^\delta)^{\mathrm{dual}}$ and $(P^n)^\delta = (P^\delta)^n$. This enables us to use the previous definition to find the σ- and π-extension of more complicated maps. If f is order-reversing, it can be seen as an order-preserving map from the order dual version of its domain to its codomain, to which the definition can be applied. Similarly, in order to deal with operations of arity greater than 1, that are in each coordinate either order-preserving or order-reversing, we regard these as maps of which the domain is a finite product of P's and order dual versions of P.

When extending a particular operation, one has to choose between the σ- and the π-extension, depending on the properties (eg. join-preserving, meet-preserving) of the particular operation.

Chapter 2

Substructural logic

A logic consists of a collection of axioms and reasoning rules. These axioms and rules define a derivability relation on a set of formulas that is built inductively from a set of variables and a set of connectives. This derivability relation is a quasi-order and is denoted by $\vdash$.

The syntax of a logic deals with the concept of derivability, whereas the semantics provides a class of mathematical structures that model the behaviour of the logic.

In 1935, Gentzen introduced a formalism to reflect logical reasoning. He came up with three structural rules that were needed for the formalism, but according to him did not have any logical content. The rules are as follows:

$$\begin{array}{rc}
\text{Commutativity} & \Gamma, A, B, \Delta \vdash C \;\Rightarrow\; \Gamma, B, A, \Delta \vdash C \\
\text{Weakening} & \Gamma, \Delta \vdash B \;\Rightarrow\; \Gamma, A, \Delta \vdash B \\
\text{Contraction} & \Gamma, A, A, \Delta \vdash B \;\Rightarrow\; \Gamma, A, \Delta \vdash B,
\end{array}$$

where A, B and C are formulas, and Γ and Δ are finite sequences of formulas. All of these structural rules are valid in classical logic. In the second half of the 20th century, logicians realised that the structural rules do have logical content and that different logics arise when not all of the structural rules are valid. If one or more of the structural rules does not hold in a logic, it is called **substructural**.

For example, the Lambek calculus introduced in [Lam58] does not satisfy the rules of commutativity, weakening and contraction. Another example of a substructural logic is linear logic, introduced in [Gir87]. In this logic, which is commutative, premises stand for resources that cannot be reused. The contraction and weakening rules do not hold in general for this logic, but are allowed on certain special formulas.

2.1 Algebraic semantics and relational semantics

The semantics of a logic is given by a class of mathematical structures that characterises the logic. A model of a logic is such a structure, together with a valuation mapping V

that assigns an element of the structure to each variable from the logic. The connectives of the logic and the derivability relation are also written in the language of the structure such that every sentence of the logic gives a statement about the model.

The mapping $\overline{V}$, defined inductively from V and the translation of the logical connectives, maps the complex formulas to elements of the structure. Since the comma should also be seen as a logical connective, the notation $\overline{V}(\Gamma)$ makes sense and denotes the element of the structure that is obtained by performing the appropriate operations to the valuations of the variables involved. The translation of the derivability relation to the structure is denoted by $\vDash$.

Overloading the symbol $\vDash$, we say that $\Gamma \vDash A$ is valid over a class of structures if, in every model in the class (i.e. in each structure in the class, with every possible valuation mapping), the statement $\overline{V}(\Gamma) \vDash \overline{V}(A)$ holds. This relation $\vDash$ should model the behaviour of the derivability relation.

To ensure that a semantics is truly characterising a logic, we demand that it is **complete**, i.e. that for every finite sequence of formulas Γ and for every formula A,

$$\Gamma \vdash A \;\Leftrightarrow\; \Gamma \vDash A.$$

The direction from left to right is called the **soundness** of the semantics, while the converse direction by itself is also called the **completeness**.

We will now make the notions described above more concrete, by discussing two types of semantics: algebraic semantics and relational semantics.

The algebraic semantics for a logic are obtained by associating a class of ordered algebras to a logic, such that the properties of the logic are related to the algebraic properties of the associated algebras. The derivability relation is identified with the order on the algebra, and the connectives from the logic are identified with the operations of the algebra. An algebra is made into a model by adding a valuation mapping V that sends the variables from the logic to elements of the algebra.

Let $\mathcal{L}$ be a logic that is axiomatised by a collection $\mathcal{E}$ of axioms. These axioms are translated to inequalities that are statements about all elements of an algebra. The ordered algebras satisfying all of these inequalities provide complete algebraic semantics for $\mathcal{L}$, in the sense that for every formula A, for every finite sequence of formulas Γ,

$$\begin{gathered}\Gamma \vdash A \text{ holds in the logic}\\ \Leftrightarrow\\ \overline{V}(\Gamma) \leqslant \overline{V}(A) \text{ holds in all models built on algebras from the associated class.}\end{gathered}$$

For example, the class of Boolean algebras provides algebraic semantics for classical logic, and the class of Heyting algebras fulfills this task for intuitionistic logic.

Besides algebraic semantics, some logics are naturally associated with a class of relational structures. For example, the relational semantics for modal logic is given by (traditional) Kripke frames. These are structures consisting of a set $\mathcal{W}$ (whose elements are called possible worlds) and a binary relation R that is called the accessibility relation. A Kripke model M is obtained by adding to the frame a valuation function V that assigns a truth value $V_w(p)$ to every variable p for each world $w \in \mathcal{W}$. From this, a mapping $\overline{V}_w(A)$ can be defined, which assigns truth values to all formulas A of the logic. For example, it is defined that $\overline{V}_w(\Box p) = 1$ if and only if for all $v \in \mathcal{W}$ such that $(w, v) \in R$, it holds that $V_v(p) = 1$. We say that $A \vDash B$ is valid over a class of Kripke frames, if for every model in the class holds at each world $w \in \mathcal{W}$ that $\overline{V}_w(A) = 1$ implies $\overline{V}_w(B) = 1$.

For intuitionistic logic, relational semantics are based on frames that are equipped with an ordering on the set of worlds: the accessibility relation is a partial order. These are demanded to have the property that for each frame and each valuation function it holds that if a formula is true in a world, it is also true in all worlds that are above it. Models that satisfy this property are said to be hereditary.

Chapter 3

Generalised Kripke frames

In [DGP05] canonical extensions of partially ordered algebras are used to obtain complete relational semantics for the basic implication-fusion fragment that we will define in Section 3.2. This is done in a purely algebraic way. The paper [Geh06] gives a translation of this work to the setting of possible world semantics, by introducing the notion of generalised Kripke frames. In this chapter we will first introduce the theory of the generalised Kripke frames. In Section 3.3 we will describe the algebraic context of this theory and the approach that is taken in [DGP05].

3.1 Generalising the set of worlds

In traditional Kripke semantics, a frame is built on a non-empty set of worlds W which does not, generally speaking, have any particular structure. That is, an interpretant of a logical formula in a frame may be any subset of the set of worlds. However, in Kripke-style semantics for intuitionistic logic, frames are equipped with an order and interpretants must be hereditary sets. The idea of the generalised Kripke frames, as suggested in [Geh06], is similar, but is tailored to the substructural setting. What we need in this setting is that an interpretant is not only described by the worlds at which it holds, but also by the 'information quanta' contained in it (so-called **co-worlds**), moreover, either of these completely determine the interpretant.

Definition 11. *A* ***polarity*** *is a triple $F = (X, Y, \leqslant)$, where X and Y are non-empty sets and $\leqslant \subseteq X \times Y$ is a binary relation from X to Y. Here X is a* ***set of worlds****, Y a* ***set of co-worlds****, and a co-world, or information quantum, $y \in Y$ is a* ***part of the world*** *$x \in X$ provided $x \leqslant y$ holds.*

Such a polarity yields a Galois connection between the powersets of X and Y:

$$\begin{array}{rcrcl} (\_)^{\leqslant} & : & \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ & & A & \mapsto & \{y \in Y \mid \forall x \in X \ (x \in A \Rightarrow x \leqslant y)\} \end{array}$$

$$\begin{array}{rcrcl} (\_)^{\geqslant} & : & \mathcal{P}(Y) & \to & \mathcal{P}(X) \\ & & B & \mapsto & \{x \in X \mid \forall y \in Y \ (y \in B \Rightarrow y \geqslant x)\} \end{array}$$

Interpretants in generalised Kripke semantics are Galois-stable subsets of X:

$$\mathcal{G}(F) = \{A \subseteq X \mid A = (A^{\leqslant})^{\geqslant}\}$$

Define $\uparrow x = \{y \in Y \mid x \leqslant y\}$ and $\downarrow y = \{x \in X \mid x \leqslant y\}$.

Definition 12. *A polarity F is called an* ***S-frame*** *(a* ***separating frame****) provided that*

$$\forall x_1, x_2 \in X \quad \Big(\uparrow x_1 = \uparrow x_2 \;\Rightarrow\; x_1 = x_2\Big) \qquad \textit{and} \qquad \forall y_1, y_2 \in Y \quad \Big(\downarrow y_1 = \downarrow y_2 \;\Rightarrow\; y_1 = y_2\Big).$$

Informally, these conditions mean that $\uparrow x$ ($\downarrow y$, respectively) completely describes the world x (the co-world y, resp.). More formally, it means that the mappings $X \to \mathcal{G}(F)$ defined as $x \mapsto (\uparrow x)^{\geqslant}$ and $Y \to \mathcal{G}(F)$ defined as $y \mapsto \downarrow y$ are injective.[1] Accordingly, we will often write x instead of $(\uparrow x)^{\geqslant}$, and we will use $\leqslant$ instead of $\subseteq$ for the order in $\mathcal{G}(F)$. Thus for $A \in \mathcal{G}(F)$, $x \in X$, and $y \in Y$ we write $x \leqslant A$ instead of $(\uparrow x)^{\geqslant} \subseteq A$, and $A \leqslant y$ instead of $A \subseteq \downarrow y$. Note that for $x \in X$ and $y \in Y$ we have $x \leqslant y$, where $\leqslant$ is the relation in the frame F, if and only if $(\uparrow x)^{\geqslant} \subseteq \downarrow y$, so this causes no confusion.

Definition 13. *An S-frame F is called* ***reduced*** *(or an* ***RS-frame****) if the following two properties hold:*

$$\forall x \; \exists y \; \Big(x \not\leqslant y \wedge \forall x' \; (x' < x \Rightarrow x' \leqslant y)\Big) \qquad \textit{and} \qquad \forall y \; \exists x \; \Big(x \not\leqslant y \wedge \forall y' \; (y < y' \Rightarrow x \leqslant y')\Big).$$

These two conditions basically say that all worlds are completely join irreducible (i.e., it is not possible that $x = \bigvee\{x' \mid \uparrow x \subset \uparrow x'\}$) and all co-worlds are completely meet irreducible. This notion of an RS-frame gives a generalisation of the usual notions of Kripke structure where an unstructured set of worlds W corresponds to the RS-frame $(W, W, \neq)$ and an intuitionistic Kripke frame $(W, \leq)$ corresponds to the RS-frame $(W, W, \not\geq)$.

Finally, continuing this generalisation of Kripke structures, let us introduce the notion of a valuation function.

Definition 14. *Given an RS-frame F and a set of variables Var, a* ***valuation*** *is a mapping $V \colon Var \to \mathcal{G}(F)$.*

3.2 Models for substructural logics

The paper [Geh06] provides complete semantics for the implication-fusion fragment of various substructural logics. The basic logic under consideration is the non-associative Lambek calculus. This thesis discusses two substructural logics that are extensions of this basic implication-fusion fragment. In Part II we will extend this basic fragment to a family of Lambek-Grishin calculi, and in Part III we will extend it to linear logic.

In the non-associative Lambek calculus, the set of formulae $\mathcal{F}$ is built from the set of variables Var with a product connective (which will be denoted as $\otimes$ here) and its two residuals denoted as $/$ and $\backslash$. The rules of the system are as follows ($A, B, C \in \mathcal{F}$):

[1] As a consequence, X and Y may be understood as subsets of $\mathcal{G}(F)$.

- an axiom scheme: $A \vdash A$
- a transitivity rule: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$
- residuation rules: $A \vdash C/B \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A \setminus C$

In order to capture this logic, we need to add a ternary relation $R_\otimes \subseteq X \times X \times Y$ to model the behaviour of the connectives. Since interpretants are Galois-stable sets, the following compatibility conditions on $R_\otimes$ are imposed:

$$\begin{array}{lrcl} \forall x_1, x_2 \in X & (R_\otimes[x_1, x_2, \_]^{\geqslant})^{\leqslant} & = & R_\otimes[x_1, x_2, \_] \\ \forall x_1 \in X, y \in Y & (R_\otimes[x_1, \_, y]^{\leqslant})^{\geqslant} & = & R_\otimes[x_1, \_, y] \\ \forall x_2 \in X, y \in Y & (R_\otimes[\_, x_2, y]^{\leqslant})^{\geqslant} & = & R_\otimes[\_, x_2, y] \end{array} \tag{3.1}$$

Here $R[\,]$ denotes a relational image. For example,

$$R_\otimes[x_1, x_2, \_] = \{y \in Y \mid R_\otimes(x_1, x_2, y)\}.$$

Definition 15. *We call $(F, R_\otimes)$ a* ***Lambek frame****, provided $F = (X, Y, \leqslant)$ is an RS-frame and $R_\otimes$ is a compatible relation.*

Definition 16. *A* ***model*** *is a triple $\mathcal{M} = (F, R_\otimes, V)$ where $(F, R_\otimes)$ is a Lambek frame and $V \colon Var \to \mathcal{G}(F)$ is a valuation of variables.*

Given such a model, relations $\Vdash \subseteq X \times \mathcal{F}$ and $\succ \subseteq Y \times \mathcal{F}$ are defined inductively:

- for $p \in Var$: $x \Vdash p \;\Leftrightarrow\; x \leqslant V(p)$ and $y \succ p \;\Leftrightarrow\; y \geqslant V(p)$
- if $x \Vdash A$, $x \Vdash B$, $y \succ A$, $y \succ B$ are defined for all $x \in X$ and $y \in Y$, the relations for complex formulae are defined as follows:

$$\begin{array}{rcl} y \succ A \otimes B & \Leftrightarrow & \forall x_1, x_2 \in X \,\Big((x_1 \Vdash A \;\wedge\; x_2 \Vdash B) \Rightarrow R_\otimes(x_1, x_2, y)\Big) \\ x \Vdash A \otimes B & \Leftrightarrow & \forall y \in Y \;(y \succ A \otimes B \Rightarrow x \leqslant y) \\ \\ x \Vdash A \setminus B & \Leftrightarrow & \forall x' \in X,\; \forall y \in Y \,\Big((x' \Vdash A \;\wedge\; y \succ B) \Rightarrow R_\otimes(x', x, y)\Big) \\ y \succ A \setminus B & \Leftrightarrow & \forall x \in X \;(x \Vdash A \setminus B \Rightarrow x \leqslant y) \\ \\ x \Vdash B \,/\, A & \Leftrightarrow & \forall x' \in X,\; \forall y \in Y \,\Big((x' \Vdash A \;\wedge\; y \succ B) \Rightarrow R_\otimes(x, x', y)\Big) \\ y \succ B \,/\, A & \Leftrightarrow & \forall x \in X \;(x \Vdash B \,/\, A \Rightarrow x \leqslant y) \end{array} \tag{3.2}$$

Given a formula A, we define the interpretation of A in a model $\mathcal{M} = (F, R_\otimes, V)$ to be

$$[A]_\mathcal{M} = \{x \in X \mid x \Vdash A\}.$$

It is not difficult to check that $[A]_\mathcal{M}$ always is a Galois closed set of X. Now a sequent $A \vdash B$ is said to hold in $\mathcal{M}$ provided $[A]_\mathcal{M} \leq [B]_\mathcal{M}$ and $A \vdash B$ is said to be valid over a class of Lambek frames if and only if it holds in every model based on a frame from the class. The following theorem (proven in [Geh06]) states soundness and completeness of the non-associative Lambek calculus with respect to the class of models based on Lambek frames:

Theorem 17. *For all formulae A and B the sequent $A \vdash B$ is derivable in the non-associative Lambek calculus if and only if it is valid over the class of all Lambek frames.*

Or, equivalently, the non-associative Lambek calculus is complete with respect to the class of Lambek frames.

3.3 Algebraic background

We will now briefly sketch the algebraic context of the theory described in the previous sections and elaborate on the connection between the algebraic semantics for the non-associative Lambek calculus and the relational semantics described above. These results come from [DGP05] and [Geh06], in which a more detailed explanation can be found. In Section 5.2 we will get back to this subject and discuss the algebraic approach to completeness results much more thoroughly for the case of the family of Lambek-Grishin calculi. This algebraic approach will then reveal itself to be particularly well-suited to deal with different extensions of a logic in a modular fashion.

Algebraic semantics for substructural logics are given by posets with additional operations on them. The link between the algebraic semantics and the relational semantics will be via the canonical extension of these expanded posets. First we will see that Lambek frames are in one-to-one correspondence to certain expanded perfect lattices, counting among them the canonical extensions of the expanded posets.

For an RS-frame $(X, Y, \leqslant)$, the complete lattice of Galois-stable subsets of X is perfect. Conversely, we can identify each perfect lattice $\mathbb{L}$ with a frame $(\mathcal{J}^\infty(\mathbb{L}), \mathcal{M}^\infty(\mathbb{L}), \leqslant)$, where for all $x \in \mathcal{J}^\infty(\mathbb{L}), y \in \mathcal{M}^\infty(\mathbb{L})$,

$$x \leqslant y \quad \Leftrightarrow \quad x \leqslant_{\mathbb{L}} y.$$

It can be shown that this frame is an RS-frame and that these assignments yield a one-to-one correspondence between the RS-frames and perfect lattices. For those familiar with duality theory, this is in fact the discrete duality between (not-necessarily distributive) perfect lattices and RS-frames.

For the non-associative Lambek-calculus, its algebraic semantics is given by residuated algebras, i.e. algebras $(\mathbb{P}, \otimes, /, \backslash)$, such that $\mathbb{P} = (P, \leqslant)$ is a poset and the rules of the non-associative Lambek-calculus are satisfied: for all $A, B, C \in P$,

$$A \leqslant C/B \;\Leftrightarrow\; A \otimes B \leqslant C \;\Leftrightarrow\; B \leqslant A \backslash C.$$

Each perfect residuated algebra (i.e. each residuated algebra of which the underlying poset is a perfect lattice) can be identified with a Lambek frame, by adding to the associated RS-frame $(\mathcal{J}^\infty(\mathbb{P}), \mathcal{M}^\infty(\mathbb{P}), \leqslant)$, a relation $R_\otimes \subseteq \mathcal{J}^\infty(\mathbb{P}) \times \mathcal{J}^\infty(\mathbb{P}) \times \mathcal{M}^\infty(\mathbb{P})$ such that for all $x_1, x_2 \in \mathcal{J}^\infty(\mathbb{P}), y \in \mathcal{M}^\infty(\mathbb{P})$,

$$R_\otimes(x_1, x_2, y) \Leftrightarrow \; x_1 \otimes x_2 \leqslant y.$$

This is a compatible relation as described in Section 3.2.

On the other hand, a Lambek frame $F = (X, Y, \leqslant, R_\otimes)$ gives a fusion $\otimes_R$ on the corresponding perfect lattice $\mathcal{G}(F)$, by defining:

$$\begin{aligned}
x_1 \otimes_R x_2 &= \bigwedge\{y \in Y \ : \ R(x_1, x_2, y)\} && \text{for all } x_1, x_2 \in X, \\
u_1 \otimes_R u_2 &= \bigvee\{x_1 \otimes_R x_2 \ : \ x_1 \leq u_1, \ x_2 \leq u_2\} && \text{for all } u_1, u_2 \in \mathcal{G}(F).
\end{aligned}$$

This operation is completely join-preserving in both coordinates, thus it is residuated.

It turns out that for every operation $\otimes$ of a perfect residuated algebra it holds that $\otimes_{R_\otimes} = \otimes$ and for every compatible relation R on an RS-frame, $R_{\otimes_R} = R$. Hence, there is a one-to-one correspondence between perfect residuated algebras and the Lambek frames.

To describe the relation between algebraic and relational semantics, it thus suffices to relate the residuated algebras that arise as the algebraic semantics for the non-associative Lambek calculus to perfect residuated algebras. In Chapter 1 we saw that the canonical extension of each poset is a perfect lattice. To find the canonical extension of a residuated algebra $(\mathbb{P}, \otimes, /, \backslash)$, we first apply the construction of Definition 7 to the poset reduct. This gives a perfect lattice, which should be expanded by an additional operation for each additional operation of the original algebra, as explained in Chapter 1. In the paper [DGP05] it is shown in general how to extend residuated maps. Applying this to the operations $\otimes, /, \backslash$, gives that $(\mathbb{P}^\delta, \otimes^\sigma, /^\pi, \backslash^\pi)$ is a perfect residuated algebra.

The above can be summarised in the following diagram:

$$\begin{tikzcd}
\text{Non-associative Lambek calculus} \arrow[d, rightsquigarrow] & & \\
\begin{array}{c}\text{Residuated algebras}\\ \mathbb{A} = (A, \otimes, /, \backslash)\end{array} \arrow[d, rightsquigarrow] & & \\
\begin{array}{c}\text{Canonical extensions}\\ \mathbb{A}^\delta \cong (\mathcal{G}(X, Y, \leqslant), \otimes^\sigma, /^\pi, \backslash^\pi)\end{array} \arrow[rr, leftrightarrow] & & \begin{array}{c}\text{Lambek frames}\\ \mathbb{F}(\mathbb{A}) = (X, Y, \leqslant, R_{\otimes^\sigma})\end{array} \arrow[d, rightsquigarrow] \\
& & \text{Relational semantics}
\end{tikzcd}$$

We will now describe the general method for obtaining relational semantics for a substructural logic that extends the non-associative Lambek calculus. Let $\mathcal{E}$ be a collection of inequalities axiomatising a logic $\mathcal{L}_\mathcal{E}$ that extends the basic logic. Let $\mathcal{A}_\mathcal{E}$ be the collection of residuated algebras satisfying these inequalities. This collection provides complete algebraic semantics for $\mathcal{L}_\mathcal{E}$, as defined in Chapter 2. To find relational semantics for $\mathcal{L}_\mathcal{E}$, we need to identify the collection of Lambek frames that correspond to those perfect residuated algebras that are in $\mathcal{A}_\mathcal{E}$. This will give complete relational semantics for $\mathcal{L}_\mathcal{E}$ provided that the collection of inequalities that hold in all these corresponding structures is $\mathcal{L}_\mathcal{E}$. The following proposition gives a sufficient condition for this.

Proposition 18. *If $\mathcal{A}_{\mathcal{E}}$, the class of residuated algebras satisfying $\mathcal{E}$, is closed under canonical extension, i.e. if $\mathbb{P} \in \mathcal{A}_{\mathcal{E}}$ implies $\mathbb{P}^\delta \in \mathcal{A}_{\mathcal{E}}$, then the collection of inequalities that hold in all perfect residuated algebras in $\mathcal{A}_{\mathcal{E}}$ is $\mathcal{L}_{\mathcal{E}}$.*

Proof. We denote the collection of inequalities holding in a class of algebras by *Th*. By assumption, for each $\mathbb{P} \in \mathcal{A}_{\mathcal{E}}$, it holds that $\mathbb{P}^\delta \in \mathcal{A}_{\mathcal{E}}$, hence $Th(\mathcal{A}_{\mathcal{E}}) \subseteq Th(\{\mathbb{P}^\delta \mid \mathbb{P} \in \mathcal{A}_{\mathcal{E}}\})$.

Conversely, each $\mathbb{P} \in \mathcal{A}_{\mathcal{E}}$ embeds into its canonical extension $\mathbb{P}^\delta$. This implies that the inequalities holding in its canonical extension will in particular hold in the algebra itself. Thus, if an inequality holds in all these canonical extensions, then it holds in all the original algebras from $\mathcal{A}_{\mathcal{E}}$. Thus, $Th(\{\mathbb{P}^\delta \mid \mathbb{P} \in \mathcal{A}_{\mathcal{E}}\}) \subseteq Th(\mathcal{A}_{\mathcal{E}})$.

Note that

$$\{\mathbb{P}^\delta \mid \mathbb{P} \in \mathcal{A}_{\mathcal{E}}\} \subseteq \{\mathbb{P} \in \mathcal{A}_{\mathcal{E}} \mid \mathbb{P} \text{ is perfect}\} \subseteq \mathcal{A}_{\mathcal{E}}.$$

By the above, $Th(\mathcal{A}_{\mathcal{E}}) = Th(\{\mathbb{P}^\delta \mid \mathbb{P} \in \mathcal{A}_{\mathcal{E}}\})$, thus we get

$$Th(\mathcal{A}_{\mathcal{E}}) = Th(\{\mathbb{P}^\delta \mid \mathbb{P} \in \mathcal{A}_{\mathcal{E}}\}) \supseteq Th(\{\mathbb{P} \in \mathcal{A}_{\mathcal{E}} \mid \mathbb{P} \text{ is perfect}\}) \supseteq Th(\mathcal{A}_{\mathcal{E}}).$$

Thus, the collection of inequalities satisfied by all perfect residuated algebras in $\mathcal{A}_{\mathcal{E}}$ is $Th(\mathcal{A}_{\mathcal{E}}) = \mathcal{L}_{\mathcal{E}}$. □

The property of a collection of axioms described in Proposition 18 is called **canonicity**.

To identify this collection of Lambek frames that provides relational semantics, we need to give necessary and sufficient conditions on the frames to ensure that their corresponding structures are in $\mathcal{A}_{\mathcal{E}}$. This part of the work is called **correspondence**.

Part II

Lambek-Grishin Calculus

Chapter 4

Introduction to Lambek-Grishin calculus

The substructural logic that we will consider in this part is the Lambek-Grishin calculus ($LG_\emptyset$). This calculus was first mentioned in [Gri83]. It extends the basic implication-fusion fragment in the following way: together with the fusion residuated family of connectives, there is a second residuated family.

The formulae of $LG_\emptyset$ are built from variables taken from a set Var with six connectives: product $\otimes$ and its upper residuals / and \, and plus $\oplus$ and its lower residuals $\obslash$ and $\oslash$. The rules are as follows:

- an axiom scheme: $A \vdash A$
- a transitivity rule: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$
- residuation rules for the product family: $A \vdash C/B \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A \backslash C$
- residuation rules for the plus family: $B \obslash C \vdash A \;\Leftrightarrow\; C \vdash B \oplus A \;\Leftrightarrow\; C \oslash A \vdash B$

This logic satisfies none of the structural rules: it is non-commutative, non-associative, resource-sensitive and sensitive to relevance of the premises.

4.1 Axiomatic extensions of $LG_\emptyset$

The basic Lambek-Grishin calculus can be extended with additional axioms. Additional axioms involving only connectives from one family are well-known from the Lambek calculus. For example, adding to the non-associative Lambek calculus an axiom $A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C$ yields the associative Lambek calculus. A similar axiom can be added to stipulate associativity within the plus family. Other examples of axioms involving only one family are the structural rules commutativity, weakening and contraction for either of the two residuated families.

$$\begin{array}{ll}
\text{Commutativity} & A \otimes B \dashv\vdash B \otimes A \\
 & A \oplus B \dashv\vdash B \oplus A \\
\text{Weakening} & A \otimes B \vdash A \\
 & A \vdash A \oplus B \\
\text{Contraction} & A \vdash A \otimes A \\
 & A \oplus A \vdash A \\
\text{Associativity} & A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C \\
 & A \oplus (B \oplus C) \dashv\vdash (A \oplus B) \oplus C
\end{array}$$

In the setting of $LG_\emptyset$, it is most interesting to consider mixed axioms, i.e. those involving connectives from both residuated families. In [Gri83] Grishin studied and classified these. He found four groups of six axioms, and showed that because of the rules of the basic calculus $LG_\emptyset$, the axioms within each group are mutually interderivable on the basis of $LG_\emptyset$. From the logical perspective, Grishin's first and fourth group of axioms are most interesting, since these involve connectives from both families, whereas the second and third group state some kind of semi-associativity within the families.

Group I

$$\begin{array}{llll}
(a) & (B \oplus C) \otimes A \vdash B \oplus (C \otimes A) & (d) & (A \obslash B) \backslash C \vdash B \backslash (A \oplus C) \\
(b) & A \oplus (C / B) \vdash (A \oplus C) / B & (e) & (C \otimes B) \oslash A \vdash C \oslash (A / B) \\
(c) & A \obslash (C \otimes B) \vdash (A \obslash C) \otimes B & (f) & A \otimes (C \backslash B) \vdash (A \oslash C) \oplus B
\end{array} \tag{4.1}$$

Group II

$$\begin{array}{llll}
(a) & (B \backslash C) \otimes A \vdash B \backslash (C \otimes A) & (d) & (A \otimes B) \backslash C \vdash B \backslash (A \backslash C) \\
(b) & A \backslash (C / B) \vdash (A \backslash C) / B & (e) & (A / B) / C \vdash A / (C \otimes B) \\
(c) & A \otimes (C \otimes B) \vdash (A \otimes C) \otimes B & (f) & A \otimes (C \backslash B) \vdash (C / A) \backslash B
\end{array} \tag{4.2}$$

Group III

$$\begin{array}{llll}
(a) & A \obslash (C \oplus B) \vdash (A \obslash C) \oplus B & (d) & B \oslash (C \oplus A) \vdash (B \oslash A) \oslash C \\
(b) & (A \oplus C) \oplus B \vdash A \oplus (C \oplus B) & (e) & A \obslash (B \obslash C) \vdash (B \oplus A) \obslash C \\
(c) & (A \obslash C) \oslash B \vdash A \obslash (C \oslash B) & (f) & (C \oslash B) \obslash A \vdash B \oplus (C \obslash A)
\end{array} \tag{4.3}$$

Group IV

$$\begin{array}{llll}
(a) & (B \backslash C) \oslash A \vdash B \backslash (C \oslash A) & (d) & (A \backslash C) \obslash B \vdash C \obslash (A \otimes B) \\
(b) & B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A & (e) & (A \oplus B) / C \vdash A / (C \oslash B) \\
(c) & A \otimes (C \oslash B) \vdash (A \otimes C) \oslash B & (f) & A \oslash (B \obslash C) \vdash (C / A) \backslash B
\end{array} \tag{4.4}$$

4.2 Enrichment of the connective type

Next to the axiomatic extensions from the previous section, one can also enrich the connective type. Here we consider adding a linear negation in the fashion of [dGL02].

Given a poset $(P, \vdash)$ with two binary operations $\otimes$ and $\oplus$, we define a unary operation $(\_)^\perp$ that links the two binary ones, such that the following is satisfied for all $A, B, C \in P$:

$$\begin{aligned} A \vdash B \;\Leftrightarrow\; B^\perp \vdash A^\perp, \\ (A \otimes B)^\perp \dashv\vdash A^\perp \oplus B^\perp, \\ (A \oplus B)^\perp \dashv\vdash A^\perp \otimes B^\perp, \\ A \vdash C \oplus B^\perp \;\Leftrightarrow\; A \otimes B \vdash C \;\Leftrightarrow\; B \vdash A^\perp \oplus C, \\ B^\perp \otimes C \vdash A \;\Leftrightarrow\; C \vdash B \oplus A \;\Leftrightarrow\; C \otimes A^\perp \vdash B. \end{aligned} \tag{4.5}$$

If we use the following abbreviations

$$A \setminus C = A^\perp \oplus C,\; C \,/\, B = C \oplus B^\perp,\; B \oslash C = B^\perp \otimes C, \text{ and } C \oslash A = C \otimes A^\perp, \tag{4.6}$$

then, clearly, (4.5) implies that $(\otimes, /, \setminus)$ is a residuated family, and that $(\oplus, \obslash, \oslash)$ is a dually residuated family. Together these families satisfy the rules for the Lambek-Grishin calculus.

4.2.1 Grishin's interaction axioms in the presence of $(\_)^\perp$

We will now explore how the link between $\otimes$ and $\oplus$, through $(\_)^\perp$, influences the behaviour of Grishin's groups of additional axioms (4.1), (4.2), (4.3), and (4.4). Whereas the four groups *a priori* have no dependencies, we will see that the presence of $(\_)^\perp$ yields that all four groups are equivalent.

Proposition 19. *On a Lambek-Grishin algebra $\mathbb{P}$ with a unary operation $(\_)^\perp$ satisfying (4.5) and (4.6), Grishin's four groups of additional axioms are equivalent.*

Proof. Since the axioms within each group are interderivable, the equivalence of two groups follows from sharing one axiom. Note that, in writing the axioms in terms of $\otimes, \oplus$ and $(\_)^\perp$, we can simply leave out the operation $(\_)^\perp$ on individual elements. We are allowed to do so since an axiom is a statement about *all* A, B and C in P, and since there will be no occurrences of both an element and its negation in the same axiom.

First group $\Leftrightarrow$ second group

The third axiom of the first group is $A \obslash (B \otimes C) \vdash (A \obslash B) \otimes C$. Writing this in terms of the operations $\otimes, \oplus$ and $(\_)^\perp$ gives $A^\perp \otimes (B \otimes C) \vdash (A^\perp \otimes B) \otimes C$. As we may leave out the operation $(\_)^\perp$ on A, this yields the axiom $A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C$, which is axiom 3 of the second group.

Third group $\Leftrightarrow$ second group

Axiom 3 of group 3 is $(A \obslash B) \oslash C \vdash A \obslash (B \oslash C)$, which is $(A^\perp \otimes B) \otimes C^\perp \vdash A^\perp \otimes (B \otimes C^\perp)$. This universal statement is equivalent to $(A \otimes B^\perp) \otimes C \vdash A \otimes (B^\perp \otimes C)$. Using the first and then the second property of (4.5), this yields $A^\perp \oplus (B \oplus C^\perp) \vdash (A^\perp \oplus B) \oplus C^\perp$, which is equivalent to $A \setminus (B \,/\, C) \vdash (A \setminus B) \,/\, C$, the second axiom of group 2.

Fourth group $\Leftrightarrow$ first group

The second axiom of group 4 is $A \setminus (B \oplus C) \vdash (A \setminus B) \oplus C$, which is $A^\perp \oplus (B \oplus C) \vdash$

$(A^\perp \oplus B) \oplus C$. This universal statement is equivalent to $A \oplus (B \oplus C^\perp) \leqslant (A \oplus B) \oplus C^\perp$, which is $A \oplus (B \,/\, C) \vdash (A \oplus B) \,/\, C$, and this is axiom 2 of the first group.

Thus, in this particular situation, Grishin's groups of additional axioms are all equivalent. □

Proposition 20. *On a Lambek-Grishin algebra with a unary operation $(\_)^\perp$ satisfying (4.5) and (4.6), Grishin's groups of additional axioms hold if and only if the connective $\otimes$ is associative (and if and only if the connective $\oplus$ is associative).*

Proof. The first group (and thus each group) implies associativity of $\otimes$, since the third axiom of this group is

$$A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C,$$

and the reverse inequality is obtained using the fifth, the third and again the fifth axiom:

$$(A \otimes B) \otimes C \vdash A \otimes (C \otimes B) \vdash (A \otimes C) \otimes B \vdash A \otimes (B \otimes C).$$

On the other hand, if $\otimes$ is associative, then in particular axiom 3 of the first group holds (thus each group must hold). From the rules in (4.5), it is clear that $\otimes$ is associative if and only if $\oplus$ is associative. □

The above findings apply to the multiplicative fragment of linear logic that we will define in Chapter 7. The connectives $\otimes, \parr$ and $(\_)^\perp$ of linear logic satisfy the equations listed in (4.5), where $\parr$ plays the role of $\oplus$. On top of that, $\otimes$ and $\parr$ are associative operations, thus Propositions 19 and 20 show that this fragment of linear logic is an example of a logic for which all Grishin's groups of interaction axioms hold.

4.3 Application in linguistics

The Lambek-Grishin calculus has applications in the field of linguistics. In [Lam58] Lambek wanted to capture the notion of grammaticality for natural languages in an algorithmic fashion. He aimed to find an algorithm for determining whether a string of words gave a grammatically correct sentence.

In the framework he invented, grammatical classes (as nouns, verbs, and prepositions) are turned into formulas of a logic, and grammaticality is turned into deducibility in this logic. Clearly, this logic should be non-commutative and non-associative, since changes in word order or phrase structure can turn a grammatically correct string into a grammatically incorrect one and vice versa. Similarly, contraction and weakening rules should not hold in the logic, since grammaticality is affected by the number of times that a word occurs.

The papers [Moo07, Moo09] show that the Lambek-Grishin calculus is a candidate for this logic. We will now briefly sketch the outline of this application of the Lambek-Grishin calculus and give some examples of its use.

We start by defining types. The set of types is built from a set of atomic types such as s (sentence), np (noun phrase), n (common noun), pp (prepositional phrase), etc., and the operations $\otimes, /, \backslash, \oplus, \obslash$, and $\oslash$. To every word in the language, a finite number of types is assigned. Note that natural language is compositional with respect to types:

the grammaticality of a sentence is not changed when one of its words is substituted by another word with the same type.

A sentence from the language is found to be grammatically correct by the proposed algorithm, if there exists a derivation, using the $LG_\emptyset$ rules, showing that the expression in the corresponding types, concatenated by the operator $\otimes$, entails the type s (sentence).

More precisely,

$$\begin{aligned} \Sigma &:= \text{ set of words from a natural language} \\ T &:= \text{ the set of all types} \end{aligned}$$

$$\phi : \Sigma \to \mathcal{P}(T), \text{ such that } \forall w \in \Sigma \;\; \phi(w) \text{ is finite.}$$

For $w_1, \ldots, w_n \in \Sigma$, the string $w_1\ w_2\ \ldots\ w_n$ is found to be grammatically correct if there exist types $b_1 \in \phi(w_1),\ \ldots,\ b_n \in \phi(w_n)$ such that there exists a derivation in $LG_\emptyset$ of $b_1 \otimes (b_2 \otimes (b_3 \otimes \ldots \otimes b_n) \ldots)) \vdash s$.

In $LG_\emptyset$, the axiom $A \backslash B \vdash A \backslash B$ implies by residuation that $A \otimes (A \backslash B) \vdash B$ holds. In this context, this means that the operator $\backslash$ can be used to express incompleteness: type $A \backslash B$ needs type A on its left to become type B. Similarly, A / B needs type B on its right to become type A. The type $(np \backslash s) / np$, for example, will be assigned to a transitive verb: if the verb receives a noun phrase on its right as its object, and a noun phrase on its left as its subject, then the total will be a correct sentence.

Example 21. *Consider the sentence* Alice talks to John. *The words are assigned the following types:* Alice, John*:* np*,* talks*:* $(np \backslash s) / pp$*,* to*:* pp / np*. The following derivation shows that the sentence is grammatically correct.*

$$\begin{array}{ccccccl} & & pp / np & & \vdash & pp / np & \Rightarrow \\ & & (pp / np) \otimes & np & \vdash & pp & \Rightarrow \\ & ((np \backslash s) / pp) \otimes & ((pp / np) \otimes & np) & \vdash & ((np \backslash s) / pp) \otimes pp & \Rightarrow \\ & ((np \backslash s) / pp) \otimes & ((pp / np) \otimes & np) & \vdash & np \backslash s & \Rightarrow \\ np \otimes & (((np \backslash s) / pp) \otimes & ((pp / np) \otimes & np)) & \vdash & np \otimes (np \backslash s) & \Rightarrow \\ np \otimes & (((np \backslash s) / pp) \otimes & ((pp / np) \otimes & np)) & \vdash & s & \\ \text{Alice} & \text{talks} & \text{to} & \text{John} & & & \end{array}$$

Example 22. *Consider the sentence* Alice thinks someone left*. The words are assigned the following types:* Alice*:* np*,* thinks*:* $(np \backslash s) / s$*,* someone*:* $s / (np \backslash s)$*,* left*:* $np \backslash s$*. The following derivation shows that the sentence is grammatically correct.*

$$\begin{array}{ccccccl} & & s / (np \backslash s) & & \vdash & s / (np \backslash s) & \Rightarrow \\ & & (s / (np \backslash s)) \otimes & (np \backslash s) & \vdash & s & \Rightarrow \\ & ((np \backslash s) / s) \otimes & ((s / (np \backslash s)) \otimes & (np \backslash s)) & \vdash & ((np \backslash s) / s) \otimes s & \Rightarrow \\ & ((np \backslash s) / s) \otimes & ((s / (np \backslash s)) \otimes & (np \backslash s)) & \vdash & np \backslash s & \Rightarrow \\ np \otimes & (((np \backslash s) / s) \otimes & ((s / (np \backslash s)) \otimes & (np \backslash s))) & \vdash & np \otimes (np \backslash s) & \Rightarrow \\ np \otimes & (((np \backslash s) / s) \otimes & ((s / (np \backslash s)) \otimes & (np \backslash s))) & \vdash & s & \\ \text{Alice} & \text{thinks} & \text{someone} & \text{left} & & & \end{array}$$

In terms of semantics for natural languages, the last example has in fact two different readings:

1. Alice thinks that there exists someone, who has left. Or, informally,

$$\texttt{thinks}\Big(\text{Alice},\ \exists x.\ (\texttt{person}(x) \text{ and } \texttt{left}(x))\Big).$$

2. There exists someone, of whom Alice thinks that he has left. Or, informally,

$$\exists x.\ \Big(\texttt{person}(x) \text{ and } \texttt{thinks}(\text{Alice}, \texttt{left}(x))\Big).$$

In the first reading, Alice thinks that there is one person less than before, and thus thinks that someone (this could be anyone according to her knowledge) has left. This is usually called the *de dicto* reading. In the second reading, Alice thinks of a specific person that he has left. This is usually called the *de re* reading.

The difference lies in the scope of the existential quantifier. While *someone* in the first reading acts locally, its scope becomes global in the second reading. In $LG_\emptyset$, a type only has an effect at the place where it occurs. It is not possible to move a type around in order to let it play a role in other parts of the sentence.

To express this difference in semantics, and to be able to grasp the non-local effects of words, Moortgat uses in [Moo07, Moo09] an extension of $LG_\emptyset$ to test the deducibility. The basic calculus is extended by the interaction axioms from Grishin's fourth group. This gives the possibility to code words with non-local scope (such as *someone* in the second reading) by types in the Lambek-Grishin calculus that are allowed to behave non-locally in this extension. He shows that the type $(s \oslash s) \oslash np$ for *someone* reflects the second reading, while its type in the first reading is $s / (np \backslash s)$. It turns out that the axioms in Grishin's fourth group are able to describe the behaviour of these non-local scope elements, and are also able to capture certain other linguistic phenomena.

Note that for the algorithm to make sense, it should be decidable whether $A \vdash B$ is deducible in the logic. Lambek showed that this is the case for the non-associative Lambek-calculus, and in [Moo09] this result is given for $LG_\emptyset$ extended with Grishin's fourth group of axioms.

Chapter 5

Relational semantics with respect to Lambek-Grishin frames

In this chapter we illustrate the theory described in Chapter 3 on the Lambek-Grishin calculus. We start with its minimal version $LG_\emptyset$, in which no interaction axioms are added. We will give complete relational semantics for this logic. Also, we will shed some more light on the algebraic theory behind this result for the basic Lambek-Grishin calculus, thereby clarifying the path to completeness results for extensions of $LG_\emptyset$. These results on extensions will be given in the next chapter.

5.1 Generalised Kripke semantics for $LG_\emptyset$

In addition to the product family, the Lambek-Grishin calculus also has a plus family of connectives of which the behaviour needs to be modelled. In order to do this, we add to the frames for the Lambek calculus another ternary relation $R_\oplus \subseteq X \times Y \times Y$ satisfying the appropriate compatibility conditions:

$$\begin{array}{llcl} \forall y_1, y_2 \in Y & (R_\oplus[\_, y_1, y_2]^{\leqslant})^{\geqslant} & = & R_\oplus[\_, y_1, y_2] \\ \forall y_1 \in Y, x \in X & (R_\oplus[x, y_1, \_]^{\geqslant})^{\leqslant} & = & R_\oplus[x, y_1, \_] \\ \forall y_2 \in Y, x \in X & (R_\oplus[x, \_, y_2]^{\geqslant})^{\leqslant} & = & R_\oplus[x, \_, y_2] \end{array} \tag{5.1}$$

Definition 23. *We call $(F, R_\otimes, R_\oplus)$ a **Lambek-Grishin frame**, provided $(F, R_\otimes)$ is a Lambek frame and $R_\oplus$ is compatible in the sense given above.*

The truth conditions for the new connectives are defined as follows:

$$
\begin{array}{lcl}
x \Vdash A \oplus B & \Leftrightarrow & \forall y_1, y_2 \in Y \ \Big((y_1 \succ A \ \wedge \ y_2 \succ B) \Rightarrow R_\oplus(x, y_1, y_2) \Big) \\
y \succ A \oplus B & \Leftrightarrow & \forall x \in X \ (x \Vdash A \oplus B \Rightarrow x \leqslant y) \\
y \succ A \oslash B & \Leftrightarrow & \forall y' \in Y, \ \forall x \in X \ \Big((y' \succ A \ \wedge \ x \Vdash B) \Rightarrow R_\oplus(x, y', y) \Big) \\
x \Vdash A \oslash B & \Leftrightarrow & \forall y \in Y \ (y \succ A \oslash B \Rightarrow x \leqslant y) \\
y \succ B \oslash A & \Leftrightarrow & \forall y' \in Y, \ \forall x \in X \ \Big((y' \succ A \ \wedge \ x \Vdash B) \Rightarrow R_\oplus(x, y, y') \Big) \\
x \Vdash B \oslash A & \Leftrightarrow & \forall y \in Y \ (y \succ B \oslash A \Rightarrow x \leqslant y)
\end{array}
\tag{5.2}
$$

Again, $[A]_{\mathcal{M}}$ is defined accordingly and a sequent $A \vdash B$ holds in a Lambek-Grishin frame provided it holds in all models based on that frame. We have the following soundness and completeness theorem for the basic Lambek-Grishin calculus $LG_\emptyset$ with respect to the class of frames described above:

Theorem 24. *For all formulae A and B of the Lambek-Grishin calculus, the sequent $A \vdash B$ is derivable in $LG_\emptyset$ if and only if it is valid over the class of all Lambek-Grishin frames $(X, Y, \leqslant, R_\otimes, R_\oplus)$.*

This theorem is a direct consequence of the completeness theorem for the non-associative Lambek calculus given in Chapter 3, since each of the residuated families separately gives rise to an isomorphic or an order dual non-associative Lambek system, and no interaction between the two is stipulated. While this basic completeness theorem for $LG_\emptyset$ is a direct consequence of the work in [Geh06], it requires some work to develop the modular correspondence theory for the interaction axioms. As in Kripke semantics for modal logics, two components are needed: canonicity and correspondence. Our results on both of these components for the various axiomatic extensions mentioned in Section 4.1 are described in Chapter 6.

5.2 Algebraic background

In this section we will apply the general theory from Section 3.3 to the situation of the Lambek-Grishin calculi. We will repeat the most important notions and we will instantiate them for this particular situation. The approach to completeness results via canonical extensions of so-called LG-algebras that is behind the results given above, is described. This approach will be applied to the axiomatic extensions of the Lambek-Grishin calculus in Chapter 6.

5.2.1 LG-algebras and their canonical extensions

Definition 25. *We call $\mathbb{A} = (A, \otimes, /, \backslash, \oplus, \oslash, \oslash)$ an LG-**algebra** provided A is a poset, and the operations of $\mathbb{A}$ satisfy the rules of $LG_\emptyset$, i.e. $/$ and $\backslash$ are upper residuals of $\otimes$, while $\oslash$ and $\oslash$ are lower residuals of $\oplus$.*

To find the canonical extension of an LG-algebra $\mathbb{A}$, we first apply the construction of Definition 7 to the poset reduct P of $\mathbb{A}$. This poset extension can then be expanded by an additional operation for each additional operation of the original $\mathbb{A}$, as explained in Chapter 1. Again, we have to choose between the σ- and the π-extension for each additional operation, depending on the properties of the particular operation. We apply the general theory of extending residuated maps as discussed in [DGP05] to our operations $\otimes, /, \backslash, \oplus, \oslash, \oslash$. This gives that if we extend an LG-algebra $(A, \otimes, /, \backslash, \oplus, \oslash, \oslash)$ as $(A^\delta, \otimes^\sigma, /^\pi, \backslash^\pi, \oplus^\pi, \oslash^\sigma, \oslash^\sigma)$, then the basic Lambek-Grishin rules will hold again on this canonical extension.

Recall from Chapter 1 that the canonical extension P^δ of an arbitrary poset P is a perfect lattice, which enabled us in Section 3.3 to identify each canonical extension with a frame: because P^δ is a perfect lattice, it can be shown that $P^\delta \cong \mathcal{G}(\mathcal{J}^\infty(P^\sigma), \mathcal{M}^\infty(P^\delta), \leqslant)$ and that the frame $(\mathcal{J}^\infty(P^\delta), \mathcal{M}^\infty(P^\delta), \leqslant)$ is an RS-frame.

5.2.2 Completeness via canonicity

Our proof method for canonicity is different from the one most frequently used in modal logic. Typically, given a propositional logic L, a so-called canonical frame model is constructed using some notion of maximal consistent theories or related notions. This is also possible for the notion of frame used here and this approach to the completeness of the Lambek calculus is given in Section 4 of [Geh06]. By contrast, the completeness of the Lambek calculus and of several of its extensions is proved in the paper [DGP05] using a combination of algebraic canonicity results and correspondence results. We will use this latter approach in the next chapter, as it is particularly well-suited for dealing with additional laws such as the interaction axioms. For those familiar with standard canonicity and correspondence proofs in modal logic, the most important divergence from the standard procedure is that our correspondence results are just for frames and their complex algebras, and we do not use correspondence results when proving canonicity: we treat correspondence and canonicity separately. For canonicity, our proof method takes advantage of developments within the algebraic theory of canonical extensions of posets expanded with further operations and is a direct generalisation of the papers [DGP05] and [GNV05].

Starting from the Lindenbaum algebra of the logic, its canonical extension provided by the algebraic theory turns out to be precisely the algebra of Galois closed sets of the canonical frame as defined in Section 2 of [Geh06]. Thus we get a simple abstract manner of working with the canonical frame. This makes it easy to treat additional operations and their interaction axioms. In particular, from $\mathbb{A} = (A, \otimes, /, \backslash, \oplus, \oslash, \oslash)$ we get $\mathbb{A}^\delta = (A^\delta, \otimes^\sigma, /^\pi, \backslash^\pi, \oplus^\pi, \oslash^\sigma, \oslash^\sigma)$ and $\mathbb{A}^\delta$ is the algebra of Galois closed sets for some frame which we denote by $\mathbb{F}(\mathbb{A}) := (X, Y, \leqslant, R_{\otimes^\sigma}, R_{\oplus^\pi})$. We will often write $R_\otimes$ resp. $R_\oplus$ in stead of $R_{\otimes^\sigma}$ resp. $R_{\oplus^\pi}$, and we will sometimes leave out the superscripts $(\ )^\pi, (\ )^\sigma$ of the operations, when it is clear that we are working in the canonical extension of an LG-algebra.

The central role of the canonical extension in the process of finding relational semantics is illustrated by the following diagram.

$$
\begin{tikzcd}
\text{Lambek-Grishin logic} \arrow[d, squiggly] & \\
\begin{array}{c} LG\text{-algebras} \\ \mathbb{A} = (A, \otimes, /, \backslash, \oplus, \oslash, \oslash) \end{array} \arrow[d, squiggly] & \\
\begin{array}{c} \text{Canonical extensions} \\ \mathbb{A}^\delta \cong (\mathcal{G}(X, Y, \leqslant), \otimes^\sigma, /^\pi, \backslash^\pi, \oplus^\pi, \oslash^\sigma, \oslash^\sigma) \end{array} \arrow[r, leftrightarrow] & \begin{array}{c} \text{Lambek-Grishin frames} \\ \mathbb{F}(\mathbb{A}) = (X, Y, \leqslant, R_{\otimes^\sigma}, R_{\oplus^\pi}) \end{array} \arrow[d, squiggly] \\
& \text{Relational semantics}
\end{tikzcd}
$$

This process works for the Lindenbaum algebra $\mathbb{A}$ of the $LG_\emptyset$-logic in the sense that the canonical frame $\mathbb{F}(\mathbb{A})$ with the interpretation given by the embedding map is a model of precisely those sequents that are deducible in $LG_\emptyset$, as the following equivalences demonstrate. Here A and B are formulas that contain variables. Recall that a sequent holds in a structure iff each assignment of the variables to elements from the structure gives a valid statement about the structure.

$$
\begin{array}{clll}
 & A \vdash B & \text{is deducible in } LG_\emptyset \\
\Leftrightarrow & A \leqslant B & \text{holds in } \mathbb{A} \\
\overset{*}{\Leftrightarrow} & [A]_{\mathbb{A}^\delta} \leqslant [B]_{\mathbb{A}^\delta} & \text{holds in } \mathbb{A}^\delta \\
\Leftrightarrow & \mathbb{F}(\mathbb{A}) \Vdash A \vdash B, &
\end{array}
$$

where the subscript $\mathbb{A}^\delta$ means that the formula is interpreted in $\mathbb{A}^\delta$.

The process of getting a canonical extension and from a canonical extension a Lambek-Grishin frame works for any LG-algebra. When dealing with a class of algebras that satisfy additional interaction axioms (e.g. one of Grishin's groups), our aim is to find out which first-order condition is imposed by these axioms on the class of Lambek-Grishin frames.

In order to prove the completeness of an axiomatic extension of $LG_\emptyset$, two components are needed:

1. We have to show that if we start with $\mathbb{A}$, the Lindenbaum algebra for some extension of $LG_\emptyset$, then for each additional axiom, the equivalence indicated by $*$ still works. Recall from Section 3.3 that this equivalence is called the **canonicity** of the additional axiom. This canonicity implies completeness with respect to the class of frames satisfying the axioms. However this algebraic part of the work does not necessarily give us a good (preferably first-order) class of frames with respect to which the logic is complete.

2. The second part of our work then, typically referred to as **correspondence**, consists in showing that the axioms holding in a frame are equivalent to appropriate first-order properties of the frames.

The completeness result for the basic calculus $LG_{\emptyset}$, as stated in Theorem 24, can now be cast in this form. The canonicity result comes from the fact that the canonical extension $(A^{\delta}, \otimes^{\sigma}, /^{\pi}, \backslash^{\pi}, \oplus^{\pi}, \oslash^{\sigma}, \oslash^{\sigma})$ of an LG-algebra $(A, \otimes, /, \backslash, \oplus, \oslash, \oslash)$ is again an LG-algebra. The canonical extension of an LG-algebra is a perfect lattice that can be identified with a frame $(\mathcal{J}^{\infty}(A^{\delta}), \mathcal{M}^{\infty}(A^{\delta}), \leqslant)$, and the correspondence result is that the relations $R_{\otimes}$ and $R_{\oplus}$ are compatible relations on this RS-frame.

Chapter 6

Relational semantics for LG with various axiomatic extensions

In this chapter we consider the axiomatic extensions of $LG_\emptyset$ introduced in Section 4.1. We show that these axioms are canonical, and we show how the axioms give rise to corresponding first-order conditions that tell us how to adapt the class of Kripke frames. These first-order correspondents are such that each additional axiom holds on a canonical extension if and only if its correspondent holds on the Lambek-Grishin frame associated with that canonical extension. In this way, the class of Lambek-Grishin frames satisfying the appropriate first-order correspondents gives a complete relational semantics for the given extension of the Lambek-Grishin calculus.

This approach of proving completeness theorems algebraically is the same as in the paper [DGP05]. There, frame conditions for some axiomatic extensions of the pure implication-fusion fragment are derived by Sahlqvist correspondence methods, and it is proven that these conditions are canonical (i.e. hold in the canonical frame for each substructural logic considered). We stress here that this approach is parallel to the one for modal logics and that this was the first modular approach to these completeness results.

Sections 6.1 and 6.2 describe in detail the results for Grishin's first and fourth group. With the other additional axioms is dealt in Section 6.3. Section 6.4 summarises these results and provides an overview of the frame conditions imposed by the various extensions of $LG_\emptyset$.

6.1 Canonicity of the interaction axioms

Our first step in proving the completeness result for a group of interaction axioms is showing that if an interaction axiom holds in an LG-algebra, then it will hold in its canonical extension. In order to do this, the operations $\otimes, /, \backslash$ and $\oplus, \oslash, \oslash$ are extended to the canonical extension, as explained in Section 5.2. This will give again an LG-algebra.

To prove canonicity of the interaction axioms in (4.4), it suffices to prove that one of them lifts to the canonical extension. The validity of one axiom together with the interderivability of the axioms will yield canonicity of the other ones.

Proposition 26. *The interaction axioms from Grishin's fourth group satisfy canonicity.*

Proof. We will prove the canonicity of the interaction axiom (b) given in (4.4). This means that we will show that for an LG-algebra $\mathbb{A}$ in which for all $a, b, c \in A$ it holds that

$$b \backslash (c \oplus a) \leqslant (b \backslash c) \oplus a,$$

this axiom will also hold for the extended operations in the canonical extension $\mathbb{A}^\delta$. First, we show that the axiom holds on certain combinations of filter and ideal elements of A^δ, and we will use this to show that it holds on all elements of A^δ. Thus, we want to show that for $x_1 \in F(A^\delta)$ and $y_2,\ y_3 \in I(A^\delta)$ it holds that

$$x_1 \backslash^\pi (y_2 \oplus^\pi y_3) \leqslant (x_1 \backslash^\pi y_2) \oplus^\pi y_3.$$

For $y_2,\ y_3 \in I(A^\delta)$

$$y := y_2 \oplus^\pi y_3 = \bigvee \Big\{a_2 \oplus a_3 : a_i \leqslant y_i, a_i \in A\Big\}. \tag{6.1}$$

Note that $y \in I(A^\delta)$. Therefore, for $x_1 \in F(A^\delta)$ we have:

$$x_1 \backslash^\pi y = \bigvee \Big\{a_1 \backslash a : a_1 \geqslant x_1, a \leqslant y, a_1, a \in A\Big\}. \tag{6.2}$$

Combining (6.1) and (6.2) gives

$$x_1 \backslash^\pi (y_2 \oplus^\pi y_3) = \bigvee \Big\{a_1 \backslash a : a_1 \geqslant x_1, a \leqslant \bigvee\{a_2 \oplus a_3 : a_i \leqslant y_i, a_i \in A\}, a_1, a \in A\Big\}. \tag{6.3}$$

We show that the set $S := \{a_2 \oplus a_3 : a_i \leqslant y_i, a_i \in A\}$ is a directed set in A^δ. For $a_2 \oplus a_3,\ a_2' \oplus a_3' \in S$, we have that $(a_2 \vee a_2') \oplus (a_3 \vee a_3') \in S$, and since $\oplus$ is order-preserving in both coordinates,

$$(a_2 \vee a_2') \oplus (a_3 \vee a_3') \geqslant a_2 \oplus a_3 \quad \text{ and } \quad (a_2 \vee a_2') \oplus (a_3 \vee a_3') \geqslant a_2' \oplus a_3',$$

thus S is directed.

We now look at the condition for the a's in (6.3): $a \leqslant \bigvee S$. Since $a \in A$, it is a filter element. Using the compactness of the canonical extension, $a \leqslant \bigvee S$ implies that there is $a_2 \oplus a_3 \in S$ such that $a \leqslant a_2 \oplus a_3$. This means that there are $a_2, a_3 \in A$ with $a_2 \leqslant y_2$, $a_3 \leqslant y_3$ and $a \leqslant a_2 \oplus a_3$.

The map $\backslash$ is the upper residual of $\otimes$ and thus it is order-preserving in its second coordinate. So for all a satisfying the condition in (6.3) there exists $a_2 \oplus a_3 \in S$ such that $a_1 \backslash a \leqslant a_1 \backslash (a_2 \oplus a_3)$, implying

$$\bigvee \Big\{a_1 \backslash a : a_1 \geqslant x_1, a \leqslant \bigvee S, a_1, a \in A\Big\} \leqslant \bigvee \Big\{a_1 \backslash (a_2 \oplus a_3) : a_1 \geqslant x_1, a_2 \leqslant y_2, a_3 \leqslant y_3, a_i \in A\Big\}. \tag{6.4}$$

Thus, from (6.3) and (6.4):

$$x_1 \setminus^\pi (y_2 \oplus^\pi y_3) \leqslant \bigvee \Big\{a_1 \setminus (a_2 \oplus a_3) : a_1 \geqslant x_1, a_2 \leqslant y_2, a_3 \leqslant y_3, a_i \in A\Big\}.$$

Now we can use the assumption on the elements of A, which yields

$$x_1 \setminus^\pi (y_2 \oplus^\pi y_3) \leqslant \bigvee \Big\{(a_1 \setminus a_2) \oplus a_3 : a_1 \geqslant x_1, a_2 \leqslant y_2, a_3 \leqslant y_3, a_i \in A\Big\}.$$

Note that $\oplus$ is order-preserving in its first coordinate and that

$$x_1 \setminus^\pi y_2 = \bigvee \Big\{a_1 \setminus a_2 : a_1 \geqslant x_1, a_2 \leqslant y_2, a_1, a_2 \in A\Big\},$$

so $a_1 \setminus a_2 \leqslant x_1 \setminus^\pi y_2$ for all a_1, a_2 in this join. Therefore

$$\begin{aligned} x_1 \setminus^\pi (y_2 \oplus^\pi y_3) &\leqslant \bigvee \Big\{(a_1 \setminus a_2) \oplus a_3 : a_1 \geqslant x_1, a_2 \leqslant y_2, a_3 \leqslant y_3, a_i \in A\Big\} \\ &\leqslant \bigvee \Big\{(x_1 \setminus^\pi y_2) \oplus^\pi a_3 : a_3 \leqslant y_3, a_3 \in A\Big\}. \end{aligned}$$

And similarly,

$$x_1 \setminus^\pi (y_2 \oplus^\pi y_3) \leqslant (x_1 \setminus^\pi y_2) \oplus^\pi y_3.$$

Thus, we have proved that axiom (b) from (4.4) holds for all combinations of an $x_1 \in F(A^\delta)$ in the first spot and $y_2,\ y_3 \in I(A^\delta)$ in the second and third spot. Using this result for filter and ideal elements, we will prove that the inequality on the elements of A implies the inequality for all elements of the canonical extension A^δ. Let $u_1, u_2, u_3 \in A^\delta$:

$$u_1 \setminus^\pi (u_2 \oplus^\pi u_3) = u_1 \setminus^\pi \Big(\bigwedge \Big\{y_2 \oplus^\pi y_3 : u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta)\Big\}\Big)$$

Since $\setminus^\pi$ is meet-preserving in the second coordinate, and by definition of $\setminus^\pi$, since $y_2 \oplus^\pi y_3 \in I(A^\delta)$, we obtain:

$$\begin{aligned} u_1 \setminus^\pi (u_2 \oplus^\pi u_3) &= \bigwedge \Big\{u_1 \setminus^\pi (y_2 \oplus^\pi y_3) : u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta)\Big\} \\ &= \bigwedge \Big\{x_1 \setminus^\pi (y_2 \oplus^\pi y_3) : u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta)\Big\}. \end{aligned}$$

Using the inequality for filter and ideal elements, the definition of $\oplus^\pi$, and then the fact that it is meet-preserving in its first coordinate, we get:

$$\begin{aligned} u_1 \setminus^\pi (u_2 \oplus^\pi u_3) &\leqslant \bigwedge \Big\{(x_1 \setminus^\pi y_2) \oplus^\pi y_3 : u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, u_3 \leqslant y_3, y_i \in I(A^\delta)\Big\} \\ &= \bigwedge \Big\{(x_1 \setminus^\pi y_2) \oplus^\pi u_3 : u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, y_2 \in I(A^\delta)\Big\} \\ &= \Big(\bigwedge \Big\{x_1 \setminus^\pi y_2 : u_1 \geqslant x_1, x_1 \in F(A^\delta), u_2 \leqslant y_2, y_2 \in I(A^\delta)\Big\}\Big) \oplus^\pi u_3 \\ &= (u_1 \setminus^\pi u_2) \oplus^\pi u_3, \end{aligned}$$

where the last step is obtained by definition of $\setminus^\pi$. Thus we have proved that for all $u_1, u_2, u_3 \in A^\delta$ holds that

$$u_1 \setminus^\pi (u_2 \oplus^\pi u_3) \leqslant (u_1 \setminus^\pi u_2) \oplus^\pi u_3.$$

□

The other three groups of Grishin's additional axioms, (4.1), (4.2) and (4.3), also satisfy canonicity. For the second and third group, this result follows from a slight adjustment of Proposition 6.7 in [DGP05]. This is discussed in Section 6.3, together with the results for the weakening, contraction, commutativity, and associativity axioms. We will now give the canonicity result for Grishin's first group, which is quite similar to the proof of Proposition 26.

Proposition 27. *The interaction axioms from Grishin's first group satisfy canonicity.*

Proof. We will prove the canonicity of the second interaction axiom from Grishin's first group. Thus, we will show that for an $LG_\emptyset$-algebra $\mathbb{A}$ in which for all $a, b, c \in A$ it holds that

$$a \oplus (c \,/\, b) \leqslant (a \oplus c) \,/\, b,$$

this axiom will also hold for the extended operations in the canonical extension $\mathbb{A}^\delta$. As before, we first show that for $y_1,\ y_2 \in I(A^\delta)$ and $x_3 \in F(A^\delta)$ holds that

$$y_1 \oplus^\pi (y_2 \,/^\pi\, x_3) \leqslant (y_1 \oplus^\pi y_2) \,/^\pi\, x_3.$$

For $y_2 \in I(A^\delta)$ and $x_3 \in F(A^\delta)$,

$$y := y_2 \,/^\pi\, x_3 = \bigvee \Big\{a_2 \,/\, a_3 : a_2 \leqslant y_2, a_3 \geqslant x_3, a_i \in A\Big\}.$$

We denote $\{a_2 \,/\, a_3 : a_2 \leqslant y_2, a_3 \geqslant x_3, a_i \in A\}$ by S. Note that $y \in I(A^\delta)$. Therefore, for $y_1 \in I(A^\delta)$ we have:

$$y_1 \oplus^\pi y = \bigvee \Big\{a_1 \oplus a : a_1 \leqslant y_1, a \leqslant \bigvee S, a_1, a \in A\Big\}. \tag{6.5}$$

We show that the set S is a directed set in A^δ. For $a_2 \,/\, a_3,\ a_2' \,/\, a_3' \in S$, we have to show that there exists $s \in S$ such that $a_2 \,/\, a_3 \leqslant s$ and $a_2' \,/\, a_3' \leqslant s$. Let s be $(a_2 \vee a_2') \,/ (a_3 \wedge a_3')$. Clearly, $s \in S$. Furthermore, we have $a_2 \leqslant a_2 \vee a_2',\ a_3 \geqslant a_3 \wedge a_3'$, and since $/$ is order-preserving in its first coordinate and order reversing in its second, this gives $a_2 \,/\, a_3 \leqslant (a_2 \vee a_2') \,/ (a_3 \wedge a_3')$. Similarly, $a_2' \,/\, a_3' \leqslant s$. Thus S is directed.

We now look at the condition for the a's in (6.5): $a \leqslant \bigvee S$. Since $a \in A$, it is a filter element. Using the compactness of the canonical extension, $a \leqslant \bigvee S$ implies that there are $a_2, a_3 \in A$ such that $a_2 \leqslant y_2$, $a_3 \leqslant y_3$ and $a \leqslant a_2 \,/\, a_3$.

This, together with the fact that $\oplus$ is order-preserving in its second coordinate, gives that

$$y_1 \oplus^\pi (y_2 \,/^\pi\, x_3) \leqslant \bigvee \Big\{a_1 \oplus (a_2 \,/\, a_3) : a_1 \leqslant y_1, a_2 \leqslant y_2, a_3 \geqslant x_3, a_i \in A\Big\}.$$

Using our assumption that the axiom holds on the elements of A yields

$$y_1 \oplus^\pi (y_2 \,/^\pi\, x_3) \leqslant \bigvee \Big\{(a_1 \oplus a_2) \,/\, a_3 : a_1 \leqslant y_1, a_2 \leqslant y_2, a_3 \geqslant x_3, a_i \in A\Big\}.$$

Note that

$$y_1 \oplus^\pi y_2 = \bigvee \Big\{a_1 \oplus a_2 : a_1 \leqslant y_1, a \leqslant y_2, a_i \in A\Big\}.$$

Thus for all $a_1, a_2 \in A$ with $a_1 \leqslant y_1, a_2 \leqslant y_2$ we have $a_1 \oplus a_2 \leqslant y_1 \oplus^\pi y_2$. Since $/^\pi$ is order-preserving in its first coordinate, this means

$$y_1 \oplus^\pi (y_2 /^\pi x_3) \leqslant \bigvee \Big\{(y_1 \oplus^\pi y_2) /^\pi a_3 : a_3 \geqslant x_3, a_3 \in A\Big\},$$

and similarly,

$$y_1 \oplus^\pi (y_2 /^\pi x_3) \leqslant (y_1 \oplus^\pi y_2) /^\pi x_3.$$

So for all $y_1, y_2 \in I(A^\delta)$ and $x_3 \in F(A^\delta)$ the axiom holds.

Using this result for filter and ideal elements, we will prove the inequality for all elements $u_1, u_2, u_3 \in A^\delta$.

By definition of $\oplus^\pi$, and then the fact that it is meet-preserving in its second coordinate, we get:

$$\begin{aligned} u_1 \oplus^\pi (u_2 /^\pi u_3) &\leqslant u_1 \oplus^\pi \bigwedge \Big\{(y_2 /^\pi y_3) : u_2 \leqslant y_2, u_3 \geqslant x_3, y_2 \in I(A^\delta), x_3 \in F(A^\delta)\Big\} \\ &= \bigwedge \Big\{u_1 \oplus^\pi (y_2 /^\pi x_3) : u_2 \leqslant y_2, u_3 \geqslant x_3, y_2 \in I(A^\delta), x_3 \in F(A^\delta)\Big\}. \end{aligned}$$

Since $y_2 /^\pi x_3 \in I(A^\delta)$, we have by definition of $\oplus^\pi$ that

$$u_1 \oplus^\pi (u_2 /^\pi u_3) \leqslant$$
$$\bigwedge \Big\{y_1 \oplus^\pi (y_2 /^\pi x_3) : u_1 \leqslant y_1, u_2 \leqslant y_2, u_3 \geqslant x_3, y_1, y_2 \in I(A^\delta), x_3 \in F(A^\delta)\Big\}.$$

Using the axiom for combinations of filter and ideal elements gives

$$u_1 \oplus^\pi (u_2 /^\pi u_3) \leqslant$$
$$\bigwedge \Big\{(y_1 \oplus^\pi y_2) /^\pi x_3 : u_1 \leqslant y_1, u_2 \leqslant y_2, u_3 \geqslant x_3, y_1, y_2 \in I(A^\delta), x_3 \in F(A^\delta)\Big\}.$$

Now by definition of $/^\pi$, this is equal to

$$\bigwedge \Big\{(y_1 \oplus^\pi y_2) /^\pi u_3 : u_1 \leqslant y_1, u_2 \leqslant y_2, y_1, y_2 \in I(A^\delta)\Big\}.$$

Using that $/^\pi$ is meet-preserving in its first coordinate, and then the definition of $\oplus^\pi$, we have

$$\begin{aligned} u_1 \oplus^\pi (u_2 /^\pi u_3) &\leqslant \Big(\bigwedge \Big\{(y_1 \oplus^\pi y_2) : u_1 \leqslant y_1, u_2 \leqslant y_2, y_1, y_2 \in I(A^\delta)\Big\}\Big) /^\pi u_3. \\ &= (u_1 \oplus^\pi u_2) /^\pi u_3. \end{aligned}$$

Thus, we have proved that for all $u_1, u_2, u_3 \in A^\delta$ holds that

$$u_1 \oplus^\pi (u_2 /^\pi u_3) \leqslant (u_1 \oplus^\pi u_2) /^\pi u_3.$$

□

6.2 Correspondence for the interaction axioms

The next step in the completeness results is to characterise the Lambek-Grishin frames that are associated to the canonical extensions of LG-algebras satisfying a specific additional axiom. Thus, we want to capture in terms of the associated dual structure what it means for a canonical extension to satisfy a given axiom. It will turn out to be a fruitful approach to try to write the axiom as a statement about elements of X and Y, the completely join resp. meet irreducibles of the canonical extension, and write this in relational terms to find a first-order statement on the Lambek-Grishin frames.

6.2.1 Frame conditions for interaction axioms

Let us illustrate how adding to the basic Lambek-Grishin calculus only one axiom of interaction between the $\otimes$ and $\oplus$ families influences the relations $R_\otimes$ and $R_\oplus$. For the group of axioms given above in (4.4), we will work on the form given as axiom (b): $B \setminus (C \oplus A) \vdash (B \setminus C) \oplus A$. The corresponding frame condition looks as follows:

$$\begin{aligned}&\forall x, x', x_2 \in X \ \forall y, y' \in Y \\ &\quad \Big[\Big(\forall x_1 \in X \ \Big[R_\otimes^{\downarrow}(x', x, x_1) \Rightarrow \ R_\oplus(x_1, y', y)\Big] \text{ and } R_\oslash^{\downarrow}(x, y, x_2)\Big) \Rightarrow \ R_\otimes(x', x_2, y')\Big]\end{aligned} \tag{6.6}$$

Here $R_\otimes^{\downarrow}$ is a ternary relation in $X \times X \times X$ which is defined as

$$(x_1, x_2, x) \in R_\otimes^{\downarrow} \ \Leftrightarrow \ \forall y \in Y \ \Big((x_1, x_2, y) \in R_\otimes \Rightarrow x \leqslant y\Big).$$

And $R_\oslash^{\downarrow}$ is a ternary relation in $X \times Y \times X$ defined as

$$(x, y, x_2) \in R_\oslash^{\downarrow} \ \Leftrightarrow \ \forall y_1 \in Y \ \Big(x \oslash y \leqslant y_1 \ \Rightarrow \ x_2 \leqslant y_1\Big).$$

Informally, $R_\otimes(x_1, x_2, y)$ reflects the fact that the information y is contained in the 'fusion of the worlds' x_1 and x_2, whereas $R_\otimes^{\downarrow}(x_1, x_2, x)$ means that the 'fusion of the worlds' x_1 and x_2 makes x possible. We note that the relation $R_\otimes^{\downarrow}$ is the one more commonly used in Kripke semantics for substructural logics.

In order to be able to give a proof of this correspondence result, we will work in the 'complex algebra' for these non-distributive structures. That is, we work in $\mathcal{G}(X, Y, \leqslant)$. Now the x's are completely join irreducible elements of $\mathcal{G}(X, Y, \leqslant)$ (atoms in the Boolean setting), the y's are completely meet irreducible elements of $\mathcal{G}(X, Y, \leqslant)$ (co-atoms), and A, B, C are general elements of $\mathcal{G}(X, Y, \leqslant)$. Also $\otimes, /, \setminus$ and $\oplus, \oslash, \oslash$ are operations on $\mathcal{G}(X, Y, \leqslant)$ specified by (3.2) and (5.2), where $A \vdash B$ means $A \leqslant B$, $x \Vdash A$ means $x \leqslant A$ and $y \succ A$ means $y \geqslant A$.

The content of the completeness theorem for the basic Lambek-Grishin calculus is, in algebraic terms, that on $\mathcal{G}(X, Y, \leqslant)$ the connectives $\otimes, /, \setminus$ and $\oplus, \oslash, \oslash$ satisfy the rules of $LG_\emptyset$.

Note that each canonical extension $\mathbb{A}^\delta$ of an LG-algebra $\mathbb{A}$ is in fact such a structure, since a canonical extension is a perfect lattice (i.e. a corresponding RS-frame can be found) and the extended operations coincide.

Proposition 28. *The axiom* $\forall A, B, C\ \Big(B \backslash (C \oplus A) \leqslant (B \backslash C) \oplus A\Big)$ *holds in an algebra* $(\mathcal{G}(X, Y, \leqslant), \otimes, /, \backslash, \oplus, \oslash, \varnothing)$ *if and only if the first-order condition (6.6) holds for the underlying frame* $(X, Y, \leqslant, R_{\otimes}, R_{\oplus})$*.*

Proof. First, we will first try to lose all second-order variables from the axiom, by rewriting it in such a way that it is purely in terms of completely join irreducibles x and completely meet irreducibles y. We are then ready to translate what it means for a complex algebra to satisfy the axiom into a first-order condition on the class of Lambek-Grishin frames.

Since the completely join irreducibles, $x \in X$, join-generate $\mathcal{G}(X, Y, \leqslant)$, the axiom holds in $\mathcal{G}(X, Y, \leqslant)$ if and only if

$$\forall x\ \forall A, B, C\ \Big(x \leqslant B \backslash (C \oplus A)\ \Rightarrow\ x \leqslant (B \backslash C) \oplus A\Big). \tag{6.7}$$

By residuation, $x \leqslant B \backslash (C \oplus A)$ is equivalent to $B \leqslant (C \oplus A) / x$, such that (6.7) holds if and only if

$$\forall x\ \forall A, B, C\ \Big(B \leqslant (C \oplus A) / x\ \Rightarrow\ x \leqslant (B \backslash C) \oplus A\Big). \tag{6.8}$$

Instantiating $B = (C \oplus A) / x$, this implies

$$\forall x\ \forall A, C\ \Big(x \leqslant \Big([(C \oplus A) / x] \backslash C\Big) \oplus A\Big). \tag{6.9}$$

On the other hand, we can get from (6.9) to (6.8) since for every B, we have for every x, A, C that

$$\begin{aligned} B \leqslant (C \oplus A) / x\ &\Rightarrow\ B \backslash C \geqslant [(C \oplus A) / x] \backslash C \\ &\Rightarrow\ (B \backslash C) \oplus A \geqslant \Big([(C \oplus A) / x] \backslash C\Big) \oplus A. \end{aligned}$$

Combining this with (6.9) yields the desired conclusion in (6.8): $x \leqslant (B \backslash C) \oplus A$, thus the axiom is equivalent to (6.9).

By residuation, the following equivalence holds:

$$x \leqslant \Big([(C \oplus A) / x] \backslash C\Big) \oplus A\ \Leftrightarrow\ \Big([(C \oplus A) / x] \backslash C\Big) \oslash x \leqslant A.$$

And since the $y \in Y$ meet-generate $\mathcal{G}(X, Y, \leqslant)$, (6.9) is equivalent to

$$\forall x, y\ \forall A, C\ \Big(y \geqslant A\ \Rightarrow\ y \geqslant \Big([(C \oplus A) / x] \backslash C\Big) \oslash x\Big). \tag{6.10}$$

Instantiating $A = y$, (6.10) implies

$$\forall x, y\ \forall C\ \Big(y \geqslant \Big([(C \oplus y) / x] \backslash C\Big) \oslash x\Big). \tag{6.11}$$

On the other hand, assuming (6.11), we have the following for every A and every x, y, C

$$\begin{aligned} y \geqslant A &\Rightarrow C \oplus A \leqslant C \oplus y \\ &\Rightarrow (C \oplus A) \,/\, x \leqslant (C \oplus y) \,/\, x \\ &\Rightarrow [(C \oplus A) \,/\, x] \setminus C \geqslant [(C \oplus y) \,/\, x] \setminus C \\ &\Rightarrow \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oslash x \leqslant \Big([(C \oplus y) \,/\, x] \setminus C\Big) \oslash x. \end{aligned}$$

Using $y \geqslant \Big([(C \oplus y) \,/\, x] \setminus C\Big) \oslash x$ from (6.11), this gives (6.10), so that the axiom is equivalent to (6.11). Note that we have now lost already one of the second-order variables.

By residuation, $y \geqslant \Big([(C \oplus y) \,/\, x] \setminus C\Big) \oslash x$ is equivalent to $x \oslash y \leqslant [(C \oplus y) \,/\, x] \setminus C$, which is equivalent to $[(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant C$. Thus (6.11) holds if and only if

$$\forall x, y \; \forall C \; \Big([(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant C\Big).$$

Equivalently, since the $y' \in Y$ meet generate $\mathcal{G}(X, Y, \leqslant)$,

$$\forall x, y, y' \; \forall C \; \Big(C \leqslant y' \;\Rightarrow\; [(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y'\Big). \tag{6.12}$$

Instantiating $C = y'$, (6.12) implies

$$\forall x, y, y' \; \Big([(y' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y'\Big). \tag{6.13}$$

On the other hand, assuming (6.13), we have that for all x, y, y', C:

$$\begin{aligned} C \leqslant y' &\Rightarrow C \oplus y \leqslant y' \oplus y \\ &\Rightarrow (C \oplus y) \,/\, x \leqslant (y' \oplus y) \,/\, x \\ &\Rightarrow [(C \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant [(y' \oplus y) \,/\, x] \otimes [x \oslash y]. \end{aligned}$$

Using $[(y' \oplus y) \,/\, x] \otimes [x \oslash y] \leqslant y'$ from (6.13), this gives (6.12), thus the axiom is equivalent to the first-order condition (6.13), which is by residuation equivalent to

$$\forall x, y, y' \; \Big((y' \oplus y) \,/\, x \leqslant y' \,/ (x \oslash y)\Big). \tag{6.14}$$

We reformulate this in order to obtain a first-order condition in terms of relations. Condition (6.14) holds if and only if

$$\forall x, x', y, y' \; \Big(x' \leqslant (y' \oplus y) \,/\, x \;\Rightarrow\; x' \leqslant y' \,/ (x \oslash y)\Big). \tag{6.15}$$

By residuation, the antecedent $x' \leqslant (y' \oplus y) \,/\, x$ is equivalent to $x' \otimes x \leqslant y' \oplus y$, which is equivalent to

$$\forall x_1 \Big[x_1 \leqslant x' \otimes x \;\Rightarrow\; x_1 \leqslant y' \oplus y\Big].$$

By definition, this is

$$\forall x_1 \Big[x_1 \leqslant x' \otimes x \;\Rightarrow\; R_{\oplus}(x_1, y', y)\Big],$$

and $x_1 \leqslant x' \otimes x$ is equivalent to $R^{\downarrow}_{\otimes}(x', x, x_1)$, since for all y such that $R_{\otimes}(x', x, y)$, it holds that $x_1 \leqslant x' \otimes x \leqslant y$. In total, (6.15) is equivalent to

$$\forall x, x', y, y' \; \Big(\forall x_1 \Big[R^{\downarrow}_{\otimes}(x', x, x_1) \Rightarrow \; R_{\oplus}(x_1, y', y)\Big] \Rightarrow \; x' \leqslant y' \,/\, (x \oslash y)\Big). \tag{6.16}$$

The conclusion $x' \leqslant y' \,/\, (x \oslash y)$ is equivalent to $x \oslash y \leqslant x' \setminus y'$, which is

$$\forall x_2 \; \Big[x_2 \leqslant x \oslash y \;\; \Rightarrow \;\; x_2 \leqslant x' \setminus y'\Big]. \tag{6.17}$$

The conclusion of (6.17), $x_2 \leqslant x' \setminus y'$, is equivalent to $x' \otimes x_2 \leqslant y'$, and this can be written as $R_{\otimes}(x', x_2, y')$.

The antecedent of (6.17) is equivalent to

$$\forall y_1 \; \Big[x \oslash y \leqslant y_1 \;\; \Rightarrow \;\; x_2 \leqslant y_1\Big],$$

which we will write as $R^{\downarrow}_{\oslash}(x, y, x_2)$. Thus, (6.17) becomes

$$\forall x_2 \Big[R^{\downarrow}_{\oslash}(x, y, x_2) \;\; \Rightarrow \;\; R_{\otimes}(x', x_2, y')\Big].$$

Inserting this into (6.16), gives the following condition in terms of relations:

$$\forall x, x', y, y' \; \Big(\forall x_1 \Big[R^{\downarrow}_{\otimes}(x', x, x_1) \Rightarrow \; R_{\oplus}(x_1, y', y)\Big] \Rightarrow \; \forall x_2 \Big[R^{\downarrow}_{\oslash}(x, y, x_2) \Rightarrow \; R_{\otimes}(x', x_2, y')\Big]\Big),$$

which is

$$\forall x, x', x_2, y, y' \; \Big[\Big(\forall x_1 \Big[R^{\downarrow}_{\otimes}(x', x, x_1) \Rightarrow \; R_{\oplus}(x_1, y', y)\Big] \text{ and } R^{\downarrow}_{\oslash}(x, y, x_2)\Big) \Rightarrow \; R_{\otimes}(x', x_2, y')\Big].$$

$\square$

The next proposition gives the correspondence result for the axioms of Grishin's first group, listed in (4.1). We will work on the form given as axiom (b) of this group: $A \oplus (C \,/\, B) \leqslant (A \oplus C) \,/\, B$. As was the case for the canonicity proofs in the previous section, the proof of this correspondence result is again quite similar to the proof of the correspondence result for Grishin's fourth group.

Proposition 29. *The axiom* $\forall A, B, C \; \Big(A \oplus (C \,/\, B) \leqslant (A \oplus C) \,/\, B\Big)$ *holds in an algebra* $(\mathcal{G}(X, Y, \leqslant), \otimes, /, \setminus, \oplus, \oslash, \oslash)$ *if and only if the first-order condition*

$$\begin{aligned} &\forall x, x', x_1 \in X \;\; \forall y, y' \in Y \\ &\qquad \Big[\Big(\forall x_2 \Big[R^{\downarrow}_{\oslash}(y, x, x_2) \Rightarrow \; R_{\otimes}(x_2, x', y')\Big] \text{ and } R^{\downarrow}_{\otimes}(x, x', x_1)\Big) \Rightarrow \; R_{\oplus}(x_1, y, y')\Big] \end{aligned}$$

holds for the underlying frame $(X, Y, \leqslant, R_{\otimes}, R_{\oplus})$.

Proof. The axiom $\forall A, B, C \; \Big(A \oplus (C \,/\, B) \leqslant (A \oplus C) \,/\, B\Big)$ is equivalent to

$$\forall x, A, B, C \; \Big(x \leqslant A \oplus (C \,/\, B) \;\; \Rightarrow \;\; x \leqslant (A \oplus C) \,/\, B\Big), \tag{6.18}$$

of which the antecedent can be written as $x \oslash (C \,/\, B) \leqslant A$. Instantiating $A = x \oslash (C \,/\, B)$, (6.18) implies

$$\forall x, B, C \ \Big(x \leqslant \Big([x \oslash (C \,/\, B)] \oplus C\Big) \,/\, B\Big). \tag{6.19}$$

On the other hand, (6.19) implies (6.18) since for every A and every x, B, C:

$$\begin{aligned} x \oslash (C \,/\, B) \leqslant A \quad &\Rightarrow \quad [x \oslash (C \,/\, B)] \oplus C \leqslant A \oplus C \\ &\Rightarrow \quad \Big([x \oslash (C \,/\, B)] \oplus C\Big) \,/\, B \leqslant (A \oplus C) \,/\, B. \end{aligned}$$

Using $x \leqslant ([x \oslash (C \,/\, B)] \oplus C) \,/\, B$ from (6.19), this gives (6.18). Thus the axiom can be reformulated to (6.19), which has one less second-order variable than the initial form has.

By residuation, $x \leqslant \Big([x \oslash (C \,/\, B)] \oplus C\Big) \,/\, B$ is equivalent to $x \otimes B \leqslant [x \oslash (C \,/\, B)] \oplus C$, which is $(x \otimes B) \oslash C \leqslant x \oslash (C \,/\, B)$. Thus (6.19) holds if and only if

$$\forall x, y, B, C \ \Big(x \oslash (C \,/\, B) \leqslant y \ \Rightarrow \ (x \otimes B) \oslash C \leqslant y\Big) \tag{6.20}$$

Multiple use of residuation yields that the antecedent of (6.20) is equivalent to $(y \obslash x) \otimes B \leqslant C$. As before, we can instantiate $C = (y \obslash x) \otimes B$ and obtain an equivalent form of the axiom in one less second-order variable (leaving out the details):

$$\forall x, y, B \ \Big((x \otimes B) \oslash [(y \obslash x) \otimes B] \leqslant y\Big),$$

which is

$$\forall x, y, B \ \Big(y \obslash (x \otimes B) \leqslant (y \obslash x) \otimes B\Big).$$

This we can write as

$$\forall x, y, y', B \ \Big((y \obslash x) \otimes B \leqslant y' \ \Rightarrow \ y \obslash (x \otimes B) \leqslant y'\Big). \tag{6.21}$$

By residuation, the antecedent of (6.21) is $B \leqslant (y \obslash x) \setminus y'$. Instantiating $B = (y \obslash x) \setminus y'$, (6.21) can be showed to be equivalent to the first-order condition

$$\forall x, y, y' \ \Big(y \obslash (x \otimes [(y \obslash x) \setminus y']) \leqslant y'\Big). \tag{6.22}$$

We proceed in order to rewrite this as a frame condition in relational terms. Multiple use of residuation yields that (6.22) is equivalent to

$$\forall x, y, y' \ \Big((y \obslash x) \setminus y' \leqslant x \setminus (y \oplus y')\Big).$$

Introducing another first-order variable gives

$$\forall x, x', y, y' \Big(x' \leqslant (y \obslash x) \setminus y' \Rightarrow \ x' \leqslant x \setminus (y \oplus y')\Big),$$

which is

$$\forall x, x', y, y' \Big(y \obslash x \leqslant y' \,/\, x' \Rightarrow \ x \otimes x' \leqslant y \oplus y'\Big). \tag{6.23}$$

The conclusion of (6.23), $x \otimes x' \leqslant y \oplus y'$, is equivalent to $\forall x_1 \Big[x_1 \leqslant x \otimes x' \Rightarrow \ x_1 \leqslant y \oplus y'\Big]$, which translates to the frame as

$$\forall x_1 \Big[R^{\downarrow}_{\otimes}(x, x', x_1) \Rightarrow \ R_{\oplus}(x_1, y, y')\Big]. \tag{6.24}$$

Similarly, the antecedent of (6.23), $y \oslash x \leqslant y' / x'$, is equivalent to

$$\forall x_2 \Big[x_2 \leqslant y \oslash x \Rightarrow \; x_2 \leqslant y' / x' \Big]. \tag{6.25}$$

The conclusion of (6.25) is by residuation equivalent to $x_2 \otimes x' \leqslant y'$, which is $R_{\otimes}(x_2, x', y')$. The antecedent of (6.25) is equivalent to

$$\forall y_1 \Big[y \oslash x \leqslant y_1 \Rightarrow \; x_2 \leqslant y_1 \Big],$$

which we will write as $R^{\downarrow}_{\oslash}(y, x, x_2)$. Putting these together gives that (6.25) is equivalent to

$$\forall x_2 \Big[R^{\downarrow}_{\oslash}(y, x, x_2) \Rightarrow \; R_{\otimes}(x_2, x', y') \Big]. \tag{6.26}$$

Inserting (6.24) and (6.26) into (6.23) gives that the axiom is equivalent to

$$\forall x, x', y, y' \Big(\forall x_2 \Big[R^{\downarrow}_{\oslash}(y, x, x_2) \Rightarrow \; R_{\otimes}(x_2, x', y') \Big] \Rightarrow \; \forall x_1 \Big[R^{\downarrow}_{\otimes}(x, x', x_1) \Rightarrow \; R_{\oplus}(x_1, y, y') \Big] \Big),$$

which is

$$\forall x, x', x_1, y, y' \Big[\Big(\forall x_2 \Big[R^{\downarrow}_{\oslash}(y, x, x_2) \Rightarrow \; R_{\otimes}(x_2, x', y') \Big] \text{ and } R^{\downarrow}_{\otimes}(x, x', x_1) \Big) \Rightarrow \; R_{\oplus}(x_1, y, y') \Big].$$

□

6.2.2 Equivalence of restricted interaction axioms

We will now take a closer look at the proof of the correspondence result for axiom (b) from Grishin's fourth group. Notice that the first-order condition (6.14) is precisely the shape of the interaction axiom (e) listed in (4.4). What has happened however is that the second-order variables A, B, C in the join-preserving coordinates of the operations have been replaced by variables from X, while the variables in the meet-preserving coordinates have been replaced by variables from Y. It is interesting that there is some kind of calculus behind this that gives the same replacements in axioms (b), (c) and (d). The resulting restricted axioms are mutually interderivable and are particularly compact descriptions but are nevertheless clearly first-order on the frames. As noted below, variants (a) and (f) have ambivalent slots and must be treated differently.

The restricted version of axiom (e) is equivalent to

$$\forall x, x', y, y' \; \Big(x' \leqslant (y' \oplus y) / x \; \Rightarrow \; x' \leqslant y' / (x \oslash y) \Big),$$

which is by residuation equivalent to

$$\forall x, x', y, y' \; \Big(x' \otimes x \leqslant y' \oplus y \; \Rightarrow \; x \oslash y \leqslant x' \setminus y' \Big). \tag{6.27}$$

Applying residuation with respect to the different operations gives that (6.27) is equivalent to the restricted versions of axioms (b), (c) and (d):

$$\forall x, x', y, y' \; \Big(x \leqslant x' \backslash (y' \oplus y) \; \Rightarrow \; x \leqslant (x' \setminus y') \oplus y \Big),$$

$$\forall x, x', y, y' \ \Big((x' \otimes x) \oslash y \leqslant y' \ \Rightarrow \ x' \otimes (x \oslash y) \leqslant y' \Big),$$

$$\forall x, x', y, y' \ \Big(y' \obslash (x' \otimes x) \leqslant y \ \Rightarrow \ (x' \setminus y') \obslash x \leqslant y \Big).$$

As for axioms (a) and (f), the replacement of the second-order variables into variables from X and Y cannot be done directly, since there are operations in these axioms that have conflicting demands concerning the join- resp. meet-preservingness for some slots. For axiom (a) we get the following semi-restricted version directly:

$$\forall x', y, C \ \Big((x' \setminus C) \oslash y \leqslant x' \setminus (C \oslash y) \Big).$$

This is equivalent to

$$\forall x, x', y, C \ \Big(x \leqslant x' \setminus C \ \Rightarrow \ x \oslash y \leqslant x' \setminus (C \oslash y) \Big),$$

which is

$$\forall x, x', y, y', C \ \Big((x \leqslant x' \setminus C \text{ and } C \oslash y \leqslant y') \ \Rightarrow \ x \oslash y \leqslant x' \setminus y' \Big).$$

The conditions in the antecedent are by residuation equivalent to $x' \otimes x \leqslant C$ and $C \leqslant y' \oplus y$, thus we have

$$\forall x, x', y, y' \ \Big(x' \otimes x \leqslant y' \oplus y \ \Rightarrow \ x \oslash y \leqslant x' \setminus y' \Big)$$

as a restricted version of axiom (a).

To find the restricted version of axiom (f), we start with

$$\forall x, B, C \ \Big(x \oslash (B \obslash C) \leqslant (C / x) \setminus B \Big).$$

Since the $x' \in X$ join-generate $\mathcal{G}(X, Y, \leqslant)$ and $\setminus$ is order reversing in its first coordinate, this is

$$\forall x, x', B, C \ \Big(x' \leqslant C / x \ \Rightarrow \ x \oslash (B \obslash C) \leqslant x' \setminus B \Big).$$

Similarly,

$$\forall x, x', y, B, C \ \Big((x' \leqslant C / x \text{ and } B \leqslant y) \ \Rightarrow \ x \oslash (B \obslash C) \leqslant x' \setminus y \Big),$$

and

$$\forall x, x', y, y', B, C \ \Big((x' \leqslant C / x, B \leqslant y, B \obslash C \leqslant y') \ \Rightarrow \ x \oslash y' \leqslant x' \setminus y \Big).$$

Applying residuation to the conditions in the antecedent yields

$$\forall x, x', y, y' \ \Big(x' \otimes x \leqslant y' \oplus y \ \Rightarrow \ x \oslash y \leqslant x' \setminus y' \Big),$$

thus (f) has the same restricted version as (a).

Note that this restricted version of (a) and (f) is exactly (6.27), so that the restricted versions of all the axioms listed in (4.4) are equivalent. Since the restricted version of axiom (c) is equivalent to the frame condition imposed by axiom (b) (and thus by our family of axioms), this means any restricted axiom can be used to find this frame condition.

The situation is the same in Grishin's other groups of axioms: within each group, the restricted versions of the axioms denoted by (b) - (e) have immediate restricted versions, while it is a bit more involved to find the restricted versions of the axioms (a) and (f). The restricted versions of the axioms within a group are mutually interderivable, and are equivalent to the general versions of the axioms from that group.

6.3 Completeness for other axiomatic extensions

6.3.1 Canonicity for other axiomatic extensions

The canonicity of the commutativity axioms $A \otimes B \dashv\vdash B \otimes A$ and $A \oplus B \dashv\vdash B \oplus A$ follows directly from Proposition 6.8 in [DGP05], since both $\otimes$ and $\oplus$ are order-preserving. Similarly, Proposition 6.9 from [DGP05] implies canonicity of the fusion contraction axiom $A \vdash A \otimes A$, and a very small adjustment to Proposition 6.10 from the same paper gives canonicity of the fusion weakening axiom $A \otimes B \vdash A$. Furthermore, the associativity axiom for the fusion operator, $A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C$, is canonical by Proposition 6.7 from [DGP05].

As for the plus connective counterparts, $A \vdash A \oplus B$, $A \oplus A \vdash A$, and $A \oplus (B \oplus C) \dashv\vdash (A \oplus B) \oplus C$, the canonicity can be derived in the same fashion as the proofs in the paper. We will illustrate this for the plus weakening axiom:

Suppose that for all $a_1, a_2 \in A$ holds that $a_1 \leqslant a_1 \oplus a_2$. Then for ideal elements $y_1, y_2 \in I(A^\delta)$, we have the following:

$$\begin{aligned} y_1 &= \bigvee \left\{a_1 : a_1 \leqslant y_1, a_1 \in A\right\} \\ &\leqslant \bigvee \left\{a_1 \oplus a_2 : a_1 \leqslant y_1, a_2 \leqslant y_2, a_i \in A\right\} \\ &= y_1 \oplus^\pi y_2, \end{aligned}$$

where the inequality holds by the assumption on the elements of A. Then for $u_1, u_2 \in A^\delta$, using the result for ideal elements,

$$\begin{aligned} u_1 &= \bigwedge \left\{y_1 : u_1 \leqslant y_1, y_1 \in I(A^\delta)\right\} \\ &\leqslant \bigwedge \left\{y_1 \oplus^\pi y_2 : u_1 \leqslant y_1, u_2 \leqslant y_2, y_i \in I(A^\delta)\right\} \\ &= u_1 \oplus^\pi u_2, \end{aligned}$$

hence the plus weakening axiom is canonical.

Grishin's second and third group, defined in (4.2) and (4.3), state a kind of semi-associativity of $\otimes$ (the second group) and of $\oplus$ (the third group). In proving canonicity of these groups, the proofs of the canonicity of the full associativity axioms can be followed and only need some minor changes. For example, the second group contains axiom (c): $A \otimes (C \otimes B) \vdash (A \otimes C) \otimes B$. The part of the proof for the full associativity that states that for all $u_1, u_2, u_3 \in A^\delta$ holds that $u_1 \otimes^\sigma (u_2 \otimes^\sigma u_3) \leqslant (u_1 \otimes^\sigma u_2) \otimes^\sigma u_3$ only uses that for all $x_1, x_2, x_3 \in F(A^\delta)$ holds that $x_1 \otimes^\sigma (x_2 \otimes^\sigma x_3) \leqslant (x_1 \otimes^\sigma x_2) \otimes^\sigma x_3$, of which the proof only uses that axiom (c) holds for elements of A. Hence we can use these parts and obtain the canonicity of Grishin's second group.

The next proposition summarises these results.

Proposition 30. *The axioms from Grishin's second and third group, and the commutativity, weakening, contraction and associativity axioms for both $\otimes$ and $\oplus$ satisfy canonicity.*

6.3.2 Correspondence for other axiomatic extensions

The first-order correspondents for the weakening and contraction axioms for the product connective, and the commutativity axioms for both connectives $\otimes$ and $\oplus$, can be found in [DGP05] and are listed in the table in Section 6.4. Also, the frame condition imposed by postulating associativity for the product connective can be found by applying Proposition 6.12 from the paper [DGP05], and looks as follows:

$$\forall x_1, x_2, x_3 \in X\ \forall y \in Y$$
$$\Big[\forall x \in X\ \Big(R_{\otimes}^{\downarrow}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_1, x, y)\Big)\ \Leftrightarrow\ \forall x \in X\ \Big(R_{\otimes}^{\downarrow}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)\Big)\Big].$$

From the way Proposition 6.12 is proved in [DGP05], it is immediately clear that the semi-associativity of the product connective stated in Grishin's second group imposes the following frame condition:

$$\forall x_1, x_2, x_3 \in X\ \forall y \in Y$$
$$\Big[\forall x \in X\ \Big(R_{\otimes}^{\downarrow}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)\Big)\ \Rightarrow\ \forall x \in X\ \Big(R_{\otimes}^{\downarrow}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_1, x, y)\Big)\Big].$$

The frame conditions for the plus connective counterparts can be derived in the same fashion and are given in the table in Section 6.4. As an illustration, we give here the derivation of the frame condition imposed by postulating the plus weakening axiom:

The axiom

$$\forall A, B\ \Big(A \leqslant A \oplus B\Big) \tag{6.28}$$

holds in $\mathcal{G}(X, Y, \leqslant)$ if and only if

$$\forall y_1, y_2 \in Y\ \Big(y_1 \leqslant y_1 \oplus y_2\Big). \tag{6.29}$$

The direction from (6.28) to (6.29) is clear. The other direction comes from the fact that $\oplus$ is meet-preserving in both coordinates, hence for $A, B \in \mathcal{G}(X, Y, \leqslant)$:

$$\begin{aligned} A &= \bigwedge \Big\{y_1 : A \leqslant y_1, y_1 \in Y\Big\} \\ &\leqslant \bigwedge \Big\{y_1 \oplus y_2 : A \leqslant y_1, B \leqslant y_2, y_i \in Y\Big\} = A \oplus B. \end{aligned}$$

Condition (6.29) is equivalent to $\forall x \in X, y_1, y_2 \in Y\ \Big(x \leqslant y_1\ \Rightarrow\ R_{\oplus}(x, y_1, y_2)\Big)$.

6.4 Results

In this chapter we first considered in detail the interaction axioms from Grishin's first and fourth group. We proved the canonicity of these axioms and provided for each group a first-order correspondent such that the axioms of that group hold on a canonical extension if and only if the correspondent holds on the Lambek-Grishin frame associated with that canonical extension. Thus we obtained complete relational semantics for those extensions of $LG_{\emptyset}$ where the additional axioms come from these groups.

We also used this approach to handle the other axiomatic extensions, and these results are summarised in the following theorem. The strength of this approach lies in the fact that these extensions work in a modular way - one can extend with any combination of additional axioms that satisfy canonicity, and will find complete relational semantics with respect to the intersection of the corresponding frame classes.

Theorem 31. *For $LG_\emptyset$ extended with any subset of the following axioms, we obtain complete relational semantics with respect to the intersection of the corresponding frame classes.*

Additional axioms	Frame class
Grishin's first group	$\forall x, x', x_1, y, y' \ \Big[\Big(\forall x_2 \ [R^{\downarrow}_{\oslash}(y, x', x_2) \Rightarrow R_{\otimes}(x_2, x, y')]$ and $R^{\downarrow}_{\otimes}(x', x, x_1)\Big) \Rightarrow R_{\oplus}(x, y, y')\Big]$
Grishin's second group	$\forall x_1, x_2, x_3, y \ \Big(\forall x \ [R^{\downarrow}_{\otimes}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)] \Rightarrow \forall x \ [R^{\downarrow}_{\otimes}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_1, x, y)]\Big)$
Grishin's third group	$\forall x, y_1, y_2, y_3 \ \Big(\forall y \ [R^{\downarrow}_{\oplus}(y, y_1, y_2) \Rightarrow R_{\oplus}(x, y, y_3)] \Rightarrow \forall y \ [R^{\downarrow}_{\oplus}(y, y_2, y_3) \Rightarrow R_{\oplus}(x, y_1, y)]\Big)$
Grishin's fourth group	$\forall x, x', x_2, y, y' \ \Big[\Big(\forall x_1 \ [R^{\downarrow}_{\otimes}(x', x, x_1) \Rightarrow R_{\oplus}(x_1, y', y)]$ and $R^{\downarrow}_{\oslash}(x, y, x_2)\Big) \Rightarrow R_{\otimes}(x', x_2, y')\Big]$
$\wedge, \vee, 0, 1,$ satisfying the bounded lattice axioms	
distributive lattice operations	$(X, Y, \leqslant) \cong (P, P, \ngeqslant)$ for some poset P
$A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C$	$\forall x_1, x_2, x_3, y \ \Big(\forall x \ [R^{\downarrow}_{\otimes}(x_1, x_2, x) \Rightarrow R_{\otimes}(x, x_3, y)] \Leftrightarrow \forall x \ [R^{\downarrow}_{\otimes}(x_2, x_3, x) \Rightarrow R_{\otimes}(x_1, x, y)]\Big)$
$A \oplus (B \oplus C) \dashv\vdash (A \oplus B) \oplus C$	$\forall x, y_1, y_2, y_3 \ \Big(\forall y \ [R^{\downarrow}_{\oplus}(y, y_1, y_2) \Rightarrow R_{\oplus}(x, y, y_3)] \Leftrightarrow \forall y \ [R^{\downarrow}_{\oplus}(y, y_2, y_3) \Rightarrow R_{\oplus}(x, y_1, y)]\Big)$
$A \otimes B \dashv\vdash B \otimes A$	$\forall x_1, x_2, y \ [R_{\otimes}(x_1, x_2, y) \Leftrightarrow R_{\otimes}(x_2, x_1, y)]$
$A \oplus B \dashv\vdash B \oplus A$	$\forall x, y_1, y_2 \ [R_{\oplus}(x, y_1, y_2) \Leftrightarrow R_{\oplus}(x, y_2, y_1)]$
$A \otimes B \vdash A$	$\forall x_1, x_2, y \ [x_1 \leqslant y \Rightarrow R_{\otimes}(x_1, x_2, y)]$
$A \vdash A \oplus B$	$\forall x, y_1, y_2 \ [x \leqslant y_1 \Rightarrow R_{\oplus}(x, y_1, y_2)]$
$A \vdash A \otimes A$	$\forall x, y \ [R_{\otimes}(x, x, y) \Rightarrow x \leqslant y]$
$A \oplus A \vdash A$	$\forall x, y \ [R_{\oplus}(x, y, y) \Rightarrow x \leqslant y]$

Part III

Linear Logic

Chapter 7

Introduction to linear logic

The substructural logic that we will consider in this part is linear logic, which was introduced by Girard in [Gir87]. Linear logic is identified as the most natural weakening of classical logic. Linear logic satisfies the structural rule of commutativity but does not in general satisfy the rules of weakening and contraction. This makes it appropriate for interpretations in the setting of resource sensitivity.

Contrary to classical logic, the focus in the field of linear logic lies on proofs rather than truths. We will see that linear logic takes certain differences in derivations into account, while these derivations are equivalent in the setting of classical logic. This makes linear logic well-suited for applications in computer science.

The structural rules weakening and contraction are allowed for certain special formulas only, as we will see in Section 7.2. In this chapter, we will introduce linear logic as a sequent calculus in the fashion of [Cur04].

7.1 Multiplicative-additive linear logic

Consider the following two derivations that are valid in classical logic:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}$$

It is clear that the first derivation together with the weakening rule implies the second, while the second derivation together with the contraction rule implies the first. However, in the setting of linear logic, the weakening and contraction rules do not hold, and the above derivations are no longer equivalent. To account for these different derivations, in linear logic the conjunction is split into two different connectives: & and $\otimes$.

For the disjunction, the situation is similar. The derivation

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

is equivalent to the combination of the following derivations

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

if and only if the weakening and contraction rules hold. Therefore, the disjunction is also split into two different connectives: $\parr$ and $\oplus$.

The introduction rules for the new connectives are as follows:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \;\&\; B, \Delta} \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Delta_1, \Delta_2}$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}$$

These connectives come with units. Since it will be more convenient in Chapter 8 and 9, our notation for these constants deviates from the notation used traditionally in linear logic. We will denote the units for $\&, \otimes, \parr, \oplus$ as $\top, 1, 0, \bot$ (in that order).

There is a linear negation $(\_)^\perp$, such that $A \dashv\vdash A^{\perp\perp}$ and $\Gamma, A \vdash \Delta \;\Leftrightarrow\; \Gamma \vdash A^\perp, \Delta$, for all A, Γ and Δ. The linear negation relates the connectives and constants as follows, for all formulas A, B:

$$\begin{array}{llll} (A \otimes B)^\perp \dashv\vdash (A^\perp) \parr (B^\perp) & (A \parr B)^\perp \dashv\vdash (A^\perp) \otimes (B^\perp) & 1^\perp \dashv\vdash 0 & 0^\perp \dashv\vdash 1 \\ (A \;\&\; B)^\perp \dashv\vdash (A^\perp) \oplus (B^\perp) & (A \oplus B)^\perp \dashv\vdash (A^\perp) \;\&\; (B^\perp) & \top^\perp \dashv\vdash \bot & \bot^\perp \dashv\vdash \top \end{array}$$

This implies that the right introduction rules given above also serve as left introduction rules, and that every sequent can be rewritten to have an empty set of premises.

There is one axiom,

$$\frac{}{\vdash A, A^\perp}$$

and a cut rule,

$$\frac{\vdash A, \Gamma_1 \quad \vdash A^\perp, \Gamma_2}{\vdash \Gamma_1, \Gamma_2}$$

It can be shown that the cut rule is not needed, however, to find derivations it comes in useful.

The linear implication, which we will denote as $\rightarrow$, is defined as $A \rightarrow B = (A^\perp) \parr B$. This implication is traditionally denoted by $\multimap$.

On the partially ordered set of formulas, the connective $\oplus$ serves as the join-operation and the connective $\&$ serves as the meet-operation. We will illustrate this briefly, since we will make use of this fact in the next chapter. The following rules imply that $A \oplus B$ is a common upper bound of A and B:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$$

Writing the introduction rule for $\&$ in an order dual way yields:

$$\frac{A^\perp \vdash \Gamma \quad B^\perp \vdash \Gamma}{(A \mathbin{\&} B)^\perp \vdash \Gamma}$$

Since $(A \mathbin{\&} B)^\perp = A^\perp \oplus B^\perp$, and the above holds for all A, B, this gives that $A \oplus B$ is below all upper bounds of A and B:

$$\frac{A \vdash \Gamma \quad B \vdash \Gamma}{A \oplus B \vdash \Gamma}$$

In a similar way one can show that & is the meet-operation.

The above completes the description of the multiplicative-additive fragment of linear logic (MALL). This system lends itself to different interpretations, of which the most well-known one is to understand formulas as resources. In this interpretation, the resources are used in the entailment, and can be used only once. In this setting, $A \vdash B$ means that A is consumed in order to get B.

7.2 Full linear logic

Full linear logic consists of MALL with an extra connective that is called the exponential, and is denoted by !. Its introduction rule is

$$\frac{\Gamma \vdash \ A}{!\Gamma \vdash \ !A}$$

A special feature of the exponential is that on formulas of the form $!A$, the structural rules weakening and contraction are allowed. Thus, in addition to the rules for MALL, in full linear logic, we also have:

$$\frac{\vdash \ !A, \ !A}{\vdash \ !A} \qquad \frac{\vdash \ !A}{\vdash \ !A, !A}$$

In the interpretation of resources, $!A$ means that A can be used and reused at will.

In Chapter 10, we will take a closer look at the properties of this exponential from the algebraic perspective, after we have introduced the necessary notions.

The algebraic semantics for MALL (resp. full linear logic) will be given in Chapter 8 (resp. Chapter 9). From the algebraic semantics, we will derive complete relational semantics using the same approach as in Part II.

Chapter 8

Relational semantics for multiplicative-additive linear logic

To derive relational semantics for the multiplicative-additive fragment of linear logic (MALL), we first describe its algebraic semantics. Secondly, we will provide canonicity and correspondence results to obtain complete relational semantics for this logic.

8.1 Algebraic semantics

The algebraic semantics for MALL is given by classical linear algebras, which are extensions of the residuated algebras studied in Chapter 3.

Definition 32. *A* classical linear algebra *(CL-algebra) is a structure* $(\mathbf{L}, \otimes, \rightarrow, \leftarrow, 1, 0)$*, where*

1. $(\mathbf{L}, \otimes, \rightarrow, \leftarrow)$ *is a residuated algebra;*
2. *the fusion* $\otimes$ *is associative and commutative and has a unit* 1*;*
3. $\mathbf{L}$ *is a bounded lattice;*
4. *for all* $a \in \mathbf{L}$*,* $(a \rightarrow 0) \rightarrow 0 = a$*.*

In linear logic, the meet operation is denoted by & (with unit $\top$), the join by $\oplus$ (with unit 0), the implication by $\multimap$ and our constant 0 is denoted by $\bot$. However, as we will refer to the literature from lattice theory we will stick to the usual lattice theoretic notation and denote meet by $\wedge$ (with unit $\top$) and join by $\vee$ (with unit $\bot$). For further details on CL-algebras the reader is referred to [Tro92], which uses a notation similar to ours.

As before, we will denote $x \rightarrow 0$ by $x^{\perp}$ and call this operation *linear negation*. Implication sends joins in the first coordinate to meets, hence $(\_)^{\perp}$ sends joins to meets. As $(\_)^{\perp}$ is a bijection, it follows that it is a (bijective) lattice homomorphism $\mathbf{L} \rightarrow \mathbf{L}^{\mathrm{dual}}$, where $\mathbf{L}^{\mathrm{dual}}$ is the lattice obtained by reversing the order in $\mathbf{L}$.

8.2 Canonicity of the MALL axioms

As explained in Section 3.3, the first step in obtaining relational semantics for MALL is checking canonicity, i.e., ensuring that the class **CL** of CL-algebras is closed under canonical extension.

Proposition 33. *The class* **CL** *is closed under canonical extension.*

Proof. Let $\mathbf{L}$ be a CL-algebra and let $\mathbf{L}^\delta$ be its canonical extension. In [DGP05] it is shown that $\mathbf{L}^\delta$ is a perfect residuated algebra. Hence, in particular, it is a bounded lattice. Furthermore, it is shown that, if $\otimes$ is associative (resp. commutative), then so is its extension $\otimes^\sigma$. We need to show that $\otimes^\sigma$ has a unit. Note that, for all $a \in \mathbf{L}^\delta$,

$$\begin{aligned}
a \otimes^\sigma 1_{\mathbf{L}} &= \bigvee\{x \mid a \geqslant x \in F(\mathbf{L}^\delta)\} \otimes^\sigma 1_{\mathbf{L}} \\
&= \bigvee\{x \otimes^\sigma 1_{\mathbf{L}} \mid a \geqslant x \in F(\mathbf{L}^\delta)\} \\
&= \bigvee\{\bigwedge\{p \otimes 1_{\mathbf{L}} \mid x \leqslant p \in \mathbf{L}\} \mid a \geqslant x \in F(\mathbf{L}^\delta)\} \\
&= \bigvee\{\bigwedge\{p \mid x \leqslant p \in \mathbf{L}\} \mid a \geqslant x \in F(\mathbf{L}^\delta)\} \\
&= \bigvee\{x \mid a \geqslant x \in F(\mathbf{L}^\delta)\} = a.
\end{aligned}$$

Hence $1_{\mathbf{L}}$, the unit of $\otimes$ in $\mathbf{L}$, serves as the unit of $\otimes^\sigma$ in $\mathbf{L}^\delta$.

It is left to show that, for all $w \in \mathbf{L}^\delta$, $(w^{\perp^\delta})^{\perp^\delta} = w$. This may be derived from the results in [Alm09], but we choose to give a direct proof here to illustrate the methods of the theory of canonical extensions. As $(\_)^\perp$ is a lattice homomorphism $\mathbf{L} \to \mathbf{L}^{\mathrm{dual}}$, its extension is a complete lattice homomorphism $\mathbf{L}^\delta \to (\mathbf{L}^{\mathrm{dual}})^\delta = (\mathbf{L}^\delta)^{\mathrm{dual}}$. Recall that every element of the canonical extension may be written as a join of meets of elements of the original lattice, and that $F(\mathbf{L}^\delta)$ is the set of elements of $\mathbf{L}^\delta$ that may be obtained as a meet of elements of $\mathbf{L}$. For $w \in \mathbf{L}^\delta$,

$$\begin{aligned}
w &= \bigvee\{x \in F(\mathbf{L}^\delta) \,|\, x \leqslant w\} \\
&= \bigvee\{\bigwedge\{a \in \mathbf{L} \,|\, x \leqslant a\} \,|\, x \in F(\mathbf{L}^\delta)\,,\, x \leqslant w\}.
\end{aligned}$$

Hence, for $w \in \mathbf{L}^\delta$,

$$\begin{aligned}
(w^{\perp^\delta})^{\perp^\delta} &= ((\bigvee\{\bigwedge\{a \in \mathbf{L} \,|\, x \leqslant a\} \,|\, x \in F(\mathbf{L}^\delta)\,,\, x \leqslant w\})^{\perp^\delta})^{\perp^\delta} \\
&= (\bigwedge\{\bigvee\{a^\perp \,|\, a \in \mathbf{L}\,,\, x \leqslant a\} \,|\, x \in F(\mathbf{L}^\delta)\,,\, x \leqslant w\})^{\perp^\delta} \\
&= \bigvee\{\bigwedge\{(a^\perp)^\perp \,|\, a \in \mathbf{L}\,,\, x \leqslant a\} \,|\, x \in F(\mathbf{L}^\delta)\,,\, x \leqslant w\} \\
&= \bigvee\{\bigwedge\{a \in \mathbf{L} \,|\, x \leqslant a\} \,|\, x \in F(\mathbf{L}^\delta)\,,\, x \leqslant w\} \\
&= w,
\end{aligned}$$

which proves the claim. $\square$

8.3 Correspondence for multiplicative-additive linear logic

Since a CL-algebra is a residuated algebra, it follows from Section 3.3 that the structures providing relational semantics for MALL are in the class of Lambek frames.

To describe the constants 1 and 0 dually, we have to extend the Lambek frames with two Galois-closed subsets, $U \subseteq X$ and $Z \subseteq Y$. Starting from a perfect CL-algebra $\mathbf{L}$, these sets are given by

$$U = \{x \in \mathcal{J}^\infty(\mathbf{L}) \mid x \leqslant 1\} \quad \text{and} \quad Z = \{y \in \mathcal{M}^\infty(\mathbf{L}) \mid 0 \leqslant y\}.^1$$

Our next step is to characterise the collection of frames $F = (X, Y, \leqslant, R_\otimes, U, Z)$ satisfying $\mathcal{G}(F) \in \mathbf{CL}$ (this constitutes the correspondence result). Our correspondence results on Lambek-Grishin frames given in Section 6.4 in particular give correspondence results for the Lambek frame part. It is this part that we are now extending. The modular set-up allows us to reuse these results here.

Recall from Chapter 6 that the fusion is associative on $\mathcal{G}(F)$ if and only if the Lambek frame part $(X, Y, \leqslant, R_\otimes)$ of F satisfies Φ_a (again, we assume that any element named x (resp. y) with any super- or subscript comes from X (resp. Y)):

$$\forall x_1, x_2, x_3\, \forall y \left[\forall x \left(R^\downarrow_\otimes(x_2, x_3, x) \Rightarrow R_\otimes(x_1, x, y)\right) \Leftrightarrow \forall x \left(R^\downarrow_\otimes(x_1, x_2, x) \Rightarrow R_\otimes(x, x_3, y)\right)\right]. \tag{Φ_a}$$

And, the fusion in $\mathcal{G}(F)$ is commutative if and only if F satisfies Φ_c:

$$\forall x_1, x_2\, \forall y \left(R_\otimes(x_1, x_2, y) \Leftrightarrow R_\otimes(x_2, x_1, y)\right). \tag{Φ_c}$$

For U to be the unit of the fusion in $\mathcal{G}(F)$ we have to ensure that $W \otimes U = W$ for all $W \in \mathcal{G}(F)$. As the fusion on $\mathcal{G}(F)$ is completely join preserving, it suffices to ensure $x \otimes U = x$ for all $x \in X (= J^\infty(\mathcal{G}(F)))$. Note that,

$$\begin{aligned} x \otimes U \leqslant y \quad &\Leftrightarrow \quad \bigvee\{x \otimes x' \mid x' \leqslant U\} \leqslant y \\ &\Leftrightarrow \quad \forall x' \in U.\, x \otimes x' \leqslant y \\ &\Leftrightarrow \quad U \subseteq R_\otimes[x, \_, y]. \end{aligned}$$

Hence, U is the unit of the fusion in $\mathcal{G}(F)$ if and only if F satisfies Φ_u:

$$\forall x\, \forall y \left(x \leqslant y \Leftrightarrow U \subseteq R_\otimes[x, \_, y]\right). \tag{Φ_u}$$

Now we have come to the last axiom: $(a \to 0) \to 0 = a$. First note that, by the adjunction property,

$$a \leqslant (a \to 0) \to 0 \quad \Leftrightarrow \quad (a \to 0) \otimes a \leqslant 0 \quad \Leftrightarrow \quad a \to 0 \leqslant a \to 0.$$

So in any case $a \leqslant (a \to 0) \to 0$. Furthermore, the mapping $a \mapsto (a \to 0) \to 0$ is completely join-preserving and therefore it again suffices to consider completely join

[1] We could also have described Z as a subset of X, however as it occurs in the axiom $(a \to 0) \to 0 = a$ and the implication is meet-preserving in the second coordinate, it is more convenient to describe it by the collection of meet-irreducibles above it, i.e. by a subset of Y.

irreducible elements. Note that, for $x' \in \mathcal{J}^\infty(\mathcal{G}(\mathbf{F}))$,

$$\begin{aligned}
x' \leqslant (x \to 0) \to 0 \quad &\Leftrightarrow \quad (x \to 0) \otimes x' \leqslant 0 \\
&\Leftrightarrow \quad x \to 0 \leqslant x' \to 0 \\
&\Leftrightarrow \quad \forall x''.\ x'' \leqslant x \to 0 \ \Rightarrow\ x'' \leqslant x' \to 0 \\
&\Leftrightarrow \quad \forall x''.\ x \otimes x'' \leqslant 0 \ \Rightarrow\ x' \otimes x'' \leqslant 0 \\
&\Leftrightarrow \quad \forall x''.\ Z \subseteq R_\otimes[x, x'', _] \ \Rightarrow\ Z \subseteq R_\otimes[x', x'', _].
\end{aligned}$$

Hence, the equation $(a \to 0) \to 0 = a$ holds in $\mathcal{G}(F)$ if and only if F satisfies Φ_{dd}:

$$\forall x, x' \left(\forall x''.\ Z \subseteq R_\otimes[x, x'', \_] \ \Rightarrow\ Z \subseteq R_\otimes[x', x'', \_]\right) \ \Rightarrow\ x' \leqslant x.^2 \qquad (\Phi_{dd})$$

The following theorem summarises the results from this chapter.

Theorem 34. *The class of extended RS-frames $F = (X, Y, \leqslant, R_\otimes, U, Z)$ satisfying Φ_a, Φ_c, Φ_u and Φ_{dd} gives complete semantics for MALL.*

[2] Note that the statement $x' \leqslant x$ uses the ordering of $\mathcal{G}(F)$. We may also write this in the language of the frame as: $\forall y.\ x \leqslant y \ \Rightarrow\ x' \leqslant y$.

Chapter 9

Relational semantics for full linear logic

We are now ready to consider full linear logic. This consists of the previously defined multiplicative-additive fragment and an exponential. As we have seen in Section 7.2, a special feature of the exponential is the fact that on formulas that are in its image, the structural rules contraction and weakening are allowed. As before, we start by describing the algebraic semantics of the exponential. The following definition is equivalent to the one given in [Tro92].

Definition 35. *Let* $\mathbf{L}$ *be a CL-algebra. An* exponential *on* $\mathbf{L}$ *is a mapping* $! : L \to L$ *such that for all* $a, b \in L$

1. $!!a = !a \leqslant a$;
2. $a \leqslant b \;\Rightarrow !a \leqslant !b$;
3. $!\top = 1$;
4. $!a \otimes !b \; = !(a \wedge b)$.

And, in this case, $(\mathbf{L}, !)$ *is called a* CLS-algebra.

To obtain relational semantics for full linear logic, we have to prove that the properties of the exponential satisfy canonicity, and we have to describe the exponentials dually.

9.1 Canonicity of the exponential

Our next step towards complete relational semantics for full linear logic is to prove that the class **CLS** of CLS-algebras is closed under canonical extension. In Proposition 33 of Section 8.2, we already showed that the canonical extension of a CL-algebra is again a CL-algebra. Using this result, we only need to prove that $!^\sigma$, the extended version of $!$, is an exponential on $\mathbf{L}^\delta$. This illustrates the benefits of the modular set-up.

Proposition 36. *The class* **CLS** *is closed under canonical extension.*

Proof. We will check that the map $!^\sigma$ on $\mathbf{L}^\delta$ satisfies the four properties of Definition 35.

1. To prove: $!^\sigma !^\sigma a = !^\sigma a \leqslant a$. We first show the inequality. Using that $!p \leqslant p$ for all $p \in \mathbf{L}$, we have, for all $x \in F(\mathbf{L}^\delta)$,

$$\begin{aligned} !^\sigma x &= \bigwedge\{!p \mid x \leqslant p \in \mathbf{L}\} \\ &\leqslant \bigwedge\{p \mid x \leqslant p \in \mathbf{L}\} \\ &= x. \end{aligned}$$

This result for filter elements implies that for all $a \in \mathbf{L}^\delta$,

$$\begin{aligned} !^\sigma a &= \bigvee\{!^\sigma x \mid a \geqslant x \in F(\mathbf{L}^\delta)\} \\ &\leqslant \bigvee\{x \mid a \geqslant x \in F(\mathbf{L}^\delta)\} \\ &= a. \end{aligned}$$

It follows that $!^\sigma !^\sigma a \leqslant !^\sigma a$, so it is left to show that $!^\sigma a \leqslant !^\sigma !^\sigma a$. Lemma 3.4 from [DGP05] implies that $!^\sigma$ sends filter elements to filter elements, hence for $x \in F(\mathbf{L}^\delta)$, we have that $!^\sigma x \in F(\mathbf{L}^\delta)$, and thus $!^\sigma !^\sigma x = \bigwedge\{!p \mid !^\sigma x \leqslant p \in \mathbf{L}\}$. For $x \in F(\mathbf{L}^\delta)$, we will prove that

$$\begin{aligned} !^\sigma !^\sigma x &= \bigwedge\{!p \mid !^\sigma x \leqslant p \in \mathbf{L}\} \\ &\geqslant \bigwedge\{!q \mid x \leqslant q \in \mathbf{L}\} \\ &= !^\sigma x, \end{aligned}$$

by showing that for every meetand of the first meet, there exists a meetand in the second meet that is below it. Let $p \in \mathbf{L}$ be such that $!^\sigma x \leqslant p$. Then

$$!^\sigma x = \bigwedge\{!q \mid x \leqslant q \in \mathbf{L}\} \leqslant p.$$

By compactness, there exist $q_1, \dots, q_n \in \mathbf{L}$ such that $x \leqslant q_i$ for $i \in \{1, \dots, n\}$, and $!q_1 \wedge \dots \wedge !q_n \leqslant p$. Then $x \leqslant q_1 \wedge \dots \wedge q_n$, and since $!$ is order-preserving we have that $!(q_1 \wedge \dots \wedge q_n) \leqslant !q_1 \wedge \dots \wedge !q_n \leqslant p$. We denote $q_1 \wedge \dots \wedge\ q_n =: q$. By assumption, $!q \leqslant !!q$. This implies together with $!q \leqslant p$ and the fact that $!$ is order-preserving that $!q \leqslant !!q \leqslant !p$. Furthermore, $!q$ is a meetand of the second meet since $x \leqslant q$ and $q \in \mathbf{L}$. Thus, we have now proved for all filter elements x that $!^\sigma x \leqslant !^\sigma !^\sigma x$.

For $a \in \mathbf{L}^\delta$, we want to show that

$$\begin{aligned} !^\sigma a &= \bigvee\{!^\sigma x \mid a \geqslant x \in F(\mathbf{L}^\delta)\} \\ &\leqslant \bigvee\{!^\sigma x \mid !^\sigma a \geqslant x \in F(\mathbf{L}^\delta)\} \\ &= !^\sigma !^\sigma a. \end{aligned}$$

We will show that for every joinand of the first join, there exists a joinand of the second join that is above it. Let x' be such that $!^\sigma x'$ is a joinand of the first join, i.e. such that $a \geqslant x' \in F(\mathbf{L}^\delta)$. Lemma 3.4 from [DGP05] states that $!^\sigma$ is order preserving, hence $!^\sigma a \geqslant !^\sigma x'$. Since $!^\sigma$ sends filter elements to filter elements we have that $!^\sigma x' \in F(\mathbf{L}^\delta)$. Thus,

$$!^\sigma !^\sigma x' \in \{!^\sigma x \mid !^\sigma a \geqslant x \in F(\mathbf{L}^\delta)\}.$$

Since $!^\sigma !^\sigma x' \geqslant !^\sigma x'$ by the previous result for filter elements, this means that there is a joinand in the second join that is above the joinand of the first join that we started with.

2. To prove: $a \leqslant b \Rightarrow !^\sigma a \leqslant !^\sigma b$. The exponential ! is order preserving, hence by Lemma 3.4 from [DGP05] $!^\sigma$ is order preserving.

3. To prove: $!^\sigma \top_{\mathbf{L}^\delta} = 1_{\mathbf{L}^\delta}$. Since $!^\sigma \top_{\mathbf{L}^\delta} = !^\sigma \top_{\mathbf{L}} = !\top_{\mathbf{L}} = 1_{\mathbf{L}}$, we should prove that $1_{\mathbf{L}} = 1_{\mathbf{L}^\delta}$, i.e. that $1_{\mathbf{L}}$ is the unit of the fusion $\otimes^\sigma$ on $\mathbf{L}^\delta$. We have already seen that this is true in the proof of Proposition 33.

4. To prove: $!^\sigma a \otimes^\sigma !^\sigma b = !^\sigma(a \wedge_{\mathbf{L}^\delta} b)$. Define $[!, !] : (a, b) \mapsto (!a, !b)$. For all $a, b \in \mathbf{L}^\delta$,

$$
\begin{aligned}
!^\sigma a \otimes^\sigma !^\sigma b &= (\otimes^\sigma \circ [!, !]^\sigma)(a, b) \\
&\overset{(1)}{=} (\otimes \circ [!, !])^\sigma(a, b) \\
&\overset{(2)}{=} (! \circ \wedge_{\mathbf{L}})^\sigma(a, b) \\
&\overset{(3)}{=} (!^\sigma \circ \wedge_{\mathbf{L}}^\sigma)(a, b) \\
&= (!^\sigma \circ \wedge_{\mathbf{L}^\delta})(a, b) = !^\sigma(a \wedge_{\mathbf{L}^\delta} b),
\end{aligned}
$$

where we use for $^{(1)}$ that $\otimes$ is join-preserving and both $[!, !]$ and $\otimes$ are order-preserving, for $^{(2)}$ the assumption for ! on $\mathbf{L}$, and for $^{(3)}$ that $\wedge_{\mathbf{L}}$ is meet-preserving, while both $\wedge_{\mathbf{L}}$ and ! are order-preserving. In $^{(1)}$ and $^{(3)}$, we make use of compositionality results that can be found in [GH01].

$\square$

9.2 Correspondence for the exponential

For the correspondence result for full linear logic, we need to extend the relational structures from Chapter 8 that give complete semantics for MALL, such that they also account for the exponential. The following lemma will help to find the duality that we will use in order to find relational semantics for full linear logic.

Lemma 37. *Let* $\mathbf{L}$ *be a CL-algebra,* $! : L \to L$ *an operation, and* $I = \mathrm{Im}(!)$*. Then the following are equivalent:*

i. ! *satisfies 35.1 and 35.2*

ii. ! *is the upper adjoint of the embedding* $I \hookrightarrow L$*.*

And, in this case, I *is a meet-semilattice in the order on* I*,* $! : L \to I$ *is meet-preserving, and* I *is closed under all (possibly infinitary) joins that exist in* $\mathbf{L}$*.*

Proof. We first prove the equivalence statement.

i. $\Rightarrow$ **ii.** We have to prove that $\forall a \in L, \forall i \in I$ [$i \leqslant a \Leftrightarrow i \leqslant !a$]. Suppose $i \leqslant a$. By 35.2, this implies $!i \leqslant !a$. Since $i \in I$, there is an element $b \in L$ such that $i = !b$. According to 35.1, $!b = !!b$, thus $i = !b = !!b = !i \leqslant !a$.
Now suppose $i \leqslant !a$. By 35.1, $!a \leqslant a$, hence $i \leqslant a$.

ii. $\Rightarrow$ **i.** Using the adjunction property, $!a \leqslant !a$ implies $!a \leqslant a$ (thus also $!!a \leqslant !a$) and $!a \leqslant !!a$, which gives 35.1: $!!a = !a \leqslant a$. Suppose $a \leqslant b$, then by the above $!a \leqslant a \leqslant b$ and applying adjunction yields $!a \leqslant !b$, which is 35.2.

By 35.2, for all $i, j \in I$, $!(i \wedge_L j)$ is a common lower bound for $!i = i$ and $!j = j$, and $!(i \wedge_L j) \in I$. For all $k \in I$ such that k is a common lower bound for i and j, $k \leqslant i \wedge_L j$, hence by the adjunction property $k \leqslant !(i \wedge_L j)$. Thus $i \wedge_I j = !(i \wedge_L j)$, hence I is a meet-semilattice in the order on I.

Since ! is the upper adjoint of the embedding, it is meet-preserving, i.e. for all $a, b \in L, !(a \wedge_L b) = !a \wedge_I !b$.

Let $S \subseteq I$ and suppose $\alpha := \bigvee_L S$ exists. For all $s \in S, s \leqslant \alpha$, hence by 35.2, $!s \leqslant !\alpha$. Since $s \in I$, we have that $!s = s$. Thus $!\alpha$ is an upper bound for S, hence $!\alpha \geqslant \bigvee_L S = \alpha$. By 35.1, $!\alpha \leqslant \alpha$, hence $\alpha = !\alpha \in I$. Thus I is closed under all joins that exist in L. $\square$

Note that Definition 35.4 implies that, for an exponential !, we have that $!a \otimes !b = !(a \wedge_L b)$. The above lemma implies that, for $I = \mathrm{Im}(!)$, $!(a \wedge_L b) = !a \wedge_I !b$, hence the meet on the image of ! is given by the fusion operation. This operation is join-preserving (and all finite joins in I coincide with the same joins taken in L), thus the image of ! turns out to be a distributive lattice.

The following proposition shows that exponentials on $\mathbf{L}$ can be characterised by certain subsets of $\mathbf{L}$. Later on we will use this result to describe the exponentials dually. In Theorem 8.18 in [Tro92], it was already shown that certain subsets give rise to exponentials. However, contrary to [Tro92], we demand that these subsets are closed under certain joins. In this way, we obtain a one-to-one correspondence between specific subsets and the exponentials.

Proposition 38. *There is a bijective correspondence between exponentials on* $\mathbf{L}$ *and collections* $I \subseteq \mathbf{L}$ *such that*

i. *for all* $a \in L$, $\bigvee\{i \in I \mid i \leqslant a\}$ *exists and is an element of* I*;*

ii. I *is closed under* $\otimes$*;*

iii. *for all* $i \in I, i \otimes i = i$*;*

iv. $1 \in I$ *and for all* $i \in I, i \leqslant 1$.

Proof. Let $I \subseteq \mathbf{L}$ satisfy (i) - (iv). Define $!_I(a) = \bigvee\{i \in I \mid i \leqslant a\}$. Now $i \leqslant a$ implies $i \leqslant \bigvee\{i \in I \mid i \leqslant a\} = !_I(a)$. Conversely, if $i \leqslant !_I(a)$, then since $!_I(a) = \bigvee\{i \in I \mid i \leqslant a\} \leqslant a$ we have that $i \leqslant a$. Thus $!_I$ is the upper adjoint of the embedding $I \hookrightarrow L$, and Lemma 37 gives that the first two properties of Definition 35 are satisfied. Furthermore, since $!_I$ is an upper adjoint, it sends $\top$ to the top of I, which is 1 according to (iv).

We still have to check property 4 of Definition 35 to show that $!_I$ is an exponential on $\mathbf{L}$. Lemma 37 says that for all $a, b \in L$, $!_I(a \wedge_L b) = !_I(a) \wedge_I !_I(b)$.

For all $a, b \in I$, we have that $!_I(a) \otimes !_I(b) \leqslant !_I(a) \otimes 1 = !_I(a)$. Similarly, $!_I(a) \otimes !_I(b) \leqslant !_I(b)$, hence $!_I(a) \otimes !_I(b)$ is a common lower bound for $!_I(a)$ and $!_I(b)$, which is in I. This implies $!_I(a) \otimes !_I(b) \leqslant !_I(a) \wedge_I !_I(b) = !_I(a \wedge_L b)$.

For the other inequality,

$$\begin{array}{rcll} !_I(a \wedge_L b) & = & \bigvee\{i \in I \mid i \leqslant a \wedge_L b\} & \\ & = & \bigvee\{i \in I \mid i \leqslant a, i \leqslant b\} & \\ & = & \bigvee\{i \otimes i \mid i \in I, i \leqslant a, i \leqslant b\} & \text{(by (iii))} \\ & \leqslant & \bigvee\{i \otimes i' \mid i, i' \in I, i \leqslant a, i' \leqslant b\} & \\ & = & \bigvee\{i \in I \mid i \leqslant a\} \otimes \bigvee\{i \in I \mid i \leqslant b\} & \\ & = & !_I(a) \otimes !_I(b). & \end{array}$$

Now, let ! be an exponential on $\mathbf{L}$. Define $I_! = \{a \in L \mid !a = a\}$. Note that $I_! = \mathrm{Im}(!)$, since for all $a \in I_!$, $a = !a \in \mathrm{Im}(!)$, and for all $!b \in \mathrm{Im}(!)$, $!!b = !b$ by (1), hence $!b \in I_!$. We have to prove that $I_!$ satisfies (i) - (iv):

i. For all $a \in L$, $!a \in I_!$ and by (2), $!a \leqslant a$, hence $!a \leqslant \bigvee\{i \in I \mid i \leqslant a\}$. For all $i \in I_!$, if $i \leqslant a$, then $i = !i \leqslant !a$, hence $\bigvee\{i \in I \mid i \leqslant a\} \leqslant !a$. Thus $\bigvee\{i \in I \mid i \leqslant a\} = !a \in \mathrm{Im}(!)$.

ii. For all $a, b \in \mathrm{Im}(!)$, $a \otimes b = !a \otimes !b = !(a \wedge b) \in \mathrm{Im}(!)$.

iii. For all $a \in \mathrm{Im}(!)$, $a \otimes a = !a \otimes !a = !(a \wedge a) = !a = a$.

iv. $1 = !\top \in \mathrm{Im}(!)$. And, for all $a \in \mathrm{Im}(!)$, $a \leqslant \top$ thus $a = !a \leqslant !\top = 1$.

The bijectiveness follows from the fact that $I = I_{!_I}$ and $! = !_{I_!}$.
For $i \in I, !_I(i) = \bigvee\{i' \in I \mid i' \leqslant i\} = i$, so $i \in I_{!_I}$. Thus $I \subseteq I_{!_I}$. For the other inclusion, let $a \in I_{!_I}$, then $a = !_I(a) = \bigvee\{i \in I \mid i \leqslant a\}$ and by (1) this join is an element of I, i.e. $a \in I$. Thus $I_{!_I} \subseteq I$.
Furthermore, $! = !_{I_!}$ since

$$\begin{array}{rcl} !_{I_!}(a) & = & \bigvee\{i \in I_! \mid i \leqslant a\} \\ & = & \bigvee\{i \in L \mid !i = i \text{ and } i \leqslant a\} \\ & = & \bigvee\{i \in \mathrm{Im}(!) \mid i \leqslant a\} \\ & = & !a. \end{array}$$

□

Remark 39. *In the setting of perfect lattices (such as canonical extensions), the first condition of Proposition 38 is equivalent to the condition that I is closed under all joins.*

Our next goal is to describe the exponentials dually, i.e. to translate the property of a canonical extension of having a certain exponential to a property of its corresponding frame. It turns out that we have to add another collection to the frame. Also, we want to see under which conditions such a collection on the frame-side gives rise to an exponential on the complex algebra-side. Proposition 38 naturally suggests collections that we could add to characterise the exponentials. However, we will see that because of the special structure of $\mathrm{Im}(!^\sigma)$, we only need to add part of such a collection to the frames.

As we have seen in Section 9.1, the canonical extension of a CLS-algebra is again a CLS-algebra. This canonical extension is a perfect lattice. Furthermore, the lattice $\mathrm{Im}(!^\sigma)$, taken with the order induced from $\mathbf{L}^\delta$, is by itself a perfect lattice since it is order-isomorphic to $(\mathrm{Im}(!))^\delta$, which is the canonical extension of a poset and thus a perfect lattice.

Recall from Theorem 34 that the class of extended RS-frames $F = (X, Y, \leqslant, R_\otimes, U, Z)$ satisfying Φ_a, Φ_c, Φ_u and Φ_{dd} gives complete semantics for MALL. To model the behaviour of the additional operation !, we add to the frame F a collection of Galois-closed sets $J_! := J^\infty_{\mathrm{Im}(!^\sigma)}(\mathrm{Im}(!^\sigma))$, consisting of all the completely join irreducibles in the lattice $\mathrm{Im}(!^\sigma)$.

The lattice $\mathcal{G}(F)$ is perfect, and $\mathrm{Im}(!^\sigma)$ is a complete join-subsemilattice of $\mathcal{G}(F)$, therefore $\mathrm{Im}(!^\sigma)$ may be obtained from $J_!$ as the $\bigvee$-closure of $J_!$ in $\mathcal{G}(F)$. Thus, starting from $!^\sigma$, we get $\mathrm{Im}(!^\sigma)$, and by taking its completely join irreducibles, we get $J_!$. Taking the $\bigvee$-closure of $J_!$, we get $\mathrm{Im}(!^\sigma)$ back, and Proposition 38 tells us that from $\mathrm{Im}(!^\sigma)$, we can get $!^\sigma$ again.

On the other hand, starting with a collection J on F, we want to know under which conditions this collection gives rise to an exponential on $\mathcal{G}(F)$. Let I_J be the $\bigvee$-closure of J in $\mathcal{G}(F)$. To give rise to a unique exponential on $\mathcal{G}(F)$, I_J should be a perfect lattice and it should satisfy the four properties of Proposition 38. We will first translate these four properties of I_J to conditions on J.

i. By Remark 39, this is automatically satisfied since I is the $\bigvee$-closure of J in $\mathcal{G}(F)$.

ii. We need I_J to be closed under $\otimes$. In $\mathcal{G}(F)$, the fusion is completely join-preserving in both coordinates, hence I_J is closed under $\otimes$ iff for all $w, w' \in J$, it holds that $w \otimes w' \in I_J$. This is the case iff for all $w, w' \in J$, $w \otimes w' = \bigvee\{v \in J \mid v \leqslant w \otimes w'\}$. Since $w \otimes w' \geqslant \bigvee\{v \in J \mid v \leqslant w \otimes w'\}$ is always true, the property is satisfied iff the converse inequality holds for all $w, w' \in J$. This can be rewritten as:

$$\begin{aligned} & w \otimes w' \leqslant \bigvee\{v \in J \mid v \leqslant w \otimes w'\} \\ & \Leftrightarrow \forall y.\ \bigvee\{v \in J \mid v \leqslant w \otimes w'\} \leqslant y \Rightarrow w \otimes w' \leqslant y \\ & \Leftrightarrow \forall y.\ \Big(\forall v \in J.\ v \leqslant w \otimes w' \Rightarrow v \leqslant y\Big) \Rightarrow w \otimes w' \leqslant y \end{aligned}$$

This can be rewritten further to obtain a second-order statement purely in the language of the frame. However, this condition gets very large, since the elements from J cannot be handled as easily on the frame as the elements from X and Y, since the ordering and the relation $R_\otimes$ are tailored to these elements. We will not incorporate the pure frame condition in this text, but leave the above as an abbreviation. The same holds for the following conditions.

Thus, I_J is closed under $\otimes$ iff F satisfies Φ_{e2}:

$$\forall y.\ \Big(\forall v \in J.\ v \leqslant w \otimes w' \Rightarrow v \leqslant y\Big) \Rightarrow w \otimes w' \leqslant y \qquad (\Phi_{e2})$$

iii. The fusion should be idempotent on I_J. Note that this implies that it should be idempotent on J, and because $\otimes$ is completely join-preserving, its idempotency on J implies that it is idempotent on I_J. Thus the condition we need is $\forall w \in J, w \otimes w = w$. In condition (iv) we will demand in particular that $w \leqslant 1$, for all $w \in J$. This gives $w \otimes w \leqslant w \otimes 1 \leqslant w$. Hence it suffices that for all $w \in J$, $w \leqslant w \otimes w$, and

$$w \leqslant w \otimes w \quad \Leftrightarrow \quad \forall y.\ w \otimes w \leqslant y \Rightarrow w \leqslant y.$$

Thus, under assumption of (iv), $\otimes$ is idempotent on I_J iff F satisfies Φ_{e3}:

$$\forall w \in J.\ \forall y.\ w \otimes w \leqslant y \Rightarrow w \leqslant y. \tag{Φ_{e3}}$$

iv. This property asks that $1 \in I_J$ and that for all $W \in I_J$, $W \leqslant 1$. Note that this is equivalent to $\bigvee_{\mathcal{G}(F)}\{w \in J\} = 1$, and

$$\begin{aligned}
&\bigvee\{w \in J\} = 1\\
&\Leftrightarrow \forall y.\ \Big[1 \leqslant y \Leftrightarrow \bigvee\{w \in J\} \leqslant y\Big]\\
&\Leftrightarrow \forall y.\ \Big[\Big(\forall x.\ x \leqslant 1 \Rightarrow x \leqslant y\Big) \Leftrightarrow \Big(\forall w \in J.\ w \leqslant y\Big)\Big]\\
&\Leftrightarrow \forall y.\ \Big[\Big(\forall x \in U.\ x \leqslant y\Big) \Leftrightarrow \Big(\forall w \in J.\ w \leqslant y\Big)\Big].
\end{aligned}$$

Thus, the fourth property is satisfied for I_J iff F satisfies Φ_{e4}:

$$\forall y.\ \Big(\forall x \in U.\ x \leqslant y\Big) \Leftrightarrow \Big(\forall w \in J.\ w \leqslant y\Big). \tag{Φ_{e4}}$$

The above conditions ensure that I_J satisfies Proposition 38, and thus that it is the image of an exponential on $\mathcal{G}(F)$. In addition, we want to get the collection J back as the set of completely join irreducible elements of I_J. Note that I_J, being the image of an exponential on $\mathcal{G}(F)$, is a completely distributive lattice. Hence it is equivalent to demand that the elements of J are completely join prime in I_J. As I_J is the $\bigvee$-closure of J, this gives the following condition for all $v \in J$:

$$\forall S \subseteq J.\ v \leqslant \bigvee S \Rightarrow \exists w \in S.\ v \leqslant w.$$

This can be rewritten in the following way:

$$\begin{aligned}
&\forall S \subseteq J.\ \Big(\forall w \in S.\ v \nleqslant w\Big) \Rightarrow v \nleqslant \bigvee S\\
&\Leftrightarrow\ \ v \nleqslant \bigvee\{w \in J \mid v \nleqslant w\}\\
&\Leftrightarrow\ \ \exists y.\ \Big(\forall w \in J.\ v \nleqslant w \Rightarrow w \leqslant y\Big) \text{ and } v \nleqslant y.
\end{aligned}$$

Thus the condition we need is Φ_{ej}:

$$\forall v \in J.\ \exists y.\ \Big(\forall w \in J.\ v \nleqslant w \Rightarrow w \leqslant y\Big) \text{ and } v \nleqslant y. \tag{Φ_{ej}}$$

Summarising our results from this chapter and the previous chapter gives the following theorem:

Theorem 40. *The class of extended RS-frames $F = (X, Y, \leqslant, R_\otimes, U, Z, J)$ satisfying Φ_a, Φ_c, Φ_u, Φ_{dd} and the above conditions Φ_{e2}, Φ_{e3}, Φ_{e4}, Φ_{ej} gives complete relational semantics for full linear logic.*

In this chapter and the previous chapter, we have computed the conditions on the relational frames corresponding to the axioms in a mechanical way, not worrying about getting the simplest possible formulation. For the multiplicative-additive fragment, the axioms could all be reduced to statements concerning only join irreducible elements, hence these mechanical translations yield first-order statements on the dual. Since the exponential, as a map from $\mathbf{L}$ to $\mathbf{L}$, is neither meet- nor join-preserving, its axioms are translated to second-order statements on the dual. This mechanical approach illustrates the strength of using duality theory in the search for relational semantics: it allows a modular and uniform treatment of additional operations and axioms. The paper [CGvR11b] that is in preparation discusses how the above conditions may be rewritten to get a cleaner representation, and how this relational semantics is related to the phase spaces that are traditionally used as semantics for linear logic.

Chapter 10

Properties of exponentials

It is well known that for a CL-algebra, there is not a unique admissible !. In this chapter, we will look at the family of admissible exponentials from the algebraic perspective.

We say that an exponential ! is *larger than* an exponential !′, if for all $a \in \mathbf{L}$, $!'a \leqslant !a$ or, equivalently, if $I_{!'} \subseteq I_!$. It is clear that every CL-algebra $\mathbf{L}$ has a smallest exponential, namely the exponential corresponding to the subset $\{\bot, 1\}$. Furthermore, every idempotent element of $\mathbf{L}$ below 1 gives rise to an exponential in the following way.

Lemma 41. *Let $\mathbf{L}$ be a CL-algebra. For all $a \in \mathbf{L}$ such that $a \otimes a = a$ and $a \leqslant 1$, the subset $I_a := \{\bot, a, 1\}$ corresponds to an exponential.*

Proof. We will check the four properties of Proposition 38.

$$\bigvee\{i \in I_a \mid i \leqslant b\} \quad = \quad \begin{cases} 1 & \text{if } b \geq 1 \\ a & \text{if } a \leqslant b \leqslant 1 \\ \bot & \text{else} \end{cases}$$

hence $\bigvee\{i \in I_a \mid i \leqslant b\} \in I_a$. Clearly, $\bot, a, 1$ are idempotents of $\otimes$, and I_a is closed under $\otimes$. By definition, $\bot, a, 1 \leqslant 1$. □

The following shows under which condition a largest exponential exists.

Lemma 42. *A CL-algebra $\mathbf{L}$ has a largest exponential if and only if the collection of idempotents of $\otimes$ below 1, i.e. $\{a \in L \mid a \otimes a = a, a \leqslant 1\}$, defines an exponential (which then is the largest exponential on $\mathbf{L}$).*

Proof. If $\{a \in L \mid a \otimes a = a, a \leqslant 1\}$ defines an exponential, and I is any collection satisfying the four properties from Proposition 38, then in particular for all $i \in I$ it holds that $i \otimes i = i$ and $i \leqslant 1$, hence $I \subseteq \{a \in L \mid a \otimes a = a, a \leqslant 1\}$. Thus $\mathbf{L}$ has a largest exponential.

Lemma 41 says that for all $a \in \mathbf{L}$ such that $a \otimes a = a$ and $a \leqslant 1$, the set $I_a := \{\bot, a, 1\}$ defines an exponential. If $\mathbf{L}$ has a largest exponential with corresponding subset I, then in particular $I_a \subseteq I$, for all $a \in \mathbf{L}$. This implies $I = \{a \in L \mid a \otimes a = a, a \leqslant 1\}$, which thus defines an exponential. □

Proposition 43. *Every complete CL-algebra* $\mathbf{L}$ *has a largest exponential.*

Proof. By Lemma 42, it suffices to prove that the subset $I := \{a \in L \mid a \otimes a = a, a \leqslant 1\}$ satisfies the properties from Proposition 38.

i. We will show that I is closed under arbitrary joins. Let $S \subseteq I$. Since $\mathbf{L}$ is complete, $\bigvee S$ exists in L. We will show that it is an element of I. For all $i \in I$, $i \leqslant 1$, thus $\bigvee S \leqslant 1$. Therefore, it also holds that $\bigvee S \otimes \bigvee S \leqslant \bigvee S \otimes 1 = \bigvee S$. On the other hand, $\otimes$ is completely join-preserving, so

$$\begin{aligned} \bigvee S \otimes \bigvee S &= \bigvee \{s_1 \otimes s_2 \mid s_1, s_2 \in S\} \\ &\geqslant \bigvee \{s \otimes s \mid s \in S\} \\ &= \bigvee S, \end{aligned}$$

where the last equality holds by assumption on the elements of I.

ii. For all $a, b \in I$, $(a \otimes b) \otimes (a \otimes b) = (a \otimes a) \otimes (b \otimes b) = a \otimes b$. And, $a \leqslant 1, b \leqslant 1$ implies that $a \otimes b \leqslant 1 \otimes 1 = 1$, thus I is closed under $\otimes$.

iii. & iv. These are clear from the definition of I.

$\square$

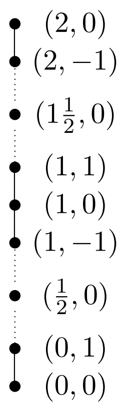
10.1 A CL-algebra without a largest exponential

Lemma 42 provides a tool to see whether a largest exponential exists on a CL-algebra. As we will see in the following, this need not always be the case. In order to construct such an algebra $\mathbf{K}$ that does not have a largest exponential, we start by defining a complete CL-algebra $\mathbf{L}$ of which $\mathbf{K}$ will be a sub CL-algebra.

Consider the poset

$$\mathbf{L} = \{(0, a) \mid a \in \mathbb{Z}^+\} \cup \{(1, b) \mid b \in \mathbb{Z}\} \cup \{(2, c) \mid c \in \mathbb{Z}^-\} \cup \{(\tfrac{1}{2}, 0), (1\tfrac{1}{2}, 0)\},$$

where the ordering is the lexicographic order on the product.



The poset $\mathbf{L}$ is a complete lattice with bottom $(0, 0)$ and a top $(2, 0)$. We define a binary fusion operation on $\mathbf{L}$ as follows,

$$(i,a)\otimes(j,b) = \begin{cases} (0,0) & \text{if } \{0\}\subseteq\{i,j\}\subseteq\{0,\frac{1}{2},1,1\frac{1}{2}\} \\ & \text{or } \{\frac{1}{2}\}\subseteq\{i,j\}\subseteq\{\frac{1}{2},1,1\frac{1}{2}\} \\ & \text{or } (i=j=1 \text{ and } a+b\leqslant 0) \\ (0,0\vee(a+b)) & \text{if } \{i,j\}=\{0,2\} \\ (\frac{1}{2},0) & \text{if } \{i,j\}=\{\frac{1}{2},2\} \\ (1,\min(a,b)) & \text{if } i=j=1 \text{ and } a+b>0 \\ (1,a) & \text{if } i=1 \text{ and } j\in\{1\frac{1}{2},2\} \\ (1,b) & \text{if } i\in\{1\frac{1}{2},2\} \text{ and } j=1 \\ (1\frac{1}{2},0) & \text{if } \{1\frac{1}{2}\}\subseteq\{i,j\}\subseteq\{1\frac{1}{2},2\} \\ (2,a+b) & \text{if } i=j=2 \end{cases}$$

Lemma 44. *The fusion on* $\mathbf{L}$ *is both associative and commutative. Its unit is* $(2,0)$.

Proof. From the definition, it follows immediately that the fusion is commutative and that $(2,0)$ is its unit. In order to prove that $\otimes$ is associative, we divide $\mathbf{L}$ into two parts, viz.

$$S := \{(0,a)\,|\,a\in\mathbb{Z}^+\}\cup\{(\tfrac{1}{2},0)\}\cup\{(1,b)\,|\,b\in\mathbb{Z}^-\},$$

$$U := \{(1,b)\,|\,b>0\}\cup\{(1\tfrac{1}{2},0)\}\cup\{(2,c)\,|\,c\in\mathbb{Z}^-\}.$$

Clearly, $(S,\otimes_S)$ is a semi-group since for all $x,y\in S$ we have that $x\otimes_S y=(0,0)$. Theorem 1 of [Jen02] can be used to see that $(U,\otimes_U)$ is a semi-group too. Now a tedious proof that we will leave out shows the associativity of the fusion defined on $\mathbf{L}$. $\square$

Lemma 45. *The fusion on* $\mathbf{L}$ *is residuated.*

Proof. As $\mathbf{L}$ is a complete lattice, the fusion is residuated iff it is completely join-preserving in both coordinates. It is clear that the fusion is order-preserving. Hence, it suffices to the check the truly infinite joins $\{(0,a)\,|\,a\in\mathbb{Z}^+\}$ and $\{(1,a)\,|\,a\in\mathbb{Z}^+\}$, whose joins are $(\frac{1}{2},0)$ and $(1\frac{1}{2},0)$ respectively.

We start by considering the first set, whose join is $(\frac{1}{2},0)$. For $i\in\{0,\frac{1}{2},1,1\frac{1}{2}\}$, $\bigvee\{(0,a)\otimes(i,b)\,|\,a\in\mathbb{Z}^+\}=\bigvee\{(0,0)\,|\,a\in\mathbb{Z}^+\}=(0,0)=(\frac{1}{2},0)\otimes(i,b)$ and $\bigvee\{(0,a)\otimes(2,b)\,|\,a\in\mathbb{Z}^+\}=\bigvee\{(0,0\vee a+b)\,|\,a\in\mathbb{Z}^+\}=(\frac{1}{2},0)=(\frac{1}{2},0)\otimes(0,b)$.

For the second set, we see that for $i\in\{0,\frac{1}{2}\}$, $\bigvee\{(1,a)\otimes(i,b)\,|\,a\in\mathbb{Z}^+\}=\bigvee\{(0,0)\,|\,a\in\mathbb{Z}^+\}=(0,0)=(1\frac{1}{2},0)\otimes(i,b)$ and $\bigvee\{(1,a)\otimes(1,b)\,|\,a\in\mathbb{Z}^+\}=(1,b)=(1\frac{1}{2},0)\otimes(1,b)$. Finally, for $i\in\{1\frac{1}{2},2\}$, $\bigvee\{(1,a)\otimes(i,b)\,|\,a\in\mathbb{Z}^+\}=\bigvee\{(1,a)\,|\,a\in\mathbb{Z}^+\}=(1\frac{1}{2},0)=(1\frac{1}{2},0)\otimes(i,b)$. $\square$

Lemma 46. *For all* $(i,a)\in\mathbf{L}$, $((i,a)\to(0,0))\to(0,0)=(i,a)$.

Proof. Recall that, for $u,v\in\mathbf{L}$, $u\to v=\bigvee\{w\,|\,u\otimes w\leqslant v\}$. It follows that

$$\begin{array}{rclrcl} (0,a)\to(0,0) & = & (2,-a) & (1\frac{1}{2},0)\to(0,0) & = & (\frac{1}{2},0) \\ (\frac{1}{2},0)\to(0,0) & = & (1\frac{1}{2},0) & (2,a)\to(0,0) & = & (0,-a), \\ (1,a)\to(0,0) & = & (1,-a) & & & \end{array}$$

which proves the claim. $\square$

Combining Lemma 44, 45 and 46 yields that $(\mathbf{L}, \otimes, \to, (2,0), (0,0))$ is a CL-algebra. Consider the subset $K = \mathbf{L} - \{(\frac{1}{2}, 0), (1\frac{1}{2}, 0)\}$. One readily checks that this set is closed under fusion and linear negation. As implication is expressible in those two operations (by $u \to v = (u \otimes v^{\perp})^{\perp}$) this implies that K is the domain of a subalgebra $\mathbf{K}$ of $\mathbf{L}$, which is then a CL-algebra as well.

We claim that $\mathbf{K}$ does not possess a largest exponential. The collection of idempotents (below $1 = \top$) of $\mathbf{K}$ is $\{(1,a) \mid a \in \mathbb{Z}^+\} \cup \{(0,0)\}$. For any element of the form $(2,b)$ with $b \neq 0$, the join of the idempotents below it does not exist. Hence the collection of all idempotents below 1 does not yield an exponential on $\mathbf{K}$ and therefore, by Lemma 42, there is no largest exponential on $\mathbf{K}$.

Just like any complete CL-algebra, $\mathbf{L}$ possesses a largest exponential corresponding to the collection of its idempotents below 1, which is

$$\mathbf{I} = \{(1,a) \mid a \in \mathbf{Z}^+\} \cup \{(1\tfrac{1}{2}, 0)\} \cup \{(0,0)\}.$$

The canonical extension of $\mathbf{L}$ may be described as

$$\mathbf{L}^{\delta} \cong \mathbf{L} \cup \{(\tfrac{1}{2}, -1), (\tfrac{1}{2}, 1), (1\tfrac{1}{2}, -1), (1\tfrac{1}{2}, 1)\},$$

in which the canonical extension of $\mathbf{I}$ embeds as

$$\mathbf{I}^{\delta} \cong \mathbf{I} \cup \{(1\tfrac{1}{2}, -1)\} \subseteq \mathbf{L}^{\delta}.$$

Note that in $\mathbf{L}^{\delta}$, both $(1\frac{1}{2}, -1)$ and $(1\frac{1}{2}, 1)$ are idempotents of $\otimes^{\delta}$. The first one is an ideal element and therefore,

$$\begin{array}{rcl} (1\frac{1}{2}, -1) \otimes^{\delta} (1\frac{1}{2}, -1) & = & \bigvee\{u \otimes v \mid u, v \in \mathbf{L} \mid u, v \leqslant (1\frac{1}{2}, -1)\} \\ & = & \bigvee\{(1,a) \mid a \in \mathbf{Z}^+\} \\ & = & (1\frac{1}{2}, -1). \end{array}$$

The element $(1\frac{1}{2}, 1)$ is a filter element, hence,

$$\begin{array}{rcl} (1\frac{1}{2}, 1) \otimes^{\delta} (1\frac{1}{2}, 1) & = & \bigwedge\{u \otimes v \mid u, v \in \mathbf{L} \mid u, v \geq (1\frac{1}{2}, 1)\} \\ & = & \bigwedge\{(2,a) \mid a \in \mathbb{Z}^-\} \\ & = & (1\frac{1}{2}, 1). \end{array}$$

However, $(1\frac{1}{2}, 1) \notin \mathbf{I}^{\delta}$. This example shows that the canonical extension of the largest exponential on $\mathbf{L}$ may not yield the largest exponential on the canonical extension $\mathbf{L}^{\delta}$.

Conclusion

The goal of this thesis was to find relational semantics for extensions of the Lambek-Grishin calculus and for full linear logic. The algebraic semantics for these substructural logics are given by classes of ordered algebras that do not have lattice operations, or for which the lattice operations are not distributive. Compared to the cases of intuitionistic logic and modal logic, this complicates the search for relational semantics. However, the representation theory for these ordered algebras provides a means to obtain relational structures that give complete semantics for the logics in question. This method of providing canonicity and correspondence results to obtain completeness results was first described in [DGP05], where it is applied to the basic implication-fusion fragment.

In this thesis, we have applied the general theory from [DGP05] to different substructural logics. We have provided the canonicity and correspondence proofs to obtain complete relational semantics for the Lambek-Grishin calculus and various extensions, and for full linear logic. While considering these substructural logics, we have taken full advantage of the fact that this approach to completeness results works in a fully modular way.

For the case of the Lambek-Grishin calculus, this enabled us to consider all relevant axiomatic extensions separately. Traditional Kripke semantics for these logics have the drawback that, as soon as additional axioms or additional connectives are present, one may have to start over to obtain semantics for such richer logics. The advantage of our approach via canonical extensions of LG-algebras is that each additional axiom that lifts to the canonical extension gives a first-order condition on the corresponding frame that can be handled in a modular way, whereas additional connectives modularly slot in as additional relational components. We have seen that all groups of axioms presented by Grishin in [Gri83] satisfy canonicity, and, for each of these axioms, we have provided a first-order frame condition that completely characterises the influence of the axiom on the logic at hand. The modular set-up allows us to augment this by the correspondence results for associativity, commutativity, weakening and contraction, given in [DGP05]. In this manner, we have obtained completeness results for a whole family of Lambek-Grishin calculi.

For each of Grishin's groups of additional axioms, we have found that the first-order frame condition imposed by such an axiom is in fact equivalent to a version of the initial axiom itself, in which the second-order variables have been replaced by completely join resp. meet irreducible elements of the complex algebra. These replacements are such that the join-preserving slots of the operations are inhabited by completely join irreducibles, and the meet-preserving slots are inhabited by completely meet irreducibles. The resulting restricted axioms are particularly compact descriptions but are nevertheless clearly first-order on the frames.

Full linear logic, on the other hand, provides an example of a logic whose axioms are not expressible in first-order conditions on the completely join resp. meet irreducible elements of the complex algebra, since the exponential is neither meet- nor join preserving. In the case of linear logic, the modularity allowed us to split up the work and focus on the multiplicative-additive fragment first. We have proved that the axioms defining this fragment satisfy canonicity and can be translated to first-order frame conditions. These conditions determine a specific class of extended RS-frames that provides relational semantics for this fragment. Secondly, we have showed that the properties of the exponential are canonical. To capture the behaviour of this additional operation, we extended the relational structures that provided semantics for MALL with an extra collection of elements of the associated complex algebra. A question that we aim to answer in the paper [CGvR11b] is how the relational semantics obtained in this way relates to the phase semantics that is traditionally used for linear logic.

In future exploration, one could further investigate the obtained frame conditions in order to gain knowledge about the logics and the different extensions. Furthermore, the approach taken in this thesis can be applied to obtain complete relational semantics for other substructural logics.

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