An energy-entanglement connection

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Abstract

We report a connection between the amount of entanglement and the local energy that a bipartite system can have with that entanglement, by explicitly finding a minimum and maximum energy pure states. We show through numerical simulations that this connection is relevant to design energetically efficient entanglement generation protocols. Moreover, our results point out formal analogies between these extremal states and thermal states, thus reinforcing the connection between entanglement and thermodynamics.

Entanglement is a key ingredient for the development of quantum technologies. Studying its relation with energy is important in order to design more efficient quantum protocols, as shown in [1].

To quantify the entanglement of a pure state $|\psi\rangle$ of a bipartite system A-B we make use of the "Entropy of entanglement": $\mathcal{E}(|\psi\rangle) = S\left(\mathrm{Tr}_{A(B)}\{|\psi\rangle\langle\psi|\}\right)$ with $S(\rho) = -\,\mathrm{Tr}\{\rho \log \rho\}$, that is regarded as the standard entanglement quantifier [2]. In our work, we found a family of minimum and maximum local energy states for a given degree of entanglement $\mathcal{E}$, for any couple of local discrete Hamiltonians. Each local Hamiltonian can be written as $H_X = \sum_{n=0}^{N_X-1} X_n |X_n\rangle\langle X_n|$, where $X = A, B$ and N_X is the dimension of the system A or B. Moreover, $X_0 \le X_1 \le \cdots \le X_{N_X-1}$ and, without loss of generality, we impose $N_A \le N_B$.

Excluding some particular cases with degeneration of the ground state or of the most excited state of $H = H_A + H_B$, a minimum energy state $|\psi_g\rangle$ and a maximum energy state $|\psi_e\rangle$ are given by [3]:

$$|\psi_g\rangle = \frac{1}{\sqrt{Z_g}} \sum_{i=0}^{N_A-1} e^{-\frac{\beta}{2}(A_i+B_i)} |A_i B_i\rangle, \quad |\psi_e\rangle = \frac{1}{\sqrt{Z_e}} \sum_{i=0}^{N_A-1} e^{\frac{\beta'}{2}(A_i+B_{i+\Delta})} |A_i B_{i+\Delta}\rangle, \tag{1}$$

where $Z_g = \sum_{i=0}^{N_A-1} e^{-\beta(A_i+B_i)}$, $Z_e = \sum_{i=0}^{N_A-1} e^{\beta'(A_i+B_{i+\Delta})}$ and $\Delta = N_B - N_A$. Lastly, β and β' are, respectively, the solutions of the equations $\left(-\beta\frac{\partial}{\partial\beta}+1\right)\log Z_g = \mathcal{E}$ and $\left(-\beta'\frac{\partial}{\partial\beta'}+1\right)\log Z_e = \mathcal{E}$. An entire family of minimum and maximum energy states can be obtained from the above ones by applying energy-preserving unitary operators of the form $U_A \otimes U_B$. Furthermore, this result can be extended to the case of mixed states for a large class of entanglement quantifiers.

In a specific bipartite system, composed of a three and a four-level system, we generated one billion of random pure states and we observed that the probability of randomly generating an extremal state is extremely low. Thus, the knowledge of minimum energy states can be useful to design optimal entanglement formation protocols from an energetic point of view.

Eventually, it is worth noting the strong formal analogies between the quantities that appear in the extremal states and thermodynamic quantities such as temperature and partition function. In particular, when $B_{N_A-1} = B_{N_A-2} = \cdots = B_0$, the state of A associated to $|\psi_g\rangle$ is equal to:

$$\rho_g^A = \frac{1}{Z_g} \sum_{i=0}^{N_A-1} e^{-\beta A_i} |A_i\rangle\langle A_i| \quad \text{with} \quad Z_g = \sum_{i=0}^{N_A-1} e^{-\beta A_i}, \tag{2}$$

which is a thermal state with respect to H_A at temperature $T = 1/(k_B\beta)$, where k_B is the Boltzmann constant. This seems to reinforce the idea of a deep connection between entanglement and thermodynamics [4].

[1] C. Bény, C. T. Chubb, T. Farrelly, and T. J. Osborne, *Energy cost of entanglement extraction in complex quantum systems*, Nat. Comm. **9**, 3792 (2018).

[2] M. B. Plenio and S. Virmani, *An introduction to entanglement measures*, Quantum Info. Comput. **7**, 1 (2007).

[3] N. Piccione, B. Militello, A. Napoli, and B. Bellomo, *An energy-entanglement connection*, in preparation.

[4] S. Popescu, A. J. Short, and A. Winter, *Entanglement and the foundations of statistical mechanics*, Nat. Phys. **2**, 754 (2006).