On Shifner-Erugin-Salakhova-Chebotarev Type Open Systems

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Abstract
In this presentation we will study the problem of integrability of open quantum systems, in particular, we will consider the Shifner-Erugin-Salakhova-Chebotarev type systems.

Unlike the closed systems, where the time-evolution is given by the von Neumann equation, the master equation of an open quantum system on an n-dimensional Hilbert space $\mathcal{H}$ is generally written as

$$\dot{\rho}_t = L(t)\,\rho_t \tag{1}$$

where $\rho_t \in \mathcal{S}(\mathcal{H}) \subset \mathcal{B}^*(\mathcal{H})$. If $L(t) = L$ is constant in time, then the closed form of the solution (linear evolution operator) of this master equation is given by $\Phi_t = \exp(Lt)$ (by assuming the initial time is $t_0 = 0$). However, if $L(t)$ is time-dependent, then no general closed form of the solution is known, and we need to assume additional conditions to solve the equation in closed forms. Famous integrable classes include (i) periodic systems [4], (ii) functionally commutative systems [3, 4] (or the Lappo-Danilevsky systems [2]), and (iii) Wu's systems [7]. In this presentation, we will forcus on another class, what we call the Shifner-Erugin-Salakhova-Chebotarev systems [6, 1, 5].

Suppose the generator $L(t)$ can be represented as

$$L(t) = P(t) + Q(t), \tag{2}$$

where $P(t), Q(t)$ satisfy the following conditions:

1. With continuous scalar functions μ_j, ν_j and constant operators P_j, Q_j,

$$P(t) = \sum_{j=1}^{p} \mu_j(t) P_j, \qquad Q(t) = \sum_{j=1}^{q} \nu_j(t) Q_j,$$

2. $[P_j, P_k] = O$, $[Q_j, Q_k] = O$ for all j, k.

Then, the following statements are known to be equivalent:

- There exist φ_j $(j = 1, \ldots, p)$ such that $[L_0, [Q_k, L_0]] = O$ $(\forall k)$, where $L_0 = \sum_{j=1}^{p} \varphi_j P_j$ has relatively prime elementary divisors.
- The linear evolution operator for the equation (1) with the generator (2) can be represented as

$$\Phi_t = e^{K(t)} e^{D(t)}, \tag{3}$$

and $\left[K(t), \dot{K}(t)\right] = O$ for all t, where

$$K(t) = \int_0^t e^{D(\tau)} L(\tau) e^{-D(\tau)} d\tau, \qquad D(t) = \int_0^t Q(\tau) d\tau.$$

A particular example is the equation such that the generator is given by $L(t) = \nu(t)P + \mu(t)Q$ with $[P, [Q, P]] = O$, which was studied in [6, 1].

In this presentation, we will review Shifner-Erugin-Salakhova-Chebotarev type open quantum systems and discuss how to compute the closed forms of the solutions.

References

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