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Quantum error correction in multi-parameter quantum metrology

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Abstract

We derive a necessary and sufficient condition for the possibility of preserving the Heisenberg scaling in general adaptive multi-parameter estimation schemes in presence of Markovian noise. In situations where the Heisenberg scaling can be preserved, we provide an efficient numerical algorithm to identify the optimal quantum error correcting (QEC) protocol that yields the best estimation precision. We provide examples of significant advantages offered by joint-parameter QEC protocols that sense all the parameters utilizing a single error-protected subspace over separate-parameter QEC protocols where each parameter is effectively sensed in a separate subspace.

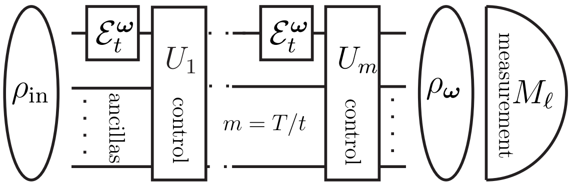

Figure 1: General adaptive mutli-parameter quantum metrological scheme, where P parameters $\boldsymbol{\omega} = [\omega_i]_{i=1}^P$ are to be estimated. Total probe system evolution time T is divided into a number m of t-long steps of probe evolution $\mathcal{E}_t^{\boldsymbol{\omega}}$ interleaved with general unitary controls U_i. In the end a general collective measurement $\{M_\ell\}$ is performed yielding estimated value of all parameters $\tilde{\boldsymbol{\omega}}(\ell)$ with probability $p(\ell) = \mathrm{Tr}(\rho_T^{\boldsymbol{\omega}} M_\ell)$.

Formulation of the model. We assume the dynamics of a d-dimensional probe system is given by a general quantum master equation:

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{k=1}^{r} (L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}), \qquad (1)$$

where the parameters to be estimated $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_P]$ enter linearly into the Hamiltonian of the evolution via Hermitian generators $\boldsymbol{G} = [G_1, \ldots, G_P]^T$ (where ^{T} denotes transpose) so that $H = \boldsymbol{\omega G} \equiv \sum_{k=1}^P \omega_k G_k$, and L_k are operators representing a general Markovian noise. $\mathcal{E}_t^{\boldsymbol{\omega}}$ represents the probe system dynamics integrated over time t, whereas the total probe interrogation time is T.

In multi-parameter case the estimator covariance matrix is the key object capturing estimation precision, defined as:

$$\Sigma_{ij} = \textstyle\sum_\ell \mathrm{Tr}(\rho_{\boldsymbol{\omega}} M_\ell)(\tilde{\omega}_i(\ell) - \omega_i)(\tilde{\omega}_j(\ell) - \omega_j), \qquad (2)$$

for $i,j = 1, \ldots, P$, where the estimator $\tilde{\boldsymbol{\omega}}(\ell)$ is a function mapping the measurement result ℓ to the parameter space, and measurement operator $M_\ell \geq 0$ and $\sum_\ell M_\ell = \mathbb{I}$. As a figure of merit we take $\mathrm{Tr}(W\Sigma)$, where W is a real positive cost matrix that determines the weight we associate with each parameter in the effective scalar cost function $\Delta_W^2 \tilde{\boldsymbol{\omega}} \equiv \mathrm{Tr}(W\Sigma)$.

Theorem 1 Heisenberg scaling for simultaneous estimation of all the parameters can be achieved in a multi-parameter estimation problem if and only if $\{(G_i)_\perp, i = 1, \ldots, P\}$ are linearly independent operators. Here $(G_i)_\perp$ are orthogonal projections of G_i onto space $\mathcal{S}^\perp$ which is the orthogonal complement of the Lindblad span

$$\mathcal{S} = \mathrm{span}_{\mathbb{R}}\{\mathbb{I}, L_k^{\mathrm{H}}, iL_k^{\mathrm{AH}}, (L_k^\dagger L_j)^{\mathrm{H}}, i(L_k^\dagger L_j)^{\mathrm{AH}}, \ \forall j,k\}, \qquad (3)$$

in the Hilbert space of Hermitian matrices under the standard Hilbert-Schmidt scalar product, whereas ^{H}, ^{AH} denote the Hermitian and anti-Hermitian part of an operator respectively.

Theorem 2 Given a cost matrix W, the minimum cost $\Delta_W^2 \tilde{\boldsymbol{\omega}}$ that can be achieved in a join quantum error correcting protocol reads

$$\min \Delta_W^2 \tilde{\boldsymbol{\omega}} = \frac{1}{T^2} \min_{|\chi_i\rangle, G_i^{\mathcal{C}}, B_i, \nu_i} \mathrm{Tr}(WV),$$

$$V_{ij} = \langle \chi_i | \chi_j |, \rangle \ 2\mathrm{Im}\left(\langle \chi_i | \, G_j^{\mathcal{C}} \, |0\rangle\right) = \delta_{ij}, \ C \geq 0,$$

$$C = \frac{\mathbb{I}}{d} + \sum_{i=1}^{P} (G_i^{\mathcal{C}})^T \otimes G_{i\perp} + \sum_{i=P+1}^{P'} \nu_i \mathbb{I} \otimes S_i + \sum_{i=P'+1}^{d^2-1} B_i \otimes R_i, \qquad (4)$$

where $\mathbb{I}/\sqrt{d}$, $\{G_i\}_{i=1}^P$, $\{S_i\}_{i=P+1}^{P'}$, $\{R_i\}_{i=P'}^{d^2-1}$ form an orthonormal basis of Hermitian operators in $\mathcal{L}(\mathcal{H}_S)$ such that $\mathcal{S} = \mathrm{span}_{\mathbb{R}}\{\mathbb{I}, (S_i)_{P+1=1}^{P'}\}$. Moreover, $G_i^{\mathcal{C}}$, B_i are Hermitian matrices in $\mathcal{L}(\mathcal{C})$ where $\mathcal{C}$ is an abstract $P+1$ dimensional code space $\mathcal{C} = \mathrm{span}\{|0\rangle, \ldots, |P\rangle\}$ and $|\chi_i\rangle = \sum_{j=1}^P \alpha_i^j |j\rangle$, $\alpha_i^j \in \mathbb{R}$. The solution of C can be used to define the optimal QEC code.

[1] W. Gorecki, S. Zhou, L. Jiang, and R. Demkowicz-Dobrzanski, *Quantum error correction in multi-parameter quantum metrology*, arxiv:1901.0089 (2019).