Hidden correlations and information-entropic inequalities in systems of qudits

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Abstract

For a single-qudit system (N-level atom), we formally construct artificial subsystems and obtain analogs of Bayes' formula for the probability distribution of one random variable and the corresponding density matrix.

In classical and quantum theories, the notion of correlations is introduced for systems without subsystems. The states of classical systems (e.g., with two subsystems) are described by joint probability distributions $P(jk)$ of two random variables j and k, determining the marginal probability distributions $\Pi(j) = \sum_k P(jk)$ and $\mathcal{P}(k) = \sum_j P(jk)$, along with the conditional probability distributions $\Pi(j \mid k)$, satisfying Bayes' formula $P(jk) = \Pi(j \mid k)\mathcal{P}(k)$. The correlations between two subsystems do not exist if $P(jk) = \Pi(j)\mathcal{P}(k)$, which is expressed by entropic inequality for the mutual information. Analogously, for quantum states the correlations do not exist if the state density operator $\hat{\rho}(j,k) = \hat{R}(j) \otimes \hat{r}(k)$, where $\hat{R}(j) = \mathrm{Tr}_k\, \hat{\rho}(j,k)$ and $\hat{r}(k) = \mathrm{Tr}_j\, \hat{\rho}(j,k)$.

Our aim is to study correlations (called hidden correlations) existing for systems without subsystems. For classical systems, the states are described by the probability distribution $P(q)$ of one random variable $q \leftrightarrow (j,k)$, where $q = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$, and $N = nm$. We write an analog of Bayes' formula for the classical system with one random variable $q \leftrightarrow (j,k)$. In the case $N = 4$, we employ the map of numbers $1 \leftrightarrow (1,1)$, $2 \leftrightarrow (1,2)$, $3 \leftrightarrow (2,1)$, $4 \leftrightarrow (2,2)$. Bayes' formula written in terms of distribution $P(1)$, $P(2)$, $P(3)$, and $P(4)$, provides, for example, the obvious identity since $\Pi(1 \mid 1)$ and $\mathcal{P}(1)$ are given by the numbers $\Pi(1 \mid 1) = \dfrac{P(1)}{P(1) + P(2)}$ and $\mathcal{P}(1) = P(1) + P(3)$.

For qudit states, the density operator $\hat{\rho}(q)$ is bijectively mapped onto the density operator $\hat{\rho}(j,k)$, where two artificial qudits are introduced. For example, the density operator of four-level atom can be mapped bijectively onto the density operator of two qubits. In view of the map constructed, after obtaining an analog of Bayes' formula for the classical system with one random variable, we are in the position to write a new information-entropic inequality for single qudit states [1]. For example, we show that the ququart-state density 4×4-matrix mapped onto the density matrix of two artificial qubits demonstrates hidden correlations expressed through the violation of formal Bell's inequalities, being given as some relations of matrix elements of the ququart-state density matrix.

Since the density matrix of an arbitrary qudit state can be mapped onto a set of probability distributions of classical-like random variables [1–3] or a single specific probability distribution of one random variable, the new information-entropic inequalities for matrix elements of the qudit density matrix can be derived. These inequalities can be checked experimentally using superconducting circuit devices based on employing Josephson junctions.

Some new information-entropic inequalities for the states of systems in thermodynamic equilibrium are obtained in [4].

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