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Sensitivity of bargaining solutions to set curvature

Thomas Streck

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Abstract

This paper studies the distortion of the asymmetric Nash and Kalai–Smorodinsky bargaining solutions that arises from differences in the shape of the bargaining set. In a two-person bargaining framework, we compare outcomes across different feasible sets for fixed bargaining power and repeat this for all possible power constellations between the players. Two arbitrary bargaining sets are selected, and the distortion of the asymmetric Nash solution, the symmetric Kalai–Smorodinsky solution, and the Nash solution with a shifted disagreement point is measured relative to the asymmetric Kalai–Smorodinsky solution, which serves as a reference. We compare pairs of bargaining problems that differ only in curvature and analyze how outcomes vary across solutions. This yields a quantitative measure of relative distortion and shows how sensitive bargaining solutions are to changes in set structure. For all solutions under observation, a weaker (stronger) player always prefers the more (less) curved bargaining set, as the distortion increases (decreases) in their favor. Indifference between two sets occurs when the distortion is equal for a given power constellation. For the asymmetric Nash and Kalai–Smorodinsky solutions, indifference occurs exactly when the solution points exhibit the same slope on their respective Pareto frontiers. Finally, the number of indifference points is always odd if one bargaining set contains the other. The results highlight how the shape of the Pareto frontier can introduce additional distortion in already unequal situations, suggesting that the structure of the bargaining set plays a crucial role in determining the fairness of outcomes.

Keywords: Asymmetric bargaining power, Nash bargaining solution, Kalai-Smorodinsky bargaining solution

JEL Classification Numbers: C78, D63

*Department of Economics, Paderborn University, Warburger Str. 100, 33098 Paderborn, Germany, thomas.streck@upb.de

1 Introduction

In bargaining situations that are characterized by asymmetric bargaining power between the negotiating agents, the outcome may strongly depend on how asymmetry is modeled, as it affects the final payoff of each player. Symmetric bargaining solutions of Nash (1950) and Kalai and Smorodinsky (1975), when combined with shifted disagreement points, have sometimes been considered as representations of asymmetric bargaining power. However, recent work by Haake and Streck (2022) has shown that using shifted disagreement points to model asymmetry leads to divergent outcomes compared to asymmetric bargaining solutions, which has important implications for the distribution of payoffs. In addition to these, asymmetric versions of both solutions, as introduced in Kalai (1977) and Dubra (2001), have been proposed to explicitly capture differences in bargaining power. Haake and Streck (2022) also addressed the question of which formalization of asymmetry benefits the weaker party (and which the stronger) by comparing different methods of incorporating asymmetric bargaining power.

Modeling two different approaches to asymmetry may lead to divergent outcomes, which can be interpreted as a distortion arising from the chosen modeling framework. In this paper, "distortion" refers to a divergence in predicted outcomes that arises solely from how asymmetric bargaining power is modeled, rather than from actual differences in preferences or power. We then investigate how this distortion varies when the structure of the bargaining set changes, revealing an additional, internal source of distortion that emerges even when preferences and bargaining power are held constant. The impact on the players' outcomes increases with the degree of distortion, which may result in an outcome perceived as excessively biased by the disadvantaged party, a bias that stems solely from the modeling choice to incorporate asymmetry. This, in turn, can reduce the willingness to engage in future negotiations and should be considered by a social planner when implementing a bargaining solution between unequally powerful players. This perspective is of practical interest in institutional settings, such as centralized wage negotiations or regulated bilateral bargaining environments, where the planner must choose not only the solution concept but also the underlying bargaining framework. Svejnar (1980, 1986), for example, applied the Nash solution to model wage bargaining in the public sector, illustrating how structural modeling assumptions, such as the shape of the bargaining set, can influence both the distributional outcome and the perceived fairness from a planner's point of view.

To illustrate such distortions in practice, one needs to compare two different bargaining situations. In real-life applications, this might occur when an agent considers cooperating with different potential partners to share the gains from joint cooperation. However, these partners give rise to different bargaining sets, resulting from divergent preferences of varying surplus potentials, which shape the respective Pareto curves. While a single bargaining set may allow conclusions based on absolute outcomes, comparing multiple sets of differing size calls for relative outcome measures. An agent would then ask the question whether their relative share in gains was higher in one bargaining situation than in another, assuming a fixed representative bargaining power across scenarios, in order to isolate the effects of the structural differences in the bargaining sets, without being influenced by changes in bargaining power. This does not imply that preferences over bargaining sets in general need to be formed, but rather that in the given bargaining situation, a clear preference exists for the set that offers the player a higher share of the gains.

The question we investigate in this paper is whether one can draw general conclusions about a bargaining set based on its shape, depending on the solution concept applied. The asymmetric KS solution serves as a comparative tool, as it enables quick identification of relative share in gains across differently sized sets. Methodologically, the distortion between two bargaining sets is analyzed across the full range of bargaining power constellations, allowing players to detect their preferred scenario in terms of relative share in gains. Across all solution concepts applied, distortion under highly unequal bargaining power increases with the degree of "concavity" of the bargaining set. In our setting, a high degree of concavity refers to sets with a more outward-curved Pareto frontier compared to a baseline (see Figure 6 for an illustration). In the asymmetric Nash solution, the weaker player benefits from the more outward-curved set, in addition to the advantage that arises from comparing the asymmetric Nash to the asymmetric KS solution (Proposition 1). When comparing two arbitrary (but different) bargaining sets, the slope of the respective Pareto curves immediately indicates which set a player prefers. If players are indifferent between the two sets because their relative share in gains is the same, the corresponding solution points coincide in slope on their respective Pareto frontiers. Given that one bargaining set contains the other, the number of indifference cases is always odd (Corollary 1). For the symmetric KS and the symmetric Nash solution with shifted disagreement points (Propositions 2 & 3), the more outward-curved bargaining set likewise additionally favors the weaker player. However, the location of indifference points depends highly on the specific shapes of the bargaining sets.

The importance of bargaining set structure and power asymmetries is also evident in empirical settings. Banerjee (2020) found the IIA axiom to be violated in settings with asymmetric bargaining power, with stronger effects of contraction as asymmetry increases. Since the contraction is implemented as a cap on one player's individual payoffs, it can be interpreted as a modification of the bargaining set relative to a baseline scenario. Draganska et al. (2010) examined the German market for coffee and found that bargaining power varies with the negotiation partner, just as the overall profitability in different manufacturer–retailer channels. They identified cases in which the relative share in profits may decrease for a party, even though overall profits increase due to a larger pie. This finding directly relates to the present paper's model, which focuses on relative comparisons across differently shaped bargaining sets while deliberately excluding absolute values.

These results illustrate that measured outcomes and power parameters in real markets depend not only on the characteristics of the negotiating party, but also on structural conditions such as the feasible set or implicit payoff constraints. This connects to methodological challenges raised by Chiappori et al. (2012), who showed that almost any observed distribution of payoffs can be rationalized as the outcome of Nash bargaining in the absence of exclusion restrictions, making empirical identification problematic. They found that without clear exclusion restrictions, it is difficult to empirically distinguish between different bargaining models, as multiple bargaining processes can rationalize the same observed outcomes. Chambers and Echenique (2014) offered a theoretical characterization of which observed allocations are consistent with classical bargaining theories, showing that only those allocations that are comonotonic, i.e., that respect the relative preferences of each party, are consistent with classical bargaining models. This result highlights the importance of considering the shape of the bargaining set when assessing the consistency of bargaining outcomes with theoretical models. Cherchye et al. (2011) further demonstrated how difficult it is to empirically distinguish between cooperative and non-cooperative models of household behavior. In their study, they showed that household consumption data could be rationalized through a cooperative Nash bargaining framework, but distinguishing between cooperation and non-cooperation was not straightforward due to the similarities in outcomes that arise under different assumptions. Collectively, these studies emphasize that reliable inference about bargaining processes requires structural clarity about the underlying model. In response to these empirical challenges, this paper aims to identify an internal source of distortion, specifically the structural differences in bargaining sets, such as their curvature, which can lead to divergent outcomes even when preferences and bargaining power are held constant. The results of this paper demonstrate how the structure of bargaining sets, particularly their shape and the asymmetric distribution of power, can significantly impact outcomes. This has practical implications for negotiations where it is important to consider the structure of the bargaining space to achieve more balanced and fair results.

From an experimental perspective, bargaining outcomes can be influenced not only by power or preferences, but also by the structural and geometric properties of the feasible set. For example, Duffy et al. (2021) showed that even moderate liquidity constraints can shift bargaining outcomes away from Nash predictions and closer to proportional solutions like the Kalai–Smorodinsky outcome. This finding reinforces the broader insight that structural conditions, such as the bargaining set's shape, play a crucial role in determining the outcome. Further experimental evidence from Anbarci and Feltovich (2018) observed that in an unstructured bargaining setting, participants respond to their dominant bargaining position as predicted by theory, whereas they under-exploit their power when a 50/50 split remained feasible. Similarly, Feltovich (2019) further detected responsiveness when subjects earned and thus comprehend the disagreement payoff, which sends a direct signal to the participants about their bargaining power. Navarro and Veszteg (2020) found that highly asymmetric disagreement points tend to shift behavior away from equality toward individually rational outcomes, which is in line with responsiveness. However, in such environments, the weaker party often refuses to reach agreement, resulting in the disagreement payoff being realized. This pattern poses a risk to a social planner, especially in distribution channels involving unequally powerful agents. Finally, Takeuchi et al. (2022) presented an experimental design where the pie is endogenously produced and bargaining cannot be solved without the outcomes in the production stage. This setting reflects a situation similar to the one modeled in this paper, as varying production functions lead to differing bargaining opportunities. These experimental results reinforce our theoretical argument by showing how changes in bargaining set structure, especially in terms of asymmetry and concavity, affect outcomes in real-world bargaining situations.

The paper is structured as follows: Section 2 reviews the two-person bargaining model, outlines different approaches to asymmetry, and introduces the comparison of two bargaining sets. Section 3 presents the comparison of relative outcomes under these sets, introduces two distortion curves, and contrasts them with alternative asymmetry models. Section 4 concludes with a discussion of limitations. All proofs of the propositions can be found in the Appendix.

2 The model

We use a classic bargaining theory model to address the following question: Given two bargaining sets and asymmetric bargaining power, which set induces more distortion under different bargaining solutions? For that, we have to go through some standard definitions first, including the Nash and the KS solution, which naturally reduces the question to a two-person bargaining problem.

Let the *two-person bargaining problem* consist of a pair (S, d), such that $S \subseteq \mathbb{R}^2$ is a nonempty, convex, compact, and comprehensive[1] set of feasible payoffs, and $d \in S$ denotes the disagreement point. The disagreement point $d \in S$ describes the players' payoffs in case of no agreement, whereas the set S reflects all possible allocations of payoffs among the two players.[2] Define the utopia point $z(S, d)$ for a bargaining problem (S, d) by $z_i(S, d) := \max\{x_i \,|\, x = (x_1, x_2) \in S, x \geq d\}$ for $i = 1, 2$. Let $\mathcal{B}$ be the set of all two-person bargaining problems, and let $\mathcal{B}_0 \subseteq \mathcal{B}$ denote the subset of problems (S, d) that are normalized such that the disagreement point is $d = (0, 0)$ and the utopia point is $z(S, d) = (1, 1)$. This normalization facilitates comparison by expressing utilities on a relative scale, independent of the original absolute payoffs. It does not entail a loss of generality, as all considered solutions are covariant under affine linear transformations[3] of the bargaining problem. For any $(S, d) \in \mathcal{B}$, we refer to the line segment connecting $(z_1(S, d), d_2)$ and $(d_1, z_2(S, d))$ as the *counter diagonal*, serving as a geometric reference line. In a normalized problem $(S, 0) \in \mathcal{B}_0$, this segment corresponds to the line $x_1+x_2 = 1$, $x \geq 0$, and thus serves as a geometric baseline for the interpretation of relative gain shares.

Allowing for an unequal distribution of bargaining power as a fundamental pillar of a negotiation, asymmetric versions of bargaining solutions are modeled next, following the formalizations introduced in Kalai (1977) and Dubra (2001). For any bargaining problem $(S, d) \in \mathcal{B}$, the asymmetric Nash bargaining solution $N^\alpha(S, d)$, with weight α for player 1 and $1 - \alpha$ for player 2, is determined by

$$N^\alpha(S, d) = \underset{x \in S, x \geq d}{\text{argmax}} (x_1 - d_1)^\alpha (x_2 - d_2)^{1-\alpha}.$$

This maximization problem assumes that x is an individually rational point, meaning that $x \geq d$. The convexity of S and the strict quasiconcavity of the objective function for $\alpha \in (0, 1)$ guarantee that there exists a unique maximizer. Specifically, strict quasiconcavity ensures that the utility function has a single global maximum in convex sets, meaning that the maximizer is unique. For $\alpha \in \{0, 1\}$, the utility function becomes linear, and thus uniqueness may fail due to the absence of strict concavity in these boundary cases. The (symmetric) Nash bargaining solution is denoted by $N = N^{1/2}$ and is given by $N(S, d) = \text{argmax}_{x \in S, x \geq d}(x_1 - d_1)^{1/2}(x_2 - d_2)^{1/2}$. Should S not contain a point x which strictly dominates d, the asymmetric Nash bargaining solution $N^\alpha(S, d)$ is defined to be the unique Pareto optimal point in S that (weakly) Pareto dominates d, in accordance with the convention introduced above.

We now turn to the KS solution. Consider a bargaining problem $(S, d) \in \mathcal{B}$ and a weight parameter $\beta \in [0, 1]$. Define $D^\beta(S, d)$ as the (possibly weakly) Pareto optimal point

[1] That is, $y \in \mathbb{R}^2$ and $y \leq x$ for some $x \in S$, implies $y \in S$.

[2] We restrict attention to individually rational points. If $S \cap \{x \geq d\}$ contains only the disagreement point d, then, by convention, the bargaining solution (when defined) is taken to be d for all $\alpha \in [0, 1]$.

[3] Covariance under affine transformations means that if a bargaining problem (S, d) is transformed linearly, the solution transforms accordingly.

in S that lies on the ray originating at d in the direction of $(\beta z_1(S,d),(1-\beta)z_2(S,d))$. Let

$$\tau^*(\beta) := \max\{\tau \mid d+\tau(\beta(z_1(S,d)-d_1),(1-\beta)(z_2(S,d)-d_2))\in S\},$$

so that $D^\beta(S,d) := d+\tau^*(\beta(z_1(S,d)-d_1),(1-\beta)(z_2(S,d)-d_2))$. The asymmetric KS solution $KS^\beta(S,d)$ with weights $\beta, 1-\beta$, following Dubra (2001), is defined as the unique Pareto optimal point in S that (weakly) Pareto dominates $D^\beta(S,d)$. The symmetric KS solution is denoted by $KS = KS^{1/2}$ and given by

$$KS(S,d) = d+\tau^*\cdot\left(\frac{1}{2}(z_1(S,d)-d_1),\frac{1}{2}(z_2(S,d)-d_2)\right).$$

As the remainder of this paper focuses exclusively on normalized bargaining problems $(S,0)\in\mathcal{B}_0$, we write $KS^\beta(S,0)$ and $KS(S,0)$ whenever applicable, analogously to $N^\alpha(S,0)$ and $N(S,0)$.

Returning to the introductory question, two different bargaining sets have to be compared such that each player exactly knows about her relative share in gains, so that the comparison is made independent of sets' sizes. For that, we do not let S contain a line segment parallel to the axes at its Pareto boundary, i.e., do not allow for weakly Pareto dominant points. This restriction is necessary, as the asymmetric KS solution $KS^\beta(S,0)$ uniquely determines a point on the Pareto frontier corresponding to a fixed share of relative gains only when $D^\beta(S,0) = KS^\beta(S,0)$, i.e., when the point $D^\beta(S,0)$ is Pareto optimal in S. Let $u := KS^\beta(S,0)$ denote the solution point. Then, from player 1's perspective, the asymmetry parameter β corresponds to the ratio $\frac{u_1}{u_1+u_2}$, which allows for a consistent comparison of relative shares across different bargaining sets. Thus, the comparison of relative shares means that we are focusing on how each player's share of the total payoff changes across different sets. Consequently, a fixed β across different bargaining sets identifies a unique direction from the origin, namely the ray with slope $\frac{1-\beta}{\beta}$, but this holds only when $D^\beta(S,d)$ is Pareto optimal in S. By definition, the asymmetric KS solution $KS^\beta(S,0)$ lies on this ray, but this geometric property applies only if the corresponding $D^\beta(S,d)$ has been established as Pareto optimal.

In what follows, we use the asymmetric KS solution as a fixed reference point to compare different representations of bargaining power. We do so because it allows for a consistent mapping of relative gain shares across different feasible sets, regardless of their sizes or shapes. To this end, we define, for a given bargaining set $S\in\mathcal{B}_0$ and fixed parameter $\beta\in(0,1)$, the distortion weight $\alpha_S^\beta\in(0,1)$ such that $N^{\alpha_S^\beta}(S,0) = KS^\beta(S,0)$. This allows us to quantify the amount of bargaining power (in the asymmetric Nash framework) required to reach the same relative share β as under the asymmetric KS solution. Likewise, when modeling asymmetry via a shifted disagreement point, we define $\Delta_S^\beta\in\mathbb{R}^2$ as the disagreement point such that $N(S,\Delta_S^\beta) = KS^\beta(S,0)$ or $KS(S,\Delta_S^\beta) = KS^\beta(S,0)$, depending on the symmetric solution concept applied. In both cases, each mapping from β to α_S^β or Δ_S^β provides a way to represent the same relative share β under different forms of asymmetry, depending on the underlying bargaining set. We now consider two sets $S_1,S_2\subseteq\mathbb{R}^2$ such that $(S_1,0),(S_2,0)\in\mathcal{B}_0$ with $S_1\neq S_2$. Fixing a value $\beta\in(0,1)$ then yields the two solution points $KS^\beta(S_1,0)$ and $KS^\beta(S_2,0)$, both of which represent the same relative gain shares across the two sets. From this point onward, we assume that all bargaining problems are normalized, i.e., $(S,d) = (S,0)\in\mathcal{B}_0$. Assume that the functions g_1 and g_2, which represent the Pareto frontiers of the bargaining sets

S_1 and S_2, are continuously differentiable for all $x_1 \in (0, \varepsilon)$, where $\varepsilon > 0$ is sufficiently small.

Having established this, we now compare the two solutions to other forms of asymmetry, i.e., if and to which extent those solutions indicate different bargaining weights and hence might provide an advantage in relative payoffs for a player towards one bargaining set over the other. Starting with the asymmetric Nash solution, we find the corresponding $\alpha^\beta_{S_1}$ and $\alpha^\beta_{S_2}$ such that $N^{\alpha^\beta_{S_1}}(S_1, 0) = KS^\beta(S_1, 0)$ and $N^{\alpha^\beta_{S_2}}(S_2, 0) = KS^\beta(S_2, 0)$. This relates the baseline weight β to the bargaining weights $\alpha^\beta_{S_1}$ and $\alpha^\beta_{S_2}$, where the asymmetric KS solution with fixed β yields the same solution point as the asymmetric Nash solution with weights $\alpha^\beta_{S_1}$ and $\alpha^\beta_{S_2}$, respectively. Figure 1 illustrates this approach while using the asymmetric KS solution with weight $\beta = 0.7$.

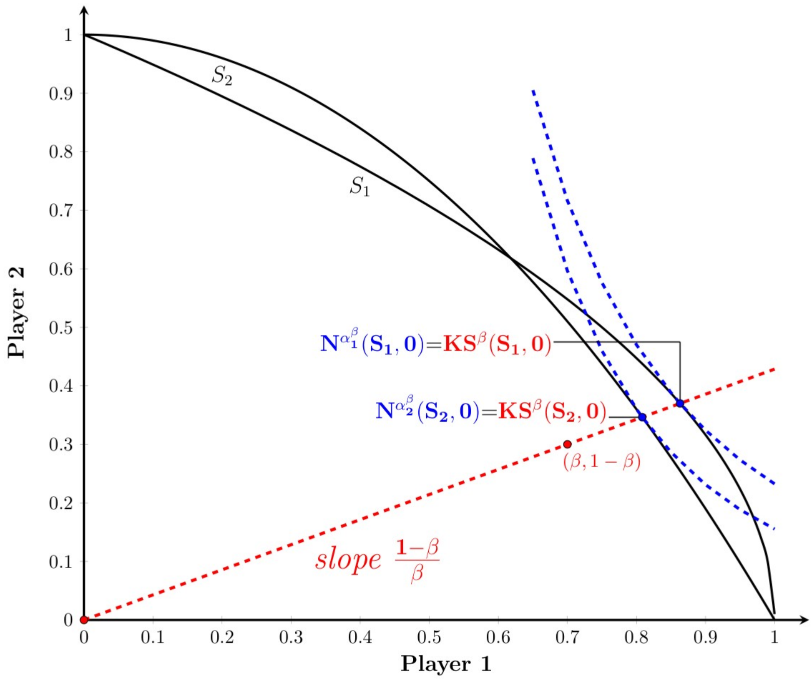

Figure 1: Retrieving $N^{\alpha^\beta_{S_1}}(S_1, 0)$ and $N^{\alpha^\beta_{S_2}}(S_2, 0)$ for $\beta = 0.7$, displayed by $\frac{1-\beta}{\beta}$. The Pareto curve of S_1 is described by the graph of $g_1 : [0, 1] \to [0, 1]$ with $g_1(a_1) = \sqrt{1 - a_1}$, and that of S_2 by $g_2 : [0, 1] \to [0, 1]$ with $g_2(b_1) = 1 - b_1^2$.

In a final step, it can be observed to which extent β translates to α^β_1 and α^β_2. This, in turn, allows us to quantitatively compare α^β_1 with α^β_2.[4] Since the asymmetric KS solution is fixed at the same β, both weights $\alpha^\beta_{S_1}$ and $\alpha^\beta_{S_2}$ correspond to the same relative gain in shares, despite being derived from different bargaining sets. Thus, player 1 would need smaller bargaining power with respect to one bargaining set compared to the other to receive a certain relative share in gains, which she would clearly prefer. More specifically, should $\alpha^\beta_{S_1} < \alpha^\beta_{S_2}$, the set S_1 is more promising for player 1 in that case, as less bargaining power in the asymmetric Nash scenario is needed to reach that certain relative share

[4]Please note at this point that the relation between some α and β already quantifies distortion between two different methods to model asymmetry, and provides general implications to the players which method to prefer (cf. Haake and Streck (2022)). Here, the focus explicitly is on two comparable bargaining solutions with the same β and its implications for a player, when two different bargaining sets are observed.

β. This situation is illustrated in Figure 1 at a given $\beta = 0.7$, where $\alpha^{\beta}_{S_1} = 0.7593 < 0.7905 = \alpha^{\beta}_{S_2}$. If $\alpha^{\beta}_{S_1} = \alpha^{\beta}_{S_2}$, player 1 (and player 2 as well) would be indifferent over the two bargaining sets under observation for that given β, when the asymmetric Nash solution is applied. We write $S_1 \succ_i S_2$ if player i prefers the bargaining set S_1 over S_2 in terms of the relative gain obtained under the solution concept under consideration. Conversely, $S_1 \prec_i S_2$ denotes that player i prefers S_2 over S_1. If the player is indifferent between the two sets, we write $S_1 \sim_i S_2$. In summary, if $\alpha^{\beta}_{S_1} < \alpha^{\beta}_{S_2}$, then $S_1 \succ_1 S_2$, whereas $S_1 \prec_2 S_2$, and vice versa.

Considering another possibility to express bargaining power, one could shift the disagreement point to $\Delta = (\Delta_1, 1-\Delta_1) = (\beta z_1(S,0), (1-\beta) z_2(S,0))$, and interpret this point as a guaranteed share of the maximally possible payoff, when $(S,0) \in \mathcal{B}_0$. Since Δ is typically not Pareto optimal in S, the portion of the set above Δ, i.e., the region that exceeds Δ in the set S, can be interpreted as the surplus to be distributed. Applying a symmetric bargaining solution to (S, Δ) thus constitutes an alternative way to model asymmetry. Having expressed asymmetric bargaining power via the disagreement point Δ, we apply the symmetric version of either the KS or the Nash solution to the surplus part of the set above Δ, which remains to be distributed among the players. With respect to the (symmetric) *KS solution with shifted disagreement point*, find $KS(S_1, \Delta) = KS^{\beta}(S_1, 0)$ and $KS(S_2, \Delta) = KS^{\beta}(S_2, 0)$ to relate the necessary bargaining power in $\Delta^{\beta}_{S_1}$ to the one in $\Delta^{\beta}_{S_2}$ to obtain equal share in gains. With Δ being on the counter diagonal in $\mathcal{B}_0$, the utopia point $z(S, \Delta)$ to calculate some $KS(S, \Delta)$ satisfies $z(S, \Delta) \leq z(S, d) = (1, 1)$. By the same procedure, a relation between $\Delta^{\beta}_{S_1}$ and $\Delta^{\beta}_{S_2}$ can be established when using the (symmetric) *Nash solution with shifted disagreement point*, i.e., find $N(S_1, \Delta) = KS^{\beta}(S_1, 0)$ and $N(S_2, \Delta) = KS^{\beta}(S_2, 0)$. In both cases, whenever $\Delta^{\beta}_{S_1} = \Delta^{\beta}_{S_2}$, a player is indifferent between the two bargaining sets under observation. However, if $\Delta^{\beta}_{S_1} \neq \Delta^{\beta}_{S_2}$, a player prefers the bargaining set, in which β translates to a quantitatively bigger Δ (that is, Δ_1 for player 1 or $1 - \Delta_1$ for player 2), as this is tantamount to less bargaining power Δ needed to obtain the same share in gains. Again, if $\Delta^{\beta}_{S_1} < \Delta^{\beta}_{S_2}$, then $S_1 \succ_1 S_2$, whereas $S_1 \prec_2 S_2$, and vice versa (with $S_1 \sim S_2$ for both players if $\Delta^{\beta}_{S_1} = \Delta^{\beta}_{S_2}$).

In the analysis, we proceed by identifying the relations between α and β, or between Δ and β, for both the KS and the Nash solution applied to shifted disagreement points, across two bargaining sets S_1 and S_2. We always treat β as a reference value on the solution side, corresponding to a fixed relative share of gains. This allows us to accommodate situations where the Pareto frontier contains non-differentiable points, such as those arising in bargaining problems over finitely many goods. Consequently, the distortion function identifies, for each bargaining set, the bargaining weight α (or disagreement point Δ) that yields the same solution as KS^{β}. Otherwise, multiple different weights α (or Δ, particularly when the Nash solution is used) could produce the same Nash solution point, each corresponding to the same fixed β.

3 Results

Comparing the asymmetric KS solution with alternative ways of modeling asymmetry, such as the asymmetric Nash solution or a shifted disagreement point, allows us to quantify the distortion that results from obtaining the same bargaining outcome via different modeling approaches to asymmetry (cf. Haake and Streck (2022)). While fixing a specific degree of asymmetry in one version, the corresponding degree in the other method, which yields the same payoff, has to be retrieved. When comparing the distortion of two different bargaining sets in a next step, consider again the use of the asymmetric KS solution as a reference value for the other methods to apply asymmetry, as it is crucial that it coincides with the relative share in gains obtained in this setting. Thus, moving along the Pareto frontier of a normalized bargaining set from top left to bottom right corresponds to increasing values of $\beta \in (0, 1)$. For each Pareto optimal point x with $x \in S$ and $x > d = 0$, there exists a unique $\beta \in (0, 1)$ such that $\beta = \frac{x_1}{x_1+x_2}$. Hence, the mapping between β and the Pareto frontier is bijective in $\mathcal{B}_0$. Therefore, two solution points on the two different bargaining sets that yield the same value β represent the same share a player would obtain in both scenarios. In other words, using β as a reference for the other methods to apply asymmetry allows us to compare the values of the weights α (or Δ, respectively) that would result in those two solution points on the two different bargaining sets. We know for each β, i.e., for each share from the total payoffs obtained, which bargaining set results in a more favorable outcome for each player under the asymmetric solution concept under consideration. In particular, the distortion mappings indicate how the weight in the asymmetric solution translates into the corresponding relative share β. If there is less bargaining power in the one set needed to obtain β than in the other, and therefore a lower weight α (or weight Δ), it will naturally be the preferred option for a player. In the analysis, we will first prove specific properties of the distortion mappings, such as monotonicity and the implications for player-specific preferences over bargaining sets. Then, we will focus on two normalized bargaining sets, where S_1 is a proper subset of S_2, i.e., $S_1 \subsetneq S_2$.

3.1 Comparing distortion between the asymmetric Nash and KS solutions

In a first step, we discuss the distortion mappings that relate the weights α and β for the asymmetric Nash and KS solutions leading to the same solution point on the respective Pareto boundaries of each bargaining set. Each distortion mapping Λ_i (for $i = 1, 2$) reflects the relationship between α and β from the perspective of one player (here: player 1) for the respective bargaining set, either S_1 or S_2. Generally, for each $(S, 0) \in \mathcal{B}_0$, define the mapping $\Lambda^S : [0, 1] \to [0, 1]$, where each $\alpha \in [0, 1]$ is assigned to a value $\Lambda^S(\alpha) = \beta$, so that $N^\alpha(S, 0) = KS^{\Lambda^S(\alpha)}(S, 0)$, provided there are no weak Pareto optimal points in S. From this point onward, we omit the superscript S when referring to Λ (or any other distortion mapping) and simply write Λ_1 and Λ_2 to denote the mappings derived from S_1 and S_2, respectively.

The left part of Figure 2 depicts the distortion mappings Λ_1 and Λ_2, which are derived from the two introductory bargaining sets S_1 and S_2, respectively (cf. Figure 1). The mappings show how the bargaining power, represented by α, relates to the relative share β for player 1 in each bargaining set. The *identity line* (dashed diagonal) marks the points where the Nash and KS solutions coincide for the same weight α, representing an equilibrium where player 1 is indifferent between the two solutions. For values of α below this line, player 1 prefers the asymmetric KS solution, while for values above, the asymmetric Nash solution provides more favorable outcomes. In both cases, the mappings lie above the identity line for small values of α, indicating that the asymmetric Nash solution is preferred over the asymmetric KS solution for players with low bargaining

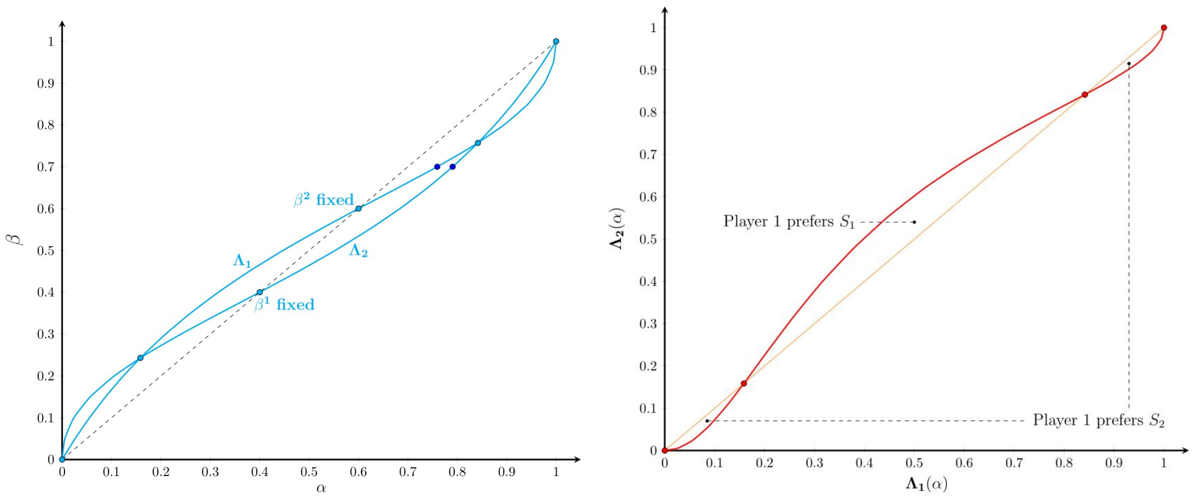

Figure 2: Left: Distortion mappings Λ_1 and Λ_2 of α to β for bargaining sets S_1 and S_2 (cf. Figure 1). Right: Comparison of distortion mappings $\Lambda_1(\alpha)$ and $\Lambda_2(\alpha)$ across the two sets.

weights. The identity line also marks the threshold where the preference between the two solutions reverses. For each mapping Λ_i, there exists a unique point β^i such that $\Lambda_i(\beta^i) = \beta^i$. Beyond this point, the mapping Λ_i lies below the identity line, which means that for sufficiently high bargaining weights $\alpha > \beta^i$, the asymmetric KS solution yields more favorable outcomes than the asymmetric Nash solution. This observation is fully in line with Proposition 3 in Haake and Streck (2022).[5]

The behavior of $\Lambda_1(\alpha)$ and $\Lambda_2(\alpha)$ already differs for values of α close to zero, i.e., on the Pareto frontier near the point $(0, 1)$. This has significant implications for both players when applying the asymmetric Nash bargaining solution. In such contexts, a player may be interested in which of the two sets requires a smaller bargaining weight α to achieve a given relative share β. Since the weight α reflects the bargaining power required to attain a particular outcome, a lower value indicates that less bargaining power is needed in the given bargaining set to achieve the same relative share. As can be observed from the distortion mappings $\Lambda_1(\alpha)$ and $\Lambda_2(\alpha)$, a higher position of the curve indicates that a smaller bargaining weight α is required to achieve a given relative share β. This is particularly relevant for small values of β, i.e., outcomes close to the point $(0, 1)$ on the Pareto frontier. Phrased differently, fixing some $\beta \in (0, 1)$, the comparison of α_1^β and α_2^β, that is, the weights such that $N^{\alpha_1^\beta}(S_1, 0) = KS^\beta(S_1, 0)$ and $N^{\alpha_2^\beta}(S_2, 0) = KS^\beta(S_2, 0)$, reveals which of the two sets requires less bargaining power to replicate the same KS outcome. Hence, the smaller α_i^β indicates the more favorable set for that particular β. Starting from small values of β close to zero (near the point $(0, 1)$ on the Pareto frontier), we observe that $\alpha_1^\beta > \alpha_2^\beta$, meaning that less bargaining power is needed in S_2 to replicate the same outcome under the asymmetric Nash solution. The right part of Figure 2 illustrates this comparison from player 1's perspective, showing for each $\beta \in [0, 1]$ which of the two sets requires a smaller value of α to reach the corresponding KS solution. This information is directly connected to the player's preference over the one or the other

[5] Proposition 3 in Haake and Streck (2022) shows that the distortion mapping Λ has at most one intersection point with and lies strictly below the identity line beyond that point, implying that for sufficiently large β, the KS solution yields higher payoffs than the Nash solution.

bargaining set for some β, whereas player 2 with bargaining power of $1-\beta$ would naturally prefer the opposite.

The fact that $\Lambda_2(\alpha)$ lies above $\Lambda_1(\alpha)$ for sufficiently small values of α can be traced back to the different slopes of the two Pareto boundaries. Let g_i denote the Pareto frontier of the bargaining set S_i for $i = 1, 2$, and let g_i'[6] refer to its slope at the point where the ray from the origin d with slope $\frac{1-\beta}{\beta}$ intersects g_i. Following such a ray, we find that $g_1' < g_2'$ implies $\Lambda_1(\alpha) < \Lambda_2(\alpha)$ in this region. This geometrically explains why less bargaining power is needed in S_2 to achieve the same relative share β under the asymmetric Nash solution, compared to S_1. The geometric configuration of the frontiers thus directly determines the relation between the distortion mappings Λ_1 and Λ_2.

Proposition 1.
Let $(S_1, 0), (S_2, 0)$ be two arbitrary bargaining problems with $S_1 \neq S_2$. For all $\alpha \in (0, 1)$, let $\beta_i := \Lambda_i(\alpha)$ for $i = 1, 2$, and let $x_i^{\beta_i}$ be the point where the ray from the origin with slope $\frac{1-\beta_i}{\beta_i}$ intersects the graph of g_i, the Pareto frontier of S_i, which is weakly Pareto optimal. Let $g_i'(x_i^{\beta_i})$ denote the slope of g_i at that point. Then:

$$\Lambda_1(\alpha) \gtreqless \Lambda_2(\alpha) \qquad \textit{if and only if} \qquad g_1'(x_1^\beta) \gtreqless g_2'(x_2^\beta).$$

The idea behind this proposition is quite intuitive and can be easily explained by assuming one of the two bargaining sets to have a linear Pareto frontier connecting $(0, 1)$ and $(1, 0)$, and therefore coinciding with the counter diagonal. Not only would the slope be consistently equal to -1, but also the resulting hyperplane game would indicate no distortion for the associated mapping of Λ. Any other feasible bargaining set being unequal to this baseline scenario requires the slope be strictly greater than -1 in a neighborhood of $(0, 1)$, triggering at least some distortion of the corresponding Λ. To reach $(1, 0)$, however, the slope should be strictly lower than -1 at some point. Being more distant from this baseline scenario is accompanied by more distortion. This geometric deviation translates into a more favorable distortion for the player with low bargaining weight, since a small α translates into more than the same value of β. This reasoning reverses for sufficiently large bargaining power reaching solution points close to $(1, 0)$. The corresponding distortion mapping Λ may exhibit greater curvature than its baseline counterpart, but it attains smaller values since the slope of the Pareto frontier must be substantially lower, specifically, strictly less than -1. Indifference between the two bargaining sets arises at the intersection of the distortion curves, which occurs precisely along the identity line where $\alpha = \beta$, that is, when the slopes of both Pareto frontiers coincide.

Coming back to the introductory example depicted in Figure 1, close to $(0, 1)$, the Pareto frontier of S_2 induces the higher distortion curve $\Lambda_2(\alpha)$ when comparing the asymmetric KS and Nash bargaining solutions, which is favorable to players with low bargaining weight, since $\Lambda_2(\alpha) > \Lambda_1(\alpha)$. This pattern holds until the curves intersect, i.e., when $g_1'(x) = g_2'(x)$. Let the Pareto frontier of S_1 be given by $g_1(x) := \sqrt{1-x}$ and that of S_2 by $g_2(x) := 1 - x^2$, both defined on the interval $x \in [0, 1]$. The equality of their derivatives, and therefore the intersection $\Lambda_1(\alpha) = \Lambda_2(\alpha)$, occurs at $\beta = \{0.243, 0.757\}$. This transition between the two solutions can be seen due to the marked dots in the right panel of Figure 2, where the two distortion curves cross the identity line at these

[6]We assume that the Pareto frontiers g_1 and g_2 are differentiable at the relevant points of intersection with the ray of slope $\frac{1-\beta}{\beta}$ in order to define the respective slopes g_1' and g_2'.

points. The intuition behind this result lies in the homotheticity of the asymmetric Nash bargaining solution, which naturally results in $\alpha_1^\beta = \alpha_2^\beta$ whenever the respective slopes at g_1 and g_2 coincide. For $0.243 < \beta < 0.757$, the slope of the Pareto frontier g_1 at the relevant point is steeper than that of g_2, i.e., $g_1'(x) > g_2'(x)$, implying that $\Lambda_1(\alpha) > \Lambda_2(\alpha)$. In this range, the bargaining set S_1 is therefore more favorable to player 1, as it yields a higher relative share β under the asymmetric Nash solution for a given bargaining weight α. This is consistent with the illustrative case $\beta = 0.7$ shown in Figure 1 and also marked in the left part of Figure 2. For $\beta > 0.757$, the order reverses: $\Lambda_1(\alpha) < \Lambda_2(\alpha)$, and the bargaining set S_2 again becomes more favorable to player 1 for the given weight α.

3.2 Comparing distortion when applying asymmetry via a shifted disagreement point

Interpreting asymmetric bargaining power as a fallback position of the negotiating partners shifts the disagreement point and might be understood as the actual starting point of a negotiation. In this subsection, we start with $S \in \mathcal{B}_0$ and shift the disagreement point to the counter diagonal between $(1, 0)$ and $(0, 1)$ at $\Delta = (\Delta_1, 1 - \Delta_1)$ and the symmetric versions of the KS and the Nash bargaining solutions are applied. Technically, the asymmetry follows the ray of the asymmetric KS solution with slope $\frac{1-\beta}{\beta} \cdot \frac{z_2(S,0)}{z_1(S,0)}$ from $d = 0$ until it intersects the counter diagonal. While for the asymmetric KS solution, the ray has to be followed unopposedly until the intersection of the Pareto frontier, the bargaining solutions $KS(S, \Delta)$ and $N(S, \Delta)$ aim to attenuate the progression of the former. It represents a concession to the "weaker" players, potentially encouraging more agreements to be reached. Intuitively, a player with sufficiently small bargaining weight would therefore always prefer the application of the symmetric bargaining solution with shifted disagreement point over the unhindered application of bargaining power, which comes along with the use of the asymmetric KS solution. When it comes to comparing the two applications to the asymmetric KS solution via $KS^\beta(S, 0) = KS(S, \Delta)$ or $KS^\beta(S, 0) = N(S, \Delta)$, respectively, the former intuition is tantamount to requiring less bargaining power Δ to achieve a certain relative share β, when the player is weak. This dampening effect is amplified at the edges of the Pareto frontier, e.g. near $(0, 1)$, the further a bargaining set initially deviates from the baseline set that coincides with the counter diagonal.

Starting with the KS application, we define distortion as a bijective mapping $\Phi^S : \mathcal{D}_S \subset \mathbb{R}^2 \to [0, 1], \quad \Delta \mapsto \Phi^S(\Delta) =: \beta_\Delta$, implicitly defined by the condition that $KS(S, \Delta) = KS^{\beta_\Delta}(S, 0)$ for each normalized bargaining problem $(S, 0) \in \mathcal{B}_0$. For example, if $\Delta_1 \in \mathcal{D}_S$, then we write $\beta_{\Delta_1} := \Phi^S(\Delta_1)$. Since Φ^S assigns to each disagreement point Δ a unique value β_Δ representing player 1's relative share, non-differentiable points along the Pareto frontier are admissible without affecting the construction.

Figure 4 shows this distortion between the asymmetric KS solution and the (symmetric) KS solution with shifted disagreement point for the introductory example (cf. Figure 1). The progression of both curves Φ_1 and Φ_2 matches the previous intuition, which is in line with Proposition 1 in Haake and Streck (2022): They show that the "weaker" player prefers application of the KS solution with shifted disagreement point, while the opposite is true for the player having a sufficiently strong bargaining position. This point of indifference, indicated by the dashed diagonal where $\Phi_i(\Delta_1) = \Delta_1$, occurs exactly once for each mapping Φ_1 and Φ_2. Starting in the neighborhood of $\Delta = (0, 1)$,

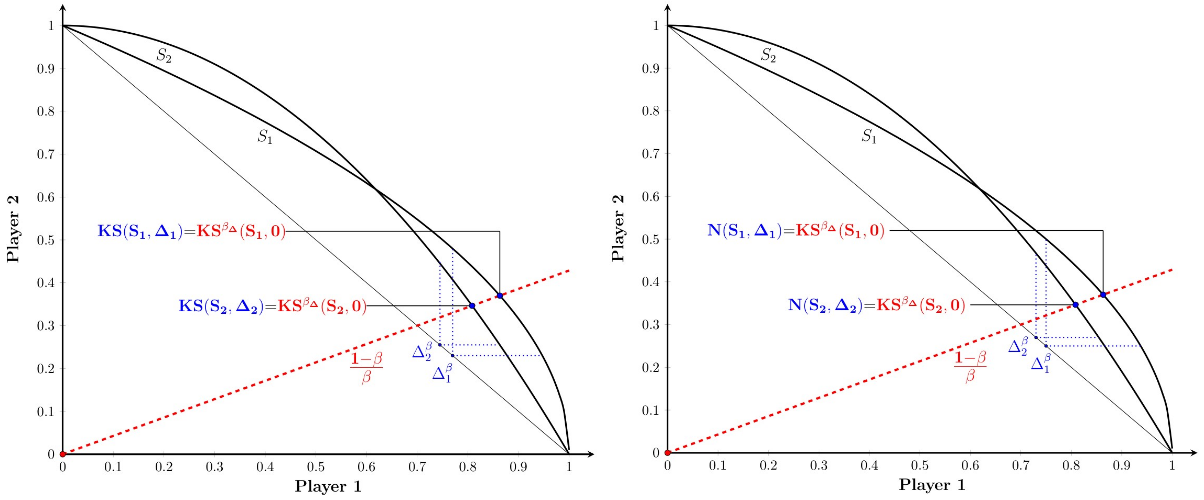

Figure 3: Left: Retrieving $KS(S_1, \Delta_1^\beta)$ and $KS(S_2, \Delta_2^\beta)$ for $\beta = 0.7$, where the points Δ_1^β and Δ_2^β on the counter diagonal are chosen such that $KS(S_i, \Delta_i^\beta) = KS^{\beta\Delta}(S_i, 0)$ for $i = 1, 2$. Right: Analogous depiction for $N(S_1, \Delta_1^\beta)$ and $N(S_2, \Delta_2^\beta)$. The Pareto frontiers of S_1 and S_2 correspond to those in Figure 1.

$\Phi_2(\Delta) > \Phi_1(\Delta)$, since the corresponding subgame that leads to $KS(S_2, \Delta)$ causes the utopia point $z(S_2, \Delta)$ to move significantly further to the right than $z(S_1, \Delta)$, due to its absolute flatter slope. Close to $\Delta = (1, 0)$, the “stronger” player would not benefit from the dampening effect of the KS solution with shifted disagreement point, so that set S_2, being closer to the baseline scenario with no distortion, is less distorted and consequently, $\Phi_2(\Delta) > \Phi_1(\Delta)$ again. For $\beta = \{0.3555, 06445\}$, $\Phi_1(\Delta) = \Phi_2(\Delta)$, so that in the specifically illustrated introductory example of $\beta = 0.7$, $\Phi_2(\Delta) > \Phi_1(\Delta)$ still holds (here, it would be $KS^{\beta\Delta}(S_1, 0) = KS(S_1, \Delta) > KS^{\beta\Delta}(S_2, 0) = KS(S_2, \Delta)$). The comparison of Φ_1 and Φ_2 with respect to which bargaining sets grants a player the greater relative share of the total amount to be distributed can be formulated with generality at sufficiently unequally distributed bargaining powers.

Proposition 2. *Let $(S_1, 0), (S_2, 0) \in \mathcal{B}_0$, where $S_1 \neq S_2$. Consider disagreement points Δ, starting from $(0, 1)$ (or from $(1, 0)$), moving along the counter diagonal toward $(1, 0)$ (or $(0, 1)$, respectively). Then $\Phi_1(\Delta) < \Phi_2(\Delta)$ holds if and only if the slope of the Pareto frontier of S_1 at the point associated with Δ is strictly smaller than that of S_2, i.e., $g_1'(\Delta) < g_2'(\Delta)$. This relation holds until the first point where $g_1'(\Delta) = g_2'(\Delta)$ is reached, at which point the difference between the two functions, $\Phi_1(\Delta)$ and $\Phi_2(\Delta)$, starts to decrease.*

The proposition resembles the former to a certain degree, though it cannot be generally specified at which point $\Phi_1(\Delta) = \Phi_2(\Delta)$ is reached, as it highly depends on the shapes of the two Pareto frontiers under observation. Nevertheless, knowledge about the preference for the more (less) distorted set for smaller (greater) bargaining weights already sheds light on how the shape of the Pareto frontier influences the players' preference over the one or the other bargaining set, when it comes to relative comparison.

With regard to observing the distortion that occurs in the comparison between applying the asymmetric KS solution and the (symmetric) Nash solution with shifted disagreement point $N(S, \Delta)$, a reasonable justification for its comparability might be missing at first glance. The distortion between $KS^\beta(S, 0)$ and $KS(S, \Delta)$ leads weak players to prefer

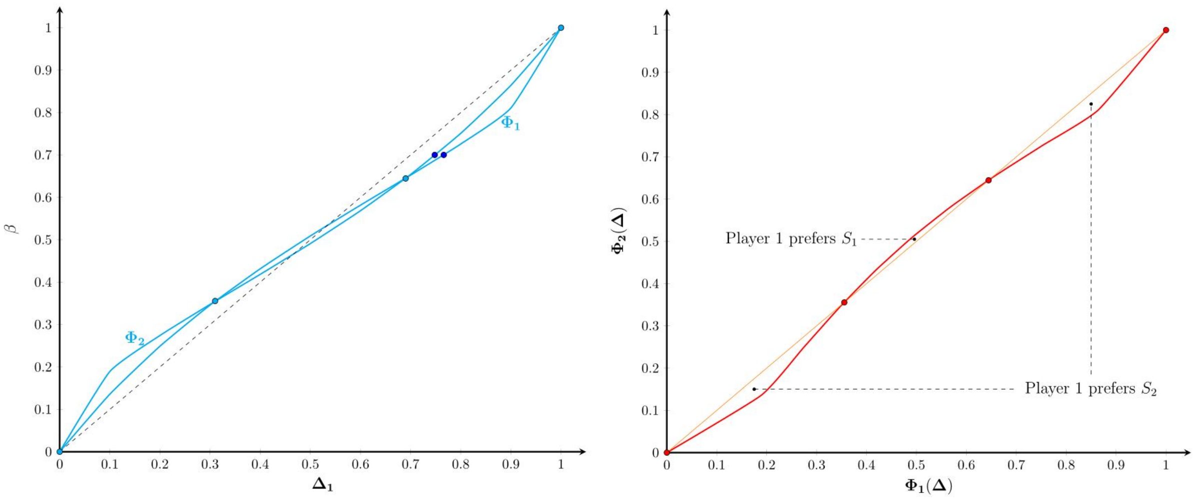

Figure 4: Left: Distortion Φ_1 and Φ_2 of Δ_1 to β with bargaining sets S_1 and S_2 from Figure 3. Right: Relating distortion of $\Phi_1(\Delta)$ to $\Phi_2(\Delta)$.

application of $KS(S,\Delta)$, while $N^\alpha(S,0)$ is the obvious favorite between applying the former or $N(S,\Delta)$. The comparison of $KS^\beta(S,0)$ or $N(S,\Delta)$ therefore is the identification of the least (most) desirable application of all four mentioned possibilities for sufficiently small (large) bargaining weights. Comparing two different bargaining sets then examines to which degree the shape of the Pareto frontier increases distortion in this particular event. Here, the distortion is described by $\Upsilon^S : [0,1] \to [0,1], \Delta \mapsto \Upsilon^S(\Delta) =: \beta_\Delta$, implicitly defined by $N(S,\Delta) = KS^{\beta_\Delta}(S,0)$ for each $(S,0) \in \mathcal{B}_0$.

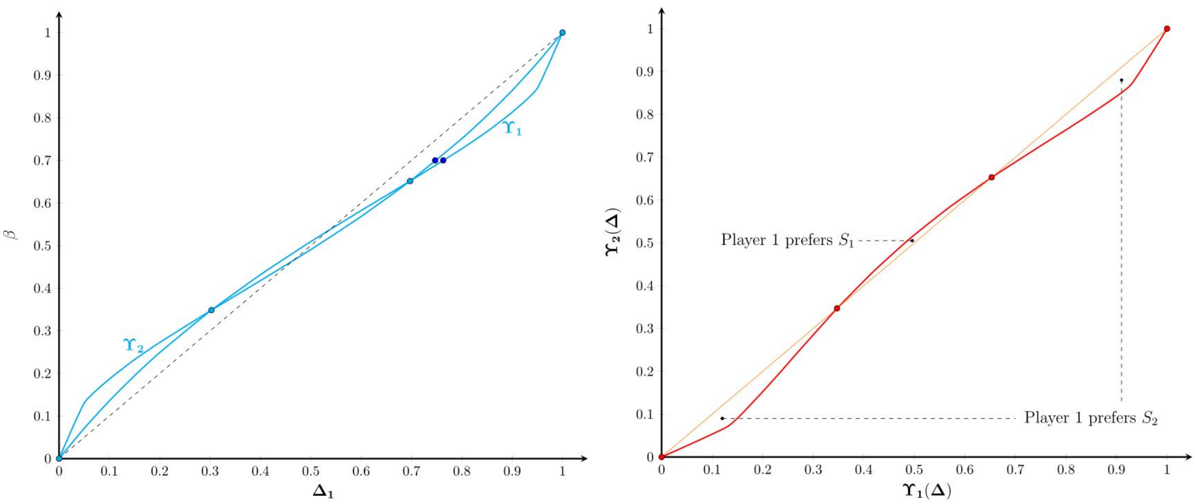

Figure 5: Left: Distortion Υ_1 and Υ_2 of Δ_1 to β with bargaining sets S_1 and S_2 from Figure 3. Right: Relating distortion of $\Upsilon_1(\Delta)$ to $\Upsilon_2(\Delta)$.

The progression of both Υ_1 and Υ_2 in Figure 5 closely mirrors the behavior observed for Φ_1 and Φ_2 in Figure 4, when the introductory example is observed. For small bargaining weights, a weak player would prefer application of the Nash solution with shifted disagreement point, since a sufficiently small Δ_1 translates into more than the same value of β. It becomes evident that when one assumes linearity of both Pareto boundaries, e.g. in the neighborhood of $(0,1)$, $N(S,\Delta)$ would coincide with $KS(S,\Delta)$.

Proposition 3. *Let $(S_1, 0), (S_2, 0) \in \mathcal{B}_0$, where $S_1 \neq S_2$. Consider disagreement points Δ, starting from $(0, 1)$ (or from $(1, 0)$), moving along the counter diagonal toward $(1, 0)$ (or $(0, 1)$, respectively). Then $\Upsilon_1(\Delta) < \Upsilon_2(\Delta)$ holds if and only if the slope of the Pareto frontier of S_1 at the point associated with Δ is strictly smaller than that of S_2, i.e., $g_1'(\Delta) < g_2'(\Delta)$. This relation holds until the first point where $g_1'(\Delta) = g_2'(\Delta)$ is reached, at which point the difference between the two functions, $\Upsilon_1(\Delta)$ and $\Upsilon_2(\Delta)$, starts to decrease.*

In the neighborhood of $(0, 1)$, the boundary of S_2 runs rather flat in comparison to the boundary of S_1 such that the value of g_1' is greater than the value of g_2', which increases distortion and makes S_2 even more desirable when the Nash solution with shifted disagreement point is applied. Close to $(1, 0)$, the preferences again reverse, so that it is the less distorted function, here $\Upsilon_2(\Delta)$, whose Pareto frontier features the less steep slope and consequently, yields the greater relative share to the player if $N(S, \Delta)$ is applied. As shown in Proposition 3, the preference between the two bargaining sets depends on the relative slopes of their Pareto frontiers, particularly near $(0, 1)$ and $(1, 0)$. At $(0, 1)$, when the slope of S_1's frontier is steeper than that of S_2, the distortion function $\Upsilon_1(\Delta)$ will be greater than $\Upsilon_2(\Delta)$, making S_1 more favorable for a weak player. As the disagreement point moves towards $(1, 0)$, the slopes of the two frontiers become closer, and eventually $g_1'(\Delta)$ will equal $g_2'(\Delta)$. From that point onward, $g_1'(\Delta)$ will become smaller than $g_2'(\Delta)$, and the difference between the two functions, $\Upsilon_1(\Delta)$ and $\Upsilon_2(\Delta)$, will start to diminish. This gradual shift indicates that S_2 becomes more favorable for the weak player as the slopes of the Pareto frontiers change.

The choice of asymmetry modeling, particularly with respect to how each bargaining set is shaped, plays a significant role in determining how these preferences evolve. In bargaining situations with highly unequal bargaining positions, this effect becomes more pronounced, as the asymmetry in the bargaining power is reflected more strongly in the resulting distortion functions. The steepness of the Pareto frontier in such extreme cases amplifies the differences in outcomes, especially when the solution method involves a shifted disagreement point, thereby affecting which bargaining set is preferred under different conditions.

3.3 Characterization of nested bargaining sets

Insights into the behavior of the distortion curves near the extremal points $(0, 1)$ and $(1, 0)$ already provide useful information on how these curves evolve further away from the endpoints. For example, when comparing the asymmetric KS with the asymmetric Nash solution in the introductory example (cf. Figure 1), we observe that $\Lambda_2 > \Lambda_1$ for values of β close to 0 and close to 1. This suggests that the distortion mappings intersect an even number of times. Indeed, the distortion curves intersect twice at $\beta = \{0.243, 0.757\}$, and within this interval, $\Lambda_1(\alpha) > \Lambda_2(\alpha)$ holds, consistent with $g_1'(\beta) > g_2'(\beta)$ in that region.

However, this pattern does not persist when S_1 is a proper subset of S_2, i.e., $S_1 \subsetneq S_2$. To formalize this, assume that both sets have differentiable Pareto boundaries and contain no linear segments parallel to the counter diagonal. This condition ensures that the distortion mappings cannot coincide over an entire interval without being identical, so that any crossing point is isolated. Figure 6 illustrates this using two representative sets: $g_1(x) = 1 - x^2$ for S_1 and $g_2(x) = 1 - x^3$ for S_2. The dashed ray with slope $\frac{1-\beta^*}{\beta^*} = \frac{4}{3}$

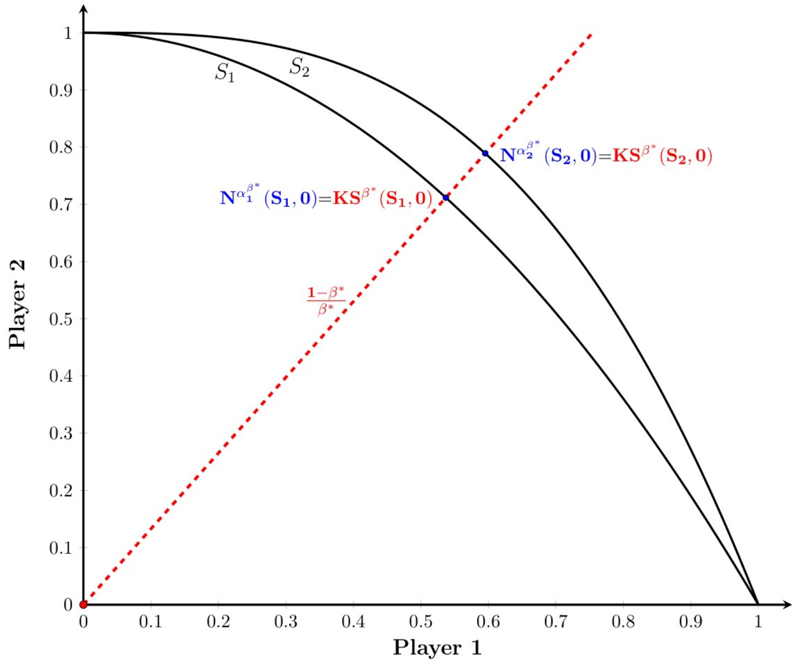

Figure 6: Retrieving $N^{\alpha_1^{\beta^*}}(S_1, 0)$ and $N^{\alpha_2^{\beta^*}}(S_2, 0)$ for $\beta^* = \frac{3}{7}$, corresponding to the ray with slope $\frac{1-\beta^*}{\beta^*} = \frac{4}{3}$. The Pareto boundaries are given by $g_1(x) = 1 - x^2$ for S_1 and $g_2(x) = 1 - x^3$ for S_2.

corresponds to the unique case where $\alpha_1^{\beta^*} = \alpha_2^{\beta^*}$, which implies that $g_1'(\beta^*) = g_2'(\beta^*)$.[7] It follows that $\Lambda_2 > \Lambda_1$ for $\beta < \frac{3}{7}$, as the slope of g_2 decreases more gradually near the top left corner compared to g_1, and $\Lambda_2 < \Lambda_1$ for $\beta > \frac{3}{7}$. The right panel of Figure 7 therefore indicates a clear preference of player 1 for S_2 at small bargaining weights, and for S_1 when endowed with sufficiently strong bargaining power.

In general, if $S_1 \subsetneq S_2$, the slope of the Pareto frontier of S_2 is flatter than that of S_1 for small values of β, while the opposite holds for large β. As a consequence, the distortion mapping Λ_2 lies above Λ_1 near $(0, 1)$ and below it near $(1, 0)$. This implies that S_2 is initially preferred by player 1 for small bargaining weights, while S_1 becomes more favorable for larger weights. This monotonic shift in relative preference yields a predictable pattern in the number of intersection points of the distortion mappings. Under mild assumptions, this relation holds for all nested bargaining sets, as captured in the following corollary:

Corollary 1. *Let $(S_1, 0), (S_2, 0) \in \mathcal{B}_0$ be two normalized bargaining problems with differentiable Pareto frontiers g_1 and g_2 that do not contain any linear segment parallel to the counter diagonal. If $S_1 \subsetneq S_2$, then the number of intersection points between $\Lambda_1(\alpha)$ and $\Lambda_2(\alpha)$ is odd.*

The assumption that the Pareto frontier contains no linear segment parallel to the counter diagonal rules out cases in which both players are indifferent over an entire interval of proportional allocations. Such configurations may arise in bargaining problems

[7]Since $g_1'(x) = -2x$ and $g_2'(x) = -3x^2$, equality $g_1'(x) = g_2'(x)$ holds if and only if $x = 0$ or $x = \frac{2}{3}$. The point $x = \frac{2}{3}$ is the only interior solution where the slopes coincide, which corresponds to a unique ray with slope $\frac{1-\beta^*}{\beta^*} = \frac{4}{3}$, i.e., $\beta^* = \frac{3}{7}$.

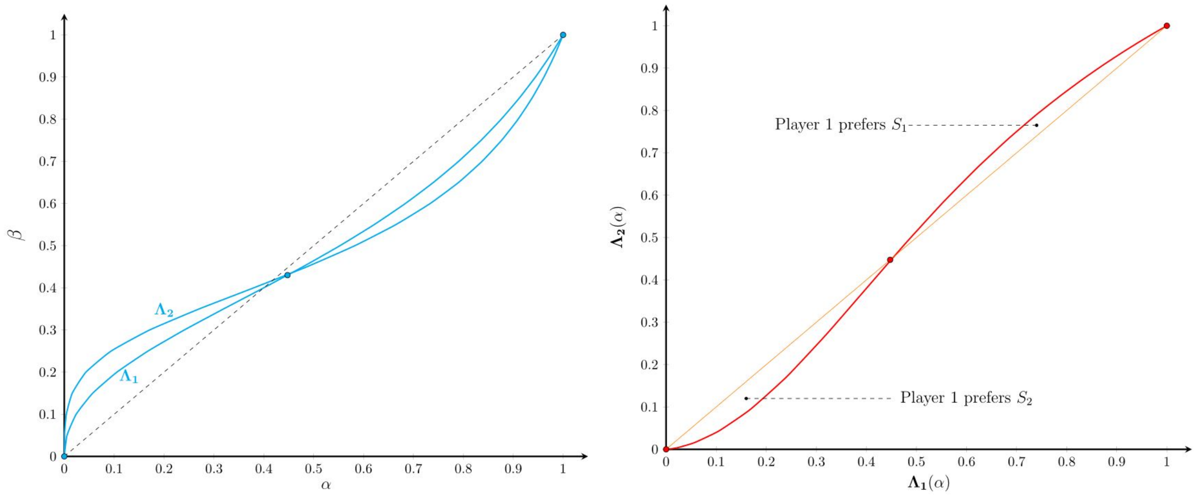

Figure 7: Left: Distortion mappings Λ_1 and Λ_2 from α to β, based on the bargaining sets S_1 and S_2 from Figure 6. Right: Relating distortion of $\Lambda_1(\alpha)$ to $\Lambda_2(\alpha)$.

involving divisible or homogeneous goods, and they can lead to distortion mappings that are no longer single-valued or continuous. In such cases, the set of intersection points between the mappings may be uncountably infinite or form entire intervals. By excluding these degenerate cases, we ensure that the distortion mappings remain continuous and well-defined on the open interval $(0, 1)$, which allows the application of the intermediate value theorem in its standard form. In particular, it guarantees that at least one intersection point exists between $\Lambda_1(\alpha)$ and $\Lambda_2(\alpha)$.

To understand why the number of intersection points is odd, note that the nested structure $S_1 \subsetneq S_2$ implies that the slope of the Pareto frontier of S_2 is initially flatter than that of S_1 near $(0, 1)$ and steeper near $(1, 0)$. This results in $\Lambda_2(\alpha) > \Lambda_1(\alpha)$ for small α and $\Lambda_2(\alpha) < \Lambda_1(\alpha)$ for large α, so the relative order of the mappings must change an odd number of times. Since the mappings are continuous and their intersection points are isolated, the set of such points is countable and consists of a finite number of isolated crossings. The order change implies that their number must be odd.

4 Conclusion

Analyzing the distortion that arises from different ways of modeling bargaining power requires close attention to the structure of the bargaining set itself. As shown throughout this paper, the shape of the set, particularly the curvature of its Pareto frontier, has a decisive impact on the resulting distortion. More concave bargaining sets lead to sharper increases in distortion, especially in highly asymmetric situations. Thus, not only the method by which bargaining power is incorporated, but also the geometry of the feasible set can favor one player over the other. This insight is particularly relevant for a social planner who selects both the bargaining solution and the method for incorporating bargaining power, with the goal of reaching an agreement.

To evaluate the implications of these solution concepts, we systematically compared their respective outcomes across a spectrum of bargaining power distributions. Using the asymmetric KS bargaining solution as a reference point and starting from $(0, 1)$, we found that in all three observed approaches, a sufficiently weak player benefits more when the absolute slope of the Pareto frontier near $(0, 1)$ is flatter. Conversely, a sufficiently strong player prefers the bargaining set that is less concave near $(1, 0)$, that is, one whose Pareto frontier has a smaller absolute slope in that region.

With regard to the asymmetric Nash bargaining solution, distortion curves Λ_1 and Λ_2 associated with any two bargaining sets intersect exactly at the point where the slopes of their respective Pareto frontiers coincide (with respect to a fixed bargaining weight β under the asymmetric KS solution). If, in addition, the frontiers are differentiable and contain no linear segments parallel to the counter diagonal, and one bargaining set is strictly contained in the other, then the number of intersection points is always odd. This structural result helps explain the predictable shift in player preferences as bargaining power varies.

In contrast, when using the symmetric Nash or symmetric KS solution with a shifted disagreement point, the intersection structure of the distortion mappings is not as rigid. Nevertheless, local preferences near the endpoints $(0, 1)$ and $(1, 0)$ still follow the same geometric reasoning: the relative steepness of the Pareto frontier remains decisive.

Taken together, these results illustrate how the interplay between asymmetry modeling and the geometry of the bargaining set systematically shapes the outcome. This raises an open question regarding the interpretation of the Pareto frontier of a bargaining set. While it is evident that any non-linear Pareto curve reflects differences in players' preferences, a formal notion and quantitative assessment of the degree of *preference divergence* (given cardinal utilities) could help further explain the observed patterns in the behavior of distortion curves. Developing such a framework might also prove valuable in applied bargaining settings involving resource allocation or divisible goods, where a better understanding of the implications of modeling choices is essential.

Acknowledgments

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Appendix

Proof of Proposition 1

Take an arbitrary $(S, 0) \in \mathcal{B}_0$. Under the assumption that S is convex and does not contain axis-parallel segments on its Pareto boundary, there exists a bijective mapping $g : [0, 1] \to [0, 1]$ such that the Pareto frontier of S coincides with the graph of g, i.e., $(x_1, g(x_1))$ lies on the Pareto boundary of S for $x_1 \in [0, 1]$. The assumption that the Pareto boundary contains no linear segment parallel to the counter diagonal implies that there exists a unique Pareto optimal point $\bar{x} \in S$ with $\bar{x}_1 \in (0,1)$ such that the vector $(\frac{1}{z_1(S,0)}, \frac{1}{z_2(S,0)})$ is the normal vector of a supporting hyperplane to S at $\bar{x}$. Since $(S,0) \in \mathcal{B}_0$, this reduces to the normal vector $(\frac{1}{z_1(S,0)}, \frac{1}{z_2(S,0)}) = (1,1)$.

We first show that there exists a unique weight $\bar{w} \in (0,1)$ (assuming g is differentiable here) such that the asymmetric Nash solution with weight α and the asymmetric KS solution with weight β coincide, i.e., $\Lambda^S(\bar{w}) = \bar{w}$. The weight $\beta_{\bar{x}}$ for which $KS^{\beta_{\bar{x}}}(S,0) = \bar{x}$ holds is given by

$$\beta_{\bar{x}} = \frac{\frac{1}{z_1(S,0)}\,\bar{x}_1}{\frac{1}{z_1(S,0)}\,\bar{x}_1 + \frac{1}{z_2(S,0)}\,\bar{x}_2} = \frac{\bar{x}_1}{\bar{x}_1 + \bar{x}_2}.$$

For a point $x \in S \cap \mathbb{R}^2_{++}$ and a vector $p = (p_1, p_2)$, define the set

$$T_p^x := \left\{ y \in \mathbb{R}^2 \,|\, py \le px \right\}.$$

Now consider the supporting hyperplane to S at $\bar{x}$ with normal vector $(\frac{1}{z_1(S,0)}, \frac{1}{z_2(S,0)})$ as the Pareto boundary of a bargaining problem $(T,0)$ with constant local transfer rates, i.e., $T = T^{\bar{x}}_{(\frac{1}{z_1(S,0)}, \frac{1}{z_2(S,0)})}$. Then $\bar{x}$ is the outcome of the asymmetric Nash bargaining solution for $(T,0)$ with weight $\alpha_{\bar{x}} = \beta_{\bar{x}}$. This follows because the objective function in the asymmetric Nash solution maximizes the weighted product of gains, and the linearity of the frontier in T with normal vector $(1,1)$ implies that the solution lies at $\bar{x}$, where the slope of the indifference curve equals the slope of the frontier. Since $S \subseteq T$, the independence of irrelevant alternatives (IIA) implies that the asymmetric Nash solution on S with weight $\alpha_{\bar{x}}$ also yields $\bar{x}$, i.e., $\bar{x} = N^{\alpha_{\bar{x}}}(S,0)$. Choosing $\bar{w} := \beta_{\bar{x}} = \alpha_{\bar{x}}$, we thus obtain the desired identity

$$KS^{\bar{w}}(S,0) = N^{\bar{w}}(S,0) = \bar{x},$$

where the slope of the supporting hyperplane at $\bar{x}$ reflects the marginal rate of substitution between the two players' payoffs, as implied by both the Nash and KS bargaining solutions. This is consistent with the fact that $\Lambda^S(\bar{w}) = \bar{w}$.

Having identified the point of intersection at $\bar{x}$ between the asymmetric KS and Nash solutions, we now investigate how these solutions diverge for bargaining weights below that point. We show that for any weight $w < \bar{w}$, player 1 prefers the asymmetric Nash solution over the asymmetric KS solution. To see this, consider any Pareto optimal point $x = (x_1, x_2) \in \partial S$ with $0 < x_1 < \bar{x}_1$. Using the normalization $z(S,0) = (1,1)$, the weight for which $x = KS^{\beta}(S,0)$ holds is given by

$$\beta_x = \frac{x_1}{x_1 + x_2}.$$

Let $p(x) = (p_1(x), p_2(x))$ be the normal vector of a separating hyperplane to S at x. Since the Pareto frontier is concave, the slope of the separating hyperplane decreases (in absolute value) as we move along the Pareto frontier towards points with smaller values of x_1, implying $\frac{p_1(x)}{p_2(x)} < \frac{\frac{1}{z_1(S,0)}}{\frac{1}{z_2(S,0)}} = 1$. The corresponding weight α_x for which $x = N^{\alpha_x}(T^x_{p(x)}, 0) \overset{IIA}{=} N^{\alpha_x}(S,0)$ holds is then:

$$\alpha_x = \frac{p_1(x)x_1}{p_1(x)x_1 + p_2(x)x_2} = \frac{x_1}{x_1 + \frac{p_2(x)}{p_1(x)}x_2} < \frac{x_1}{x_1 + x_2} = \beta_x.$$

Hence, for the weights smaller than $\bar{w}$, the asymmetric Nash solution assigns a higher share to player 1 than the asymmetric KS solution.

As a consequence, for the same value of β, the associated weights in the asymmetric Nash solution satisfy

$$\alpha_1^\beta > \alpha_2^\beta,$$

since a stronger curvature of the Pareto frontier results in a higher bargaining power required by the first player (with lower x_1) to obtain the same relative share of the total payoff. This confirms that stronger curvature of the Pareto frontier amplifies the distortion for small bargaining weights.

To reinforce this finding, we now compare two bargaining sets with different degrees of curvature, focusing on the effect near the origin. We show that the difference between the two solutions increases when the bargaining set is more outward-curved. Let S_1 and S_2 be two bargaining sets with differentiable Pareto boundaries given by functions g_1 and g_2, respectively. Assume that for all $x_1 \in (0, \varepsilon)$, where $\varepsilon > 0$ is sufficiently small, the functions g_1 and g_2, which represent the Pareto frontiers of the bargaining sets S_1 and S_2, are continuously differentiable, and that

$$1 > g_1'(x_1) > g_2'(x_1) > 0.$$

Consider the points $x^{(1)} = (x_1, g_1(x_1)) \in \partial S_1$ and $x^{(2)} = (x_1, g_2(x_1)) \in \partial S_2$, and let $p^{(1)}$ and $p^{(2)}$ be the normal vectors of supporting hyperplanes at these points. Then, the relatively steeper slope of g_1 implies

$$\frac{p_1^{(1)}}{p_2^{(1)}} > \frac{p_1^{(2)}}{p_2^{(2)}}.$$

As a consequence, for the same value of β, the associated weights in the asymmetric Nash solution satisfy

$$\alpha_1^\beta > \alpha_2^\beta.$$

This confirms that for small bargaining weights, the distortion between the two solutions is more pronounced when the Pareto frontier of one bargaining set exhibits a stronger outward curvature, particularly in the regions near the axes where player 1's or player 2's payoff is small (e.g., in case of player 1, for small x_1). A stronger outward curvature of one set means that the slope of its Pareto frontier becomes less steep as we move towards these regions, which increases the difference between the two bargaining solutions. $\square$

Proof of Proposition 2

Let $(S, 0) \in \mathcal{B}_0$ with a differentiable Pareto frontier $g : [0, 1] \to [0, 1]$ and inverse $h := g^{-1}$. For any Pareto efficient point $x \in S$, we have $x = (x_1, g(x_1)) = (h(x_2), x_2)$.

We analyze the distortion function $\Phi(\Delta_1)$ defined by the identity

$$KS(S, (\Delta_1, 1 - \Delta_1)) = KS^{\Phi(\Delta_1)}(S, 0),$$

and consider the behavior of its derivative for $\Delta_1 \to 0$ and $\Delta_1 \to 1$. As both limits are symmetric, we detail the argument for $\Delta_1 \to 0$. The case $\Delta_1 \to 1$ follows analogously.

To this end, define $k_1(\Delta_1)$ as the first coordinate of the symmetric KS solution with disagreement point $(\Delta_1, 1-\Delta_1)$, i.e.,

$$k_1(\Delta_1) := KS_1(S, (\Delta_1, 1-\Delta_1)),$$

so that the solution is given by $KS(S, (\Delta_1, 1-\Delta_1)) = (k_1(\Delta_1), g(k_1(\Delta_1)))$. Using the characterization of the asymmetric KS solution, we obtain

$$\Phi(\Delta_1) = \frac{k_1(\Delta_1)}{k_1(\Delta_1) + g(k_1(\Delta_1))}.$$

Differentiating this expression with respect to Δ_1, we apply the quotient and chain rule to obtain

$$\Phi'(\Delta_1) = \frac{k_1'(\Delta_1)\,[g(k_1(\Delta_1)) - k_1(\Delta_1)g'(k_1(\Delta_1))]}{(k_1(\Delta_1) + g(k_1(\Delta_1)))^2}.$$

To determine the limiting behavior of $\Phi'(\Delta_1)$ as $\Delta_1 \to 0$, we analyze the expression for $\Phi'(\Delta_1)$ using the fact that g is continuously differentiable. Observe that

$$\lim_{\Delta_1 \to 0} \Phi'(\Delta_1) = \lim_{\Delta_1 \to 0} \frac{k_1'(\Delta_1)[\overbrace{g(k_1(\Delta_1))}^{\to 1} - \overbrace{k_1(\Delta_1)g'(k_1(\Delta_1))}^{\to 0}]}{\underbrace{(k_1(\Delta_1) + g(k_1(\Delta_1)))^2}_{\to 1}} = \lim_{\Delta_1 \to 0} k_1'(\Delta_1).$$

since $g(k_1(\Delta_1)) \to 1$, $k_1(\Delta_1) \to 0$, and $g'(k_1(\Delta_1))$ remains bounded.

To evaluate $\lim_{\Delta_1 \to 0} \Phi'(\Delta_1)$, we approximate the Pareto frontier in a neighborhood of $(0, 1)$ by a linear function. This simplification allows for a closed-form expression while preserving the relevant first order behavior. Specifically, assume

$$g(x) = 1 - \eta x \quad \text{for } x \in (0, \varepsilon),$$

with $\eta > 0$, so that $g'(0) = -\eta < 0$. The same linearization argument is used in Proposition 1 in Haake and Streck (2022). The linearization is illustrated in Figure 8 and serves as a local approximation to the more general case shown in Figure 4.

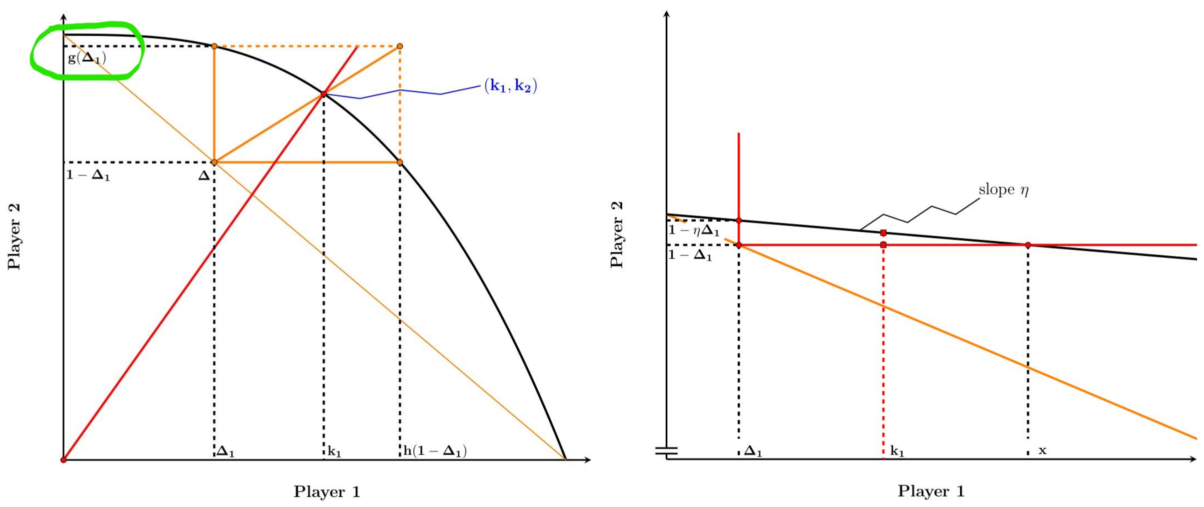

Figure 8: Left: Reconsidering the KS case with necessary notation. Right: Close-up of the linear Pareto frontier in the neighborhood of (0, 1).

In this setup, the local utopia point becomes $\left(\frac{\Delta_1}{\eta}, 1-\Delta_1\right)$, and the Pareto point corresponding to the disagreement payoff is $(\Delta_1, g(\Delta_1)) = (\Delta_1, 1-\eta\Delta_1)$, which is the lower endpoint of the individually rational segment. The segment between these endpoints is linear, so the KS solution lies at the midpoint:

$$k_1(\Delta_1) = \frac{1}{2}\left(\Delta_1 + \frac{\Delta_1}{\eta}\right).$$

Differentiating yields

$$k_1'(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta},$$

and thus

$$\lim_{\Delta_1 \to 0} \Phi'(\Delta_1) = k_1'(0) = \frac{1}{2} + \frac{1}{2\eta} > 1.$$

This shows that the derivative of the distortion function Φ satisfies $\lim_{\Delta_1 \to 0^+} \Phi'(\Delta_1) > 1$. As $\eta \to 0$, the derivative $k_1'(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta}$ increases without bound, indicating that the inequality $\Phi'(0^+) > 1$ persists even in the limiting case $\eta = 0$, by continuity of the expression with respect to η. This argument is based on a local linear approximation of the Pareto frontier near $(0,1)$ and reflects the limiting behavior of $\Phi'(\Delta_1)$ as $\Delta_1 \to 0$. Under the standing assumption that g is continuously differentiable, and since $k_1'(\Delta_1)$ is continuous in Δ_1, the result extends to general bargaining sets with smooth and strictly decreasing frontiers.

Now consider two bargaining sets S_1 and S_2 with linear Pareto frontiers in the neighborhood of $(0,1)$ and slopes $\eta_1 > \eta_2 > 0$. The corresponding derivatives are:

$$k_1'^{(1)}(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta_1}, \quad k_1'^{(2)}(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta_2}.$$

Since $\eta_1 > \eta_2$, we conclude

$$\Phi_2'(\Delta_1) > \Phi_1'(\Delta_1),$$

so that $\Phi_2(\Delta_1) > \Phi_1(\Delta_1)$ for sufficiently small $\Delta_1 > 0$. This inequality between distortion functions persists until, along the ray with slope $\frac{1-\beta}{\beta}$ from the origin, the slope of the Pareto frontier of S_1 exceeds that of S_2. This condition is necessary for the inequality between the distortion functions Φ_1 and Φ_2 to reverse.

The same reasoning applies at the other end of the Pareto frontier. By symmetry, swapping the roles of players and considering $\Delta_1 \to 1$ (i.e., $\Delta_2 \to 0$) yields:

$$\lim_{\Delta_1 \to 1} \Phi'(\Delta_1) > 1.$$

This completes the proof. □

Proof of Proposition 3

Let $(S, 0) \in \mathcal{B}_0$ with a differentiable Pareto frontier $g : [0,1] \to [0,1]$ and inverse $h := g^{-1}$. For any Pareto efficient point $x \in S$, we have $x = (x_1, g(x_1)) = (h(x_2), x_2)$.

We define the distortion function $\Upsilon(\Delta_1)$ defined by the identity

$$N(S, (\Delta_1, 1-\Delta_1)) = KS^{\Upsilon(\Delta_1)}(S, 0),$$

and examine its derivative behavior near the extremes $\Delta_1 \to 0$ and $\Delta_1 \to 1$. Due to symmetry, it suffices to analyze the case $\Delta_1 \to 0$. To this end, define $n_1(\Delta_1)$ as the first coordinate of the symmetric Nash solution with disagreement point $(\Delta_1, 1-\Delta_1)$:

$$n_1(\Delta_1) := N_1(S, (\Delta_1, 1-\Delta_1)),$$

so that the solution is given by $N(S, (\Delta_1, 1-\Delta_1)) = (n_1(\Delta_1), g(n_1(\Delta_1)))$. Using the characterization of the asymmetric KS solution, we obtain

$$\Upsilon(\Delta_1) = \frac{n_1(\Delta_1)}{n_1(\Delta_1) + g(n_1(\Delta_1))}.$$

Differentiating this expression with respect to Δ_1, we apply the quotient and chain rule to obtain

$$\Upsilon'(\Delta_1) = \frac{n_1'(\Delta_1)\left[g(n_1(\Delta_1)) - n_1(\Delta_1)g'(n_1(\Delta_1))\right]}{\left(n_1(\Delta_1) + g(n_1(\Delta_1))\right)^2}.$$

To determine the limiting behavior of $\Upsilon'(\Delta_1)$ as $\Delta_1 \to 0$, we observe that $n_1(\Delta_1) \to 0$, $g(n_1(\Delta_1)) \to 1$, and $g'(n_1(\Delta_1))$ remains bounded. Hence,

$$\lim_{\Delta_1 \to 0} \Upsilon'(\Delta_1) = \lim_{\Delta_1 \to 0} n_1'(\Delta_1).$$

To evaluate this limit, we approximate the Pareto frontier in a neighborhood of $(0,1)$ by a linear function:

$$g(x) = 1 - \eta x \quad \text{for } x \in (0, \varepsilon), \quad \eta > 0,$$

so that $g'(0) = -\eta < 0$. This linearization corresponds to the same local setting as in Proposition 2 and allows for an explicit expression of the Nash solution.

In this setup, the local utopia point becomes $\left(\frac{\Delta_1}{\eta}, 1-\Delta_1\right)$, and the Pareto point corresponding to the disagreement payoff is $(\Delta_1, 1-\eta\Delta_1)$. As the segment between these two points is linear, the Nash solution lies at the midpoint:

$$n_1(\Delta_1) = \frac{1}{2}\left(\Delta_1 + \frac{\Delta_1}{\eta}\right).$$

Differentiating yields

$$n_1'(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta},$$

and thus

$$\lim_{\Delta_1 \to 0} \Upsilon'(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta} > 1.$$

This shows that the derivative of the distortion function Υ satisfies $\lim_{\Delta_1 \to 0^+} \Upsilon'(\Delta_1) > 1$. As $\eta \to 0$, the derivative $n_1'(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta}$ increases without bound, indicating that the inequality $\Upsilon'(0^+) > 1$ persists even in the limiting case $\eta = 0$, by continuity of the expression with respect to η. This argument is based on a local linear approximation of the Pareto frontier near $(0,1)$ and reflects the limiting behavior of $\Upsilon'(\Delta_1)$ as $\Delta_1 \to 0$. Under the standing assumption that g is continuously differentiable, and since $n_1'(\Delta_1)$ is continuous in Δ_1, the result extends to general bargaining sets with smooth and strictly decreasing frontiers.

Now consider two bargaining sets S_1 and S_2 with linear Pareto frontiers in the neighborhood of $(0,1)$ and slopes $\eta_1 > \eta_2 > 0$. The corresponding derivatives are:

$$n_1'^{(1)}(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta_1}, \quad n_1'^{(2)}(\Delta_1) = \frac{1}{2} + \frac{1}{2\eta_2}.$$

Since $\eta_1 > \eta_2$, we conclude

$$\Upsilon_2'(\Delta_1) > \Upsilon_1'(\Delta_1),$$

so that $\Upsilon_2(\Delta_1) > \Upsilon_1(\Delta_1)$ for sufficiently small $\Delta_1 > 0$. This inequality between distortion functions persists until, along the ray with slope $\frac{1-\beta}{\beta}$ from the origin, the slope of the Pareto frontier of S_1 exceeds that of S_2. This condition is necessary for the inequality between the distortion functions Υ_1 and Υ_2 to reverse.

By symmetry, we obtain the analogous result as $\Delta_1 \to 1$, concluding that

$$\lim_{\Delta_1 \to 1} \Upsilon'(\Delta_1) > 1.$$

This completes the proof. $\square$