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Abstract

Suppose some individuals are allowed to engage in different groups at the same time and they generate a certain welfare by cooperation. Finding appropriate ways for distributing this welfare is a non-trivial issue. The purpose of this work is to analyze two-stage allocation procedures where first each group receives a share of the welfare which is then, subsequently, distributed among the corresponding members. To study these procedures in a structured way, cooperative games and network games are combined in a general framework by using mathematical hypergraphs. Moreover, several convincing requirements on allocation procedures are discussed and formalized. Thereby it will be shown, for example, that the Position Value and iteratively applying the Myerson Value can be characterized by similar axiomatizations.

Keywords: Allocation Rules, Economic and Social Networks, Hypergraphs, Myerson Value, Position Value

JEL Classification: C71, D85, L22

1 Introduction

Ever since the beginnings of economic theory, finding appropriate and convincing ways for distributing welfare has been one of the fundamental problems. In the context of cooperation between individuals, there are two branches, namely the Theory of Cooperative Games and, more recently, the Theory of Economic Networks, that especially focus on this question and propose several solutions, which are called values or allocation rules (e.g., de Clippel and Serrano, 2008; Herings *et al.*, 2010; McQuillin, 2009; Navarro, 2010). This study relates to both approaches. More specifically, the main purpose is to formalize and examine two-stage allocation procedures.

To gain an idea of this issue, consider situations where individuals cooperate in several groups or coalitions which may overlap, like in departments of organizations, for example. The main motivation of two-stage allocation procedures is that the generated welfare is not distributed to the

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individuals directly but first to the groups and then, in a second step, within each group. In fact, for modeling this is it necessary to extend the aforementioned approaches, Network Theory and Cooperative Game Theory, to a more general model.

To understand the difference between Network Theory and Cooperative Game Theory it is necessary to analyze the view of cooperation in both branches. Cooperative games with transferable utility (TU games) provide a theoretical framework for modeling environments where the individuals are partitioned in coalition structures, i.e., some individuals act together in groups to achieve a common goal but standardly it is assumed that nobody is a member of different groups at once (see, e.g., Aumann and Drèze, 1974). In Network Theory, on the other hand, coalitions are replaced by networks which represent bilateral relationships between individuals (see, e.g., Jackson, 2008). Indeed, one of the most characteristic features of networks is that they exhibit some overlapping structure. However, in these models, cooperation generically takes place only within pairs of individuals, which is a limitation as well. Consider, for example, the models from Radner (1993), Bolton and Dewatriopont (1994), or Arenas *et al.* (2010). Here, the authors use networks for representing the structure of organizations. Within each network, the information is shared along the links: that is, the information is exchanged among pairs of individuals. In general, however, employees also work together within larger groups, such as projects, departments, or divisions, and in many environments these groups may overlap, which neither could be depicted by using bilateral networks nor by using TU games.

A convenient tool for modeling overlapping coalition structures are mathematical hypergraphs which can be interpreted as a kind of overlapping coalition structure. Up to now, there is no general notion for these structures. Depending on the context they are called "conference structures" (e.g., Myerson, 1980), "affiliation networks" (e.g., Wasserman and Faust, 1994), "overlapping coalitions" (e.g., Chalkiadakis *et al.*, 2010), "clubs" (e.g., Fershtman and Persitz, 2012) or simply "hypergraphs" (e.g., Jorzik, 2012; van den Nouweland *et al.*, 1992). To stress both, the origins from Network Theory as well as from Cooperative Game Theory, in this paper they will be called coalitional networks.

Formalizing two-stage allocation procedures is in line with several other works from literature which analyze and discuss particular examples like the Aumann-Drèze Value, Owen's Value or the Position Value (e.g., Aumann and Drèze, 1974; Borm *et al.*, 1992; Hart and Kurz, 1983; Owen, 1977). Another contribution stems from Winter (1989) who analyzes cooperative games with exogenously-given hierarchical structures. In this context, he extends the Aumann-Drèze Value and Owen's Value to arbitrary numbers of stages. However, these publications do not provide a general framework for discussing two-stage allocation procedures in a structured way. In the context of coalitional networks, this can be done tractably by exploiting the existence of a unique dual network which can be interpreted as some kind of dual game where individuals and coalitions (or links, respectively) interchange roles. This allows not only modeling two stages explicitly but also formulating convincing properties for solution concepts which characterize them. Indeed, it is possible to integrate the aforementioned examples into the extended setting presented here. As it shall be proven in this study, these solution concepts can be characterized by using similar axiomatizations. In particular, the second part of the paper is strongly motivated by van den Brink (2007). In his work the author shows that the Shapley Value (Shapley, 1953) and Equal Division Solutions can be described by means of similar axioms. This leitmotiv will be found twice in the present study: Not only by showing that the aforementioned two-stage allocation procedures satisfy similar axioms, but also by extending the characterizations from van den Brink (2007) to coalitional networks.

Some further publications should be mentioned in the context of this study. One of the first authors theorizing overlapping coalition structures was Myerson (1980). His study is motivated by TU games and the focus is on conference structures which are modeled by means of hypergraphs. However, Myerson's approach differs from the present work in at least one important aspect. He uses overlapping coalition structures only for restricting the possible lines of cooperation between the individuals, but the generated value or welfare, respectively, does not depend on the underlying communication structure directly. This assumption is maintained in most of the works built on Myerson's publication (e.g., Albizuri *et al.*, 2005; van den Nouweland *et al.*, 1992). In recent years, computer scientists also tended to pay more and more attention to Cooperative Game Theory. They use TU games to model the behavior of autonomous agents, and in this context they also investigate overlapping coalition structures. However, most of the corresponding publications (e.g., Chalkiadakis *et al.*, 2010) focus on extensions of the core which, will not be discussed here. Although several notions and concepts used in this work are adopted from Network Theory (e.g., from Jackson, 2005; Jackson and Wolinsky, 1996), overlapping coalition structures have only been attempted by few scholars in this field. Among them are Jorzik (2012) who studies allocation rules for hypergraphs and Fershtman and Persitz (2012) who extend the Connections Model to the more general framework. A further interesting approach comes from Caulier *et al.* (2012) who study network formation for situations where the set of individuals is already partitioned into (disjoint) coalitions. In this context the authors examine the stability and efficiency of networks. Moreover, from a technical point of view the model introduced here also relates to many-to-many matchings (see, e.g., Sotomayor, 2004). In fact, there exists a canonical bijection between many-to-many matchings and coalitional networks. However, virtually all publications from this field do not address two-stage allocation rules but have different objectives.

The remainder of the paper proceeds as follows: In the next section, the model will be introduced formally. This includes the definition of coalitional networks as well as the extension of value functions and characteristic functions to the framework analyzed here. The objective of Section 3 is to formalize allocation rules and to characterize two-stage allocation procedures in general. Section 4 is devoted to particular examples, like the Position Value, for instance. Finally, Section 5 briefly summarizes and presents an agenda for further research.

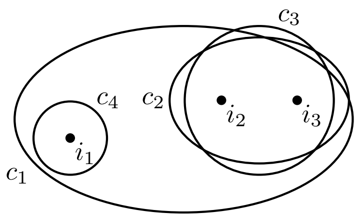
2 The Model

Let $N = \{i_1, \dots, i_n\}$ be a finite set of *players* or *individuals* who are able to generate some value or profit by cooperation. The cooperation takes place within groups which may overlap. For this, let a finite set $M = \{c_1, \dots, c_m\}$ of *connections* or *coalitions* be given. The elements of M are interpreted as projects or departments of an organization, for example, and the players are members of these projects.

Definition 1. Let $h : M \longrightarrow 2^N$ be an arbitrary mapping, where 2^N is the power set of N. The tuple (N, M, h) is a *coalitional network*.

If a coalitional network represents the structure of an organization, for instance, then the mapping h assigns to each project of the organization $c \in M$ the employees $h(c) \subseteq N$ that are working on it. In particular, from a mathematical point of view, the coalitional network (N, M, h) is simply a hypergraph (see, e.g., Berge, 1989) and the set of all coalitional networks is denoted by $\mathcal{H}$. Note that $|\mathcal{H}| = 2^{nm}$. For ease of handling, coalitional networks are also simply called networks in the following. But it should be mentioned that in literature *bilateral networks* (i.e., each connection contains exactly two players) are usually termed this way. Moreover, since N and M are not altered most of the time, a coalitional network (N, M, h) will be frequently identified with h only if no confusions can result.

Example 1. *As an example let $N = \{i_1, i_2, i_3\}$ and $M = \{c_1, \ldots, c_4\}$, i.e., there are three players and four connections. Suppose all players are contained in c_1, the players i_2 and i_3 are moreover contained in c_2 and c_3, while c_4 only contains i_1.*



$$
\begin{array}{c} \\ i_1 \\ i_2 \\ i_3 \end{array}
\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}
$$

Figure 1: An illustrative example

A coalitional network (N, M, h) describing this situation formally would be given by

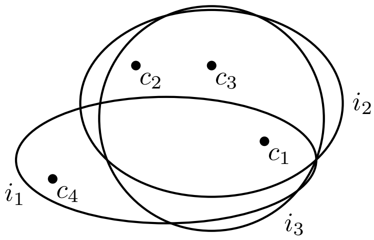
$$
h(c) = \begin{cases} \{i_1, i_2, i_3\}, & \text{if } c = c_1 \\ \{i_2, i_3\}, & \text{if } c \in \{c_2, c_3\} \\ \{i_1\}, & \text{if } c = c_4. \end{cases}
$$

The main difference to Myerson's Communication Structures (Myerson, 1980) is that here for $c \in M$ also $|h(c)| = 1$ is allowed: that is, it is possible that a connection contains only one player. If $c \in M$ represents a project or the like and the players are employees, then it is of course possible that an employee is working on the project alone. Therefore, these structures are not excluded a priori.[1] A further important special case is $|h(c)| = 0$, i.e., it is also possible that a connection is empty or, in other words, no worker is assigned to the project. The coalitional network $h^{\varnothing}$ with $h^{\varnothing}(c) = \varnothing$ for all $c \in M$ is called *empty network*.

One of the most basic and important insights in Hypergraph Theory is the existence of a unique dual hypergraph or dual network, respectively (see, e.g., Berge, 1989).

Definition 2. Let $(N, M, h) \in \mathcal{H}$ be a coalitional network. The corresponding *dual network* $\big(M, N, h^*\big)$ is given by $h^*(i) = \{m \in M \mid i \in h(m)\}$.

[1] The main arguments for assuming that each link has to contain at least two players are often based on communication issues, like that at least two players are needed to confer, for example. However, in this case it is still possible to adjust the incentives for forming connections of size one accordingly. For instance, by assuming that these connections or departments, respectively, generate no or even negative profit (see Section 3 for a formal introduction of players' payoffs). For an extensive discussion of this issue see Aumann and Drèze (1974).



$$\begin{array}{cc} & \begin{array}{ccc} i_1 & i_2 & i_3 \end{array} \\ \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

Figure 2: The dual network in Example 1

For given $h \in \mathcal{H}$, the dual network assigns to each player the set of connections she is contained in. As the name implies, it is a coalitional network, too, and similar to h it will be simply denoted by h^* in the following. Let $\mathcal{H}^*$ denote the set of dual networks. Note that for all $i \in N$ and $c \in M$, $c \in h^*(i)$ if and only if $i \in h(c)$. This implies $(h^*)^* = h$ and, thus, the dual network is uniquely determined. Phrased differently, the dual network is simply an alternative way of representing the network structure. However, the salient point is that from an economic perspective the players and connections interchange roles. In the dual network, the players link the connections and vice versa.

The following notions and definitions are variations or straightforward extensions from Network Theory (cf. Jackson, 2008). A network $h \in \mathcal{H}$ is a *subnetwork* of $h' \in \mathcal{H}$ (denoted by $h \subseteq h'$) if each connection in h is contained in the corresponding connection of h', i.e., $h(c) \subseteq h'(c)$ for all $c \in M$. An important special case of subnetworks is the *restriction* to a certain set of individuals. Given $S \subseteq N$, the subnetwork $h|_S$ is obtained from h by deleting in all connections all players who are not contained in S. Formally: $h|_S(c) = h(c) \cap S$ for all $c \in M|_{h,S}$. Given $h \in \mathcal{H}$, the *size* η_h of the coalitional network is the aggregated cardinality of all connections: i.e., $\eta_h := \sum_{c \in M} |h(c)|$. Two players $i, j \in N$ are called *neighbors* or *adjacent* if there exists $c \in M$ with $i, j \in h(c)$. Furthermore, the *degree* $\deg_i(h)$ of the player is the number of connections she is contained in: i.e., $\deg_i(h) = |\{c \in M \mid i \in h(c)\}|$. Note that the aggregated degree equals exactly the size of the network, that is, $\sum_{i \in N} \deg_i(h) = \eta_h$. If $i \in N$ has a degree of zero, she is said to be *isolated* and a player who is not isolated is *active*. The set of active players is $N(h) = \{i \in N \mid \deg_i(h) \geq 1\}$.
For two players $i, j \in N$ an *i-j-walk* in the coalitional network $h \in \mathcal{H}$ is a sequence of players $(i_0, \ldots, i_k)$ with $i = i_0, j = i_k$ and i_l is adjacent to i_{l+1} for all $0 \leq l < k$. A nonempty set of players $S \subseteq N$ is said to be *connected* if for any pair $i, j \in S$ there exists a walk between them. Furthermore, S is called *association* if it is maximally connected: i.e., it is connected and for all i, j with $i \in S$ and $j \notin S$ there is no i-j-walk in h. A nonempty network $\bar{h} \in \mathcal{H}$ is called *component* of $h \in \mathcal{H}$ if there exists an association $A \subseteq N$ with $\bar{h} = h|_A$. In the following, let $\mathcal{A}(h)$ be the set of all associations and $\mathcal{C}(h)$ the set of all components. Note that an association may consist of only one player who may have a degree of zero.

Definition 3. A *value function* is a mapping $v : \mathcal{H} \longrightarrow \mathbb{R}$ assigning a real number to each network $h \in \mathcal{H}$, where $v(h)$ is interpreted as the *worth* of the network. Thereby it is assumed that each value function is *normalized*, i.e., $v(h^{\varnothing}) = 0$.

The worth of a network is the total value of cooperation. If $h \in \mathcal{H}$ represents the structure of some organization or the like, $v(h)$ might be the organization's profit or budget. Let $\mathcal{V}$ denote the set of value functions. For each $v \in \mathcal{V}$ the corresponding dual value function $v^* : \mathcal{H}_{N,2^M} \longrightarrow \mathbb{R}$ is obtained by assigning to each dual network h^* the worth of its "primal" one: $v^*(h^*) := v\big((h^*)^*\big) = v(h)$. This induces, in particular, a bijection between $\mathcal{V}$ and the set of the dual value functions $\mathcal{V}^*$. For avoiding notational inconveniences, in the following $\mathcal{V}$ and $\mathcal{V}^*$ will be identified with each other. That is, no distinction will be made between the both sets and both of them will just be denoted by $\mathcal{V}$. Phrased differently, $v \in \mathcal{V}$ implies the value function is the primal one if v is applied to a network $h \in \mathcal{H}$ and it implies v is the dual value function if v is applied to the dual network $h^* \in \mathcal{H}$. However, because both networks receive the same worth no confusions should arise.

A specific example are *basic value functions* (e.g., Jackson, 2005). Given $h \in \mathcal{H} \setminus \{h^{\varnothing}\}$, the corresponding basic value function $1^h \in \mathcal{V}$ is defined by

$$1^h(h') = \begin{cases} 1, & \text{if } h \subseteq h' \\ 0, & \text{if } h \nsubseteq h'. \end{cases}$$

The interpretation is that a coalitional network $h' \in \mathcal{H}$ generates a positive worth if and only if it contains the specific structure h.

Let $\mathcal{CA} \subseteq \mathcal{V}$ denote the special class of component-additive value functions. A value function $v \in \mathcal{CA}$ is *component-additive* if $\sum_{\bar{h}\in\mathcal{C}(h)} v\left(\bar{h}\right) = v(h)$ for all coalitional networks $h \in \mathcal{H}$. Here, the total productivity of a coalitional network is simply the aggregated worth of all its components (cf. Jackson and Wolinsky, 1996). The productivity of a component is therefore not exogenously influenced by the other components.

Remark 1. Traditionally, a TU game is a pair (N, γ) consisting of the set of players N and a *characteristic function* γ assigning a value to each *coalition* $S \subseteq N$, i.e., $\gamma : 2^N \longrightarrow \mathbb{R}$ (see, e.g., Roth, 1988). On the other hand, a (bilateral) network game is a pair (N, w) where w assigns a value to each *bilateral network*, i.e., $w : 2^G \longrightarrow \mathbb{R}$ with $G = \{S \subseteq N \,|\, |S| = 2\}$ (see, e.g., Jackson, 2008). The main idea for combining both approaches within the extended framework considered here is to exploit the fact that both domains, coalitions and bilateral networks, can be embedded into the set of coalitional networks. For instance, if $|M| = 1$, i.e., if there is only one connection, this connection can be interpreted as a coalition and value functions can be identified with characteristic functions. Thus, the model is indeed an extension of TU games. Moreover, a network game (N, w) can also be represented in a canonical way: let $M = G$ and for each $g \in 2^G$ define

$$h^g(\{i,j\}) = \begin{cases} \{i,j\}, & \text{if } \{i,j\} \in g \\ \varnothing, & \text{if } \{i,j\} \notin g. \end{cases}$$

That is, i and j are adjacent in h^g if and only if they are linked in g. Note that the value or worth, respectively, of each bilateral network only depends on whether certain players are linked but not on in which way they are connected. Therefore, if $v^w \in \mathcal{V}$ satisfies not only $v^w(h^g) = w(g)$ for all $g \in 2^G$ but also $v^w(h^g \circ \pi) = v^w(h^g)$ for each permutation $\pi : M \longrightarrow M$, this value function de facto represents w.[2]

Remark 2. Basic value functions are a straightforward extension of unanimity games (cf. Shapley, 1953). It is well-known from literature that these games form a basis of the space of characteristic

[2]As usual, $\circ$ denotes the composition of the mappings h^g and π.

functions. Analogously it can be shown that the set of basic value functions forms a basis of $\mathcal{V}$. More precisely, each $v \in \mathcal{V}$ can be written as

$$v = \sum_{h \in \mathcal{H} \setminus \{h^{\emptyset}\}} \Delta^h(v) 1^h, \quad \text{where the} \quad \Delta^h(v) = \sum_{\bar{h} \subseteq h} (-1)^{\eta_h - \eta_{\bar{h}}} v(\bar{h})$$

are the Harsanyi coefficients (cf. Harsanyi, 1959). This representation of value functions will be helpful in the proofs.

Remark 3. Note that $\mathcal{CA} = \{v \in \mathcal{V} \mid v \text{ is component-additive}\}$ is a subspace of $\mathcal{V}$ and $1^h \in \mathcal{CA}$ if and only if $|\mathcal{C}(h)| = 1$ (cf. van den Nouweland and Slikker, 2012). Moreover, since the operator $* : \mathcal{V} \longrightarrow \mathcal{V}$ with $v \longmapsto v^*$ is not only bijective but also linear and $|\mathcal{C}(h)| = 1$ is equivalent to $|\mathcal{C}(h^*)| = 1$, $v \in \mathcal{CA}$ is component-additive if and only if $v^* \in \mathcal{CA}$ has this feature, too.

3 Allocation Rules

Given the productivity of cooperation, the next step is to allocate the worth among the players. Therefore, the pair (N, M) is an *allocation problem* (note that $\mathcal{V} = \mathcal{V}(N, M)$ is uniquely determined by N and M). Most of the solutions analyzed in Cooperative Game Theory or Network Theory distribute the worth in a player-based way, i.e., the payoff is associated to players directly.

Definition 4. A *player-based allocation rule* X assigns to each allocation problem (N, M) a tuple of mappings $(X_i)_{i \in N}$ with $X_i : \mathcal{V} \longrightarrow \mathcal{V}$ and $\sum_{i \in N} X_i(v) = v$ for all $v \in \mathcal{V}$.[3]

A player-based allocation rule X (or $(X_i)_{i \in N}$, respectively) decomposes each value function $v \in \mathcal{V}$ into individual payoff functions $v_i^X := X_i(v) \in \mathcal{V}$. Phrased differently, given the total productivity $v(h)$ of a network h, each player receives her share of the payoff $v_i^X(h) = (X_i(v))(h)$. The central assumption in Definition 4 is that allocation rules are *balanced*, i.e., $\sum_{i \in N} v_i^X(h) = v(h)$ for all $h \in \mathcal{H}$ and $v \in \mathcal{V}$. Thus, the value of cooperation is always distributed completely among the players.[4] The sets of all player-based allocation rules will be denoted by $\mathcal{X}$. Note that each allocation rule $Z \in \mathcal{X}$ assigns payoff functions $\left(v_c^Z\right)_{c \in M}$ to the dual problem (M, N), too. This is due to the fact that in the dual network the connections are considered as players and vice versa.

Example 2. *A possible way to allocate the worth of a network is to distribute it equally among all active players:*

$$v_i^{ED}(h) = (ED_i(v))(h) := \begin{cases} \frac{v(h)}{|N(h)|}, & \textit{if } i \in N(h) \\ 0, & \textit{if } i \notin N(h), \end{cases}$$

[3]In their seminal contribution, Jackson and Wolinsky (1996) introduce (player-based) allocation rules in a slightly different way. They define an allocation rule A to be a mapping $A : \mathcal{H} \times \mathcal{V} \longrightarrow \mathbb{R}^n$ with $\sum_{i \in N} A_i(h, v) = v(h)$ for all $h \in \mathcal{H}$ and $v \in \mathcal{V}$. The interpretation is, of course, the same. Given a network $h \in \mathcal{H}$ and a value function $v \in \mathcal{V}$, $A_i(h, v)$ is agent i's share of the network's value $v(h)$. Using the formulation introduced here is just for technical reasons and will be helpful in the following.

[4]In contrast to efficiency from Cooperative Game Theory (an allocation rule is said to be efficient if it always distributes the value that could be reached by the grand coalition N (see, e.g., van den Brink, 2007)), it is not implicitly assumed that the grand coalition or the complete network, respectively, generically generates the highest value. Therefore, the assumption of balancedness is less restrictive and more natural.

where $h \in H_{M,2^N}$ and $v \in \mathcal{V}$. For instance, in Example 1 this yields $v_i^{ED}(h) = \frac{v(h)}{4}$ for all $i \in N$. Analogously, in the corresponding dual network h^* each connection $c \in M$ receives $v_c^{ED}(h^*) = \frac{v(h^*)}{3} = \frac{v(h)}{3}$. The allocation rule $ED \in \mathcal{X}$ will be called Equal Division Solution *in the following.*

Distributing the worth in two stages cannot be modeled explicitly by means of player-based allocation rules since, as mentioned before, in this case it is first distributed among the connections and subsequently within each connection. Thus, it is not associated to the players directly but, if the connections represent projects, for example, each player receives her share of each project's budget. Capturing this aspect requires a refined concept of allocation rules.

Definition 5. A *position* is a pair $ic := (i,c) \in N \times M$. A *position-based allocation rule* Y assigns a family of functions $Y = (Y_{ic})_{ic \in N \times M}$ to each allocation problem (N,M) with $Y_{ic} : \mathcal{V} \longrightarrow \mathcal{V}$ and $\sum_{c \in M, i \in N} Y_{ic}(v) = v$ for all $v \in \mathcal{V}$. Moreover, given the dual problem (M,N), each Y is assumed to satisfy $(Y_{ci}(v^*))^* = Y_{ic}(v)$ for all $v \in V$.

The interpretation is basically the same as the interpretation of player-based allocation rules. But in addition there is a further symmetry axiom imposed: it requires that each position in the dual problem receives the same payoff as in the primal one. The set of position-based allocation rules is $\mathcal{Y}$. The salient point is that every position-based allocation rule $Y \in \mathcal{Y}$ induces two player-based allocation rules $X^Y \in \mathcal{X}$ and $Z^Y \in \mathcal{X}$ in a straightforward way. Given $v \in \mathcal{V}$, each player/connection receives the aggregated payoff of all her/its positions:

1. $\left(X_i^Y\right)_{i \in N}$ is given by $v_i^Y := X_i^Y(v) = \sum_{c \in M} Y_{ic}(v)$ and
2. $\left(Z_c^Y\right)_{c \in M}$ by $v_c^Y := Z_c^Y(v) = \sum_{i \in N} Y_{ic}(v)$.

3.1 A Specific Two-stage Procedure

Position-based allocation rules allow for more flexibility in tracking how the worth of a network is distributed among the players. In particular, they allow modeling two-stage allocation procedures explicitly. To fix ideas, let a pair of player-based allocation rules $(X,Z) \in \mathcal{X} \times \mathcal{X}$ be given and for each allocation problem (N,M) define $Y \in \mathcal{Y}$ via $Y_{ic} := X_i \circ Z_c$ for all $ic \in N \times M$. The interpretation is as follows:

1. The allocation rule Z specifies the payoff at the first stage. Given a network $h \in \mathcal{H}$, the connections play a dual game in h^* and they are paid according to Z. That is, connection $c \in M$ receives $v_c^Z(h^*) = (Z_c(v))(h^*)$.

2. Then, in a second step, for each $c \in M$ the value $v_c^Z(h^*)$ is distributed among the players according to $X \in \mathcal{X}$. More specifically, player i's share of connection c is $v_{ic}^Y(h) = \left(X_i\left(v_c^Z\right)\right)(h) = (X_i(Z_c(v)))(h) = ((X_i \circ Z_c)(v))(h)$.

Consequently, a position-based allocation rule $Y \in \mathcal{Y}_{M,2^N}$ is said to be *induced by two player-based allocation rules* if there exists a pair $(X,Z) \in \mathcal{X} \times \mathcal{X}$ with $Y_{ic} := X_i \circ Z_c$ for each allocation problem (N,M) and all $ic \in N \times M$. Let $\mathcal{Y}^{2ply} := \{Y \in \mathcal{Y} \mid \mathcal{Y} \text{ is induced by two player-based allocation rules}\}$. As will be shown later, this is just a special case of a broader class of two-stage allocation procedures. Nevertheless, it possesses striking and remarkable characteristics which are worth studying in more detail.

Lemma 1. Let an allocation problem (N, M) be given. If a position-based allocation rule $Y \in \mathcal{Y}^{2ply}$ is induced by two player-based allocation rules $X \in \mathcal{X}$ and $Z \in \mathcal{X}$, then $Z_c = Z_c^Y$ for all $c \in M$.

Proof. This is a straightforward implication of the balancedness assumption. In particular, in the second step the players distribute exactly the first-stage value among each other:

$$v_c^Y = \sum_{i \in N} Y_{ic}(v) = \sum_{i \in N} \left(X_i\left(Z_c(v)\right)\right) = Z_c(v) = v_c^Z \quad \text{for all } i \in N \text{ and } v \in \mathcal{V}.$$

□

Thus, if $Y \in \mathcal{Y}^{2ply}$ is induced by two player-based allocation rules, the first-stage allocation Z and the connections' values determined by Z^Y do not differ. Generically, this is not necessarily true for X and X^Y. Redistributing the value at the first stage might have a significant impact on the payoff of the players. However, there is an important class of allocation rules where both ways of allocating the worth coincide.

Definition 6. A player-based allocation rule $X \in \mathcal{X}$ is *additive* (ADD) if $X(v + v') = X(v) + X(v')$ for all (N, M) and $v, v' \in \mathcal{V}$. Additivity for position-based allocation rules is defined analogously.

Additive allocation rules exhibit a non-externalities property. The payoff generated by a value function $v \in \mathcal{V}$ is not influenced by other value functions.

Proposition 1. Let an allocation problem (N, M) be given and suppose $Y \in \mathcal{Y}^{2ply}$ is induced by two player-based allocation rules $X \in \mathcal{X}$ and $Z \in \mathcal{X}$. If X is additive, $X_i = X_i^Y$ for all $i \in N$.

Proof. If X is additive, then

$$v_i^Y = \sum_{c \in M} Y_{ic}(v) = \sum_{c \in M} \left(X_i\left(Z_c(v)\right)\right) = X_i \circ \left(\sum_{c \in M} Z_c(v)\right) = X_i(v) = v_i^X \quad \text{for all } i \in N \text{ and } v \in \mathcal{V}.$$

□

Proposition 1 shows that the final outcome obtained by integrating an additive allocation rule $X \in \mathcal{X}$ into a two-stage allocation procedure as introduced above does not differ from applying X in a player-based way. Moreover, this is completely independent of the first-stage allocation rule $Z \in \mathcal{X}$. It does not matter in which way the value is redistributed at the first stage, the final outcome is always the same. In particular, these considerations allow some interesting implications: Suppose a coalitional network represents a complexly structured firm or organization and it is more convenient to pay the players for each project separately instead of calculating the payoff of all players as a function of the organization's revenue. If an additive allocation rule is chosen, the players' payoff is not changed. This motivates the following definition:

Definition 7. A player-based allocation rule $X \in \mathcal{X}$ is *invariant under redistribution* if for all allocation problems (N, M) and $v \in \mathcal{V}$,

$$X_i(v) = \sum_{c \in M} X_i\left(Z_c(v)\right) \quad \text{for all } Z \in \mathcal{X}. \tag{1}$$

As it has been shown in Proposition 1, additive allocation rules satisfy this criterion. It turns out that these are indeed the only ones. The proof of the following result can be found in the appendix.

Proposition 2. A player-based allocation rule $X \in \mathcal{X}$ is invariant under redistribution if and only if it is additive.

Although Proposition 2 might be slightly surprising at first sight, again it is just a direct implication of the balancedness assumption. This together with additivity immediately implies Equation (1). However, even if Proposition 2 is not very complicated from a mathematical point of view, in the context of coalitional networks it gives rise to an alternative interpretation of additivity. It allows for arbitrary redistributions of the worth without changing the final outcome of the players. In fact, many allocation rules discussed in the literature are additive, such as the Shapley Value, the Equal Division Solution or the Position Value, to name but a few.

Before completing this subsection there remains a further question which requires special attention. So far the discussion concentrated on properties of allocation rules in $\mathcal{Y}^{2ply}$ without explicitly characterizing this particular class. In other words: Which position-based allocation rules are actually induced by two player-based allocation rules? In order to approach this question, let $Y \in \mathcal{Y}$ and suppose there exist $X \in \mathcal{X}$ and $Z \in \mathcal{X}$ with $Y_{ic} = X_i \circ Z_c$ for all allocation problems (N, M). This, in particular, implies that each position's value only depends on the corresponding connection's value generated at the first-stage but not on the connection itself. That is, if the first-stage outcomes v_c^X and $\bar{v}_{\bar{c}}^X$ of two connections $c, \bar{c} \in M$ for two allocation rules $v, \bar{v} \in \mathcal{V}$ coincide, the payoff player i receives for both positions has to be the same as well, i.e, $X_i\left(v_c^X\right) = X_i\left(\bar{v}_{\bar{c}}^X\right)$. By applying Lemma 1 this independence property can be restated as follows: If the values of two connections coincide, each player receives the same payoff from both of them. In fact, this property is not only necessary but also sufficient.

Proposition 3. A position-based allocation rule $Y \in \mathcal{Y}$ is induced by two player-based allocation rules if and only if it satisfies the following independence property: Given an allocation problem (N, M), if $v_c^Y = \bar{v}_{\bar{c}}^Y$ for $c, \bar{c} \in M$ and $v, \bar{v} \in \mathcal{V}$, then $v_{ic}^Y = \bar{v}_{i\bar{c}}^Y$ for all $i \in N$.

The proof can be found in the appendix. A consequence of this independence property is that players might receive payoff from connections they are not a member of. Consider for example the Equal Division Solution $ED \in \mathcal{X}$ which is additive and let an allocation problem (N, M) and a network $h \in \mathcal{H}$ be given. If $Z \in \mathcal{X}$ is a first-stage allocation rule, generically it is possible that $\left(\left(ED_i \circ Z_c\right)(v)\right)(h) \neq 0$ even if $i \notin h(c)$. This is due to the fact that v_c^Z depends on the whole network and not only on c. Of course, in certain situations it is quite reasonable that players receive payoff from connections they are not a member of, like in the case of externalities, for example. However, in many environments it seems to be more convincing that the value of a connection is distributed only among the corresponding members. This aspect is not captured so far and it requires a higher degree of flexibility. That is, instead of applying the same second-stage allocation rule $X \in \mathcal{X}$ for all $c \in M$ it is necessary to use for each connection a specifically designed $X^c \in \mathcal{X}$.

3.2 Reduced Allocation Problems

Consider a similar situation as before. Again, the connections first receive a certain payoff, but now the players are supposed to face reduced problems of allocating the value within each connection. The main idea is that the corresponding members take the rest of the network as given and compare the prospects if only their connection changes. To define this formally, assume that there is given a coalitional network $h \in \mathcal{H}$, a value function $v \in \mathcal{V}$, and a first-stage allocation rule $Z \in \mathcal{X}$. For each set of players $S \subseteq N$ the network $h|_{c,S} \in \mathcal{H}$ is obtained from h by deleting all players in $h(c)$

that are not contained in S while the other links remain the same, i.e.,

$$h|_{c,S}(c') = \begin{cases} h(c'), & \text{if } c' \neq c \\ h(c) \cap S, & \text{if } c' = c \end{cases}$$

for all $c, c' \in M$. Furthermore, let $h|_{c,S}^*$ denote the corresponding dual network. This substructure allows designing a reduced allocation problem $(N, \{c\})$ within connection $c \in M$. Each subset of players $S \subseteq N$ can be interpreted as a coalitional network with only one connection (cf. Remark 1), and the corresponding value function $w_{v^Z}^{c,h} \in \mathcal{V}_{\{c\},2^N}$ of the reduced problem is given by $w_{v^Z}^{c,h}(S) := v_c^Z\left(h|_{c,S}^*\right)$. The players only take into account in which way the first-stage payoff changes if deviations within c occur while the other connections remain the same. In order to guarantee that $w_{v^Z}^{c,h} \in \mathcal{V}_{\{c\},2^N}$ is well-defined, that is, to guarantee that $w_{v^Z}^{c,h}$ is normalized, in the following it will always be required that $Z \in \mathcal{X}$ satisfies the active players axiom.

Definition 8. A player-based allocation rule $X \in \mathcal{X}$ satisfies the *inactive players axiom* (IPLY) if for each allocation problem (N, M), $i \notin N(h)$ implies $v_i^X(h) = 0$ for all $h \in \mathcal{H}$ and $v \in \mathcal{V}$.

If a player-based allocation rule satisfies this axiom, players who are not active get zero payoff. In the dual game considered here, this means that empty connections receive a worth of zero.
Now suppose $X^{re} \in \mathcal{X}$ is an allocation rule applied to the reduced problem $(N, \{c\})$. This induces finally a second-stage allocation rule $X^c \in \mathcal{X}$ via $\left(X_i^c\left(v_c^Z\right)\right)(h) := \left(X_i^{re}\left(w_{v^Z}^{c,h}\right)\right)(h(c))$ for all $i \in N$.[5]
Note that although the main motivation was to construct a position-based allocation rule which gives zero payoff to positions not contained in the network, up to here it is not required that this condition is indeed satisfied. However, assuming additionally for all connections $c \in M$ that the allocation rule applied to $(N, \{c\})$ satisfies the inactive players axiom is sufficient to achieve the desired effect.
Of course, the main idea of considering reduced problems of allocating the value is similar to Subsection 3.1, but the main difference is that now the allocation rules in the second step do not necessarily have to be the same for all connections. Therefore, the corresponding two-stage procedure has the following form:

1. Given a value function $v \in \mathcal{V}$, the value of $c \in M$ is determined according to a first-stage allocation rule $Z \in \mathcal{X}$.
2. Then v_c^Z is allocated according to $X^c \in \mathcal{X}$ for each $c \in M$, i.e., player i receives $X_i^c\left(v_c^Z\right)$.

In the following, a position-based allocation rule $Y \in \mathcal{Y}$ is said to be *implemented in two stages* if for each allocation problem (N, M) there exist $X^c \in \mathcal{X}$ for each $c \in M$ and $Z \in \mathcal{X}$ such that $Y_{ic} = X_i^c \circ Z_c$ for all $i \in N$ and $c \in M$. Let $\mathcal{Y}^{2stg} \subseteq \mathcal{Y}$ be the set of position-based allocation rules satisfying this criterion, i.e., $\mathcal{Y}^{2stg} = \{Y \in \mathcal{Y} \mid Y \text{ is implemented in two stages}\}$. Note that $\mathcal{Y}^{2ply} \subseteq \mathcal{Y}^{2stg}$.

Lemma 2. Let an allocation problem (N, M) be given and assume a position-based allocation rule $Y \in \mathcal{Y}^{2stg}$ is implemented in two stages, where $Z \in \mathcal{X}$ is the first-stage allocation rule. Then $Z_c = Z_c^Y$ for all $c \in M$.

Proof. The reasoning is completely analog to Lemma 1. □

[5]For further allocation problems $(N', M') \neq (N, M)$ it would be possible to proceed analogously.

In particular, it does not matter whether for each connection a specifically designed allocation rule is used or whether it is always the same one. The aggregated value is the same in any case due to the balancedness assumption. Moreover, although $\mathcal{Y}^{2stg}$ is a broader class of allocation rules than $\mathcal{Y}^{2ply}$, it is possible to characterize it by a similar independence property.

Proposition 4. A position-based allocation rule $Y \in \mathcal{Y}$ is implemented in two stages if and only if it satisfies the following independence property: Given an allocation problem (N, M), if $v_c^Y = \bar{v}_c^Y$ for $c \in M$ and $v, \bar{v} \in \mathcal{V}$, then $v_{ic}^Y = \bar{v}_{ic}^Y$ for all $i \in N$.

The proof proceeds analogously to the one of Proposition 3 and the interpretation is similar as well. The difference is that here, the payoff of each position has to be independent only of how the connection's value is generated but not of the connection itself. This is, of course, due to the fact that the second-stage allocation rule now may depend on c.
The main relevance of the previous proposition is that it characterizes all two-stage allocation procedures in a natural way. Therefore, implicitly it also provides a tool for checking whether position-based allocation rules belong to this class or not. The remainder of this paper is devoted to the study of particular examples, like the Position Value, for instance. It also includes a general discussion of properties of allocation rules.

4 Some Particular Allocation Rules

Before analyzing some specific two-stage allocation procedures, first the focus lies on player-based allocation rules. The economic implications of this intermediate step are actually just straightforward extensions from Network Theory and Cooperative Game Theory. Nevertheless, they strongly support the intuition of the two-stage allocation procedures analyzed later in Subsection 4.2 and provide a technical substructure for them.

4.1 Player-based Allocation Rules

One specific player-based allocation rule, the Equal Division Solution, has already been introduced in Example 2. Another well-known example from literature is the Myerson Value which has been established by Myerson (1977) and extended to bilateral networks by Jackson and Wolinsky (1996). The main motivation of this allocation rule is to pay each player her expected marginal contribution to a (bilateral) network. Clearly, this idea can be adopted completely analogously to the setting considered here. Formally this means that given an allocation problem (N, M), for each $h \in \mathcal{H}$ and $v \in \mathcal{V}$ the *Myerson Value* (for coalitional networks) $MV\mathcal{X}$ is defined by

$$v_i^{MV}(h) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-(|S|+1))!}{n!} \cdot \left(v(h|_{S+i}) - v(h|_S)\right) \quad \text{for all } i \in N.[6]$$

At first sight the Equal Division Solution and the Myerson Value seem to be quite different and a priory it is not clear which one is appropriate for which situation. To gain a better understanding, it is necessary to analyze which properties both allocation rules have and in which way they differ.

[6] The Myerson Value is, roughly speaking, an extension of the well-known *Shapley Value* (Shapley, 1953). In fact, if Φ denotes the Shapley Value, $v_i^{MV}(h) = \Phi_i\left(w_v^h\right)$ where $w_v^h(S) := v(h|_S)$ for all $S \subseteq N$.

Definition 9. Let an arbitrary allocation problem (N, M) be given. A player-based allocation rule $X \in \mathcal{X}$ satisfies ...

(CBAL) ... *component-balancedness* if always $\sum_{i \in A} v_i^X(h) = v(h|_A)$ for every network $h \in \mathcal{H}$, every component-additive value function $v \in \mathcal{CA}$, and all associations $A \in \mathcal{A}(h)$.

(SYM) ... *symmetry of active players* if for $i, j \in N(h)$, $v(h|_{S+i}) = v(h|_{S+j})$ for all $S \subseteq N(h) \setminus \{i, j\}$ always implies $v_i^X(h) = v_j^X(h)$.

(NULI) ... the *nullifying player property* if $v(h|_{S+i}) = 0$ for all $S \subseteq N(h) \setminus \{i\}$ implies $v_i^X(h) = 0$.

(NULL) ... the *null player property* if $v(h|_{S+i}) - v(h|_S) = 0$ for all $S \subseteq N(h)$ implies $v_i^H(h) = 0$.

The previous notions are standard in the literature of Cooperative Game Theory and Network Theory and therefore the interpretations do not change. Given a *component-additive* value function, an allocation rule satisfies component-balancedness if it distributes the value of each component only among its members. *Symmetry of active players* requires that two players receive the same payoff whenever their contributions to the network are equal. An allocation rule satisfies the *nullifying player property* if a player receives no payoff whenever her presence in a network induces a value of zero. Deegan and Packel (1978) call these individuals zero players. In contrast to this, the *null player property*, also known as the dummy axiom, focuses on the marginal contribution of a player to the network. The usual interpretation is that a player has no bargaining power if her marginal contribution to the network is always zero and, thus, she should receive no payoff. Note that this implies the inactive players axiom.

In the context of Cooperative Game Theory, van den Brink (2007) proved that the Shapley Value and the Equal Division Solution satisfy similar axioms. In fact, it is possible to extend this insight to the more general setting considered here.

Proposition 5. $X \in \mathcal{X}$ satisfies (ADD), (SYM), (IPLY), and (NULI) if and only if $Y = ED$.

For the proof (and also for the proofs of the following results) refer to the appendix. In principle it proceeds in a standard fashion. More precisely, given an allocation problem (N, M), due to additivity it is sufficient to focus on basic value functions which form a basis of $\mathcal{V}$ (recall Remark 2).

Proposition 6. $X \in \mathcal{X}$ satisfies (ADD), (SYM), and (NULL) if and only if $Y = MV$.

Since the Myerson Value is an extension of the Shapley Value (Shapley, 1953), it can be characterized by similar axioms. Analogously to the characterization of the Equal Division Solution, the proof of Proposition 6 also exploits additivity and concentrates on basic value functions.
The previous results show that if an allocation rule is supposed to satisfy additivity, symmetry of active players, and the inactive players axiom, the final outcome strongly depends on the decision whether more importance is attached to nullifying or to null players. In one case the Myerson Value and in the other the Equal Division Solution is obtained.

In fact, in Cooperative Game Theory the null player property is often used to characterize solutions. Besides the Myerson and the Shapley Value, also Owen's Value (Owen, 1977), for example, satisfies this axiom. It allows some noteworthy implications which should not be neglected here:

Lemma 3. If $X \in \mathcal{X}$ satisfies (ADD) and (NULL), it also satisfies (CBAL).

Proposition 6 and Lemma 3 immediately imply the following result.

Proposition 7. The Myerson Value is component-balanced.

The intuition of Lemma 3 is straightforward. If a value function is component-additive, the value of a component does not depend on the others. Therefore, the marginal contribution of a player to other components is always zero. The result then follows from the null player property and because the individual payoff functions can be decomposed appropriately by additivity.[7]

4.2 Position-based Allocation Rules

All the properties introduced in the previous subsection focus on the players directly but not on the positions. The marginal contribution, for instance, is determined only for subnetworks where player $i \in N$ completely drops out of the network. In this case, she totally stops cooperating. In economic life, however, if the connections represent projects or the like, it might happen that a player only leaves some of her projects but not all of them. Translated to the model this would mean that she stops cooperating only partially. To fix ideas, consider the position-based variations of the properties introduced in the previous sections:

Definition 10. Let an arbitrary allocation problem (N, M) be given. A position-based allocation rule $Y \in \mathcal{Y}$ satisfies ...

(CBAL') ... *component-balancedness* if always $\sum_{i \in A} \sum_{c:h(c) \subseteq A} v_{ic}^Y(h) = v(h|_A)$ for all $h \in \mathcal{H}$, every component-additive value function $v \in \mathcal{CA}$, and all associations $A \in \mathcal{A}(h)$.

(IPOS) ... the *inactive positions axiom* if $i \notin h(c)$ always implies $v_{ic}^Y(h) = 0$ for all $v \in \mathcal{V}$.

(cwSYM) ... *symmetry of active players connection-wise* if for $c \in M$ and $i, j \in h(c)$ from $v_c^Y\left(h|_{c,S\cup\{i\}}^*\right) = v_c^Y\left(h|_{c,S\cup\{j\}}^*\right)$ for all $S \subseteq h(c) \smallsetminus \{i, j\}$ always $v_{ic}^Y(h) = v_{jc}^Y(h)$ follows.

(cwNULI) ... *connection-wise* the *nullifying player property* if for $c \in M$ from $v_c^Y\left(h|_{c,S}^*\right) = 0$ for all $S \subseteq N(h) \smallsetminus \{i\}$ always $v_{ic}^Y(h) = 0$ follows.

(cwNULL) ... *connection-wise* the *null player property* if for $c \in M$ from $v_c^Y\left(h|_{c,S\cup\{i\}}^*\right) = v_c^Y\left(h|_{c,S}^*\right)$ for all $S \subseteq h(c) \smallsetminus \{i\}$ always $v_{ic}^Y(h) = 0$ follows.

component-balancedness and the *inactive positions axiom* are analogously defined as the axioms for player-based allocation rules and therefore the interpretations are the same as well. However, here the focus lies on the positions but not on the players. For example, *(cwNULL)* concentrates on the marginal contributions of the players to single connections. Given this axiom, player $i \in N$ receives nothing from $c \in M$ if her marginal contribution to the connection is zero. Similar considerations also apply if the position-based allocation rule satisfies *connection-wise the nullifying player property* or *symmetry of active players*.

[7]For bilateral networks Jackson and Wolinsky (1996), who refer to Myerson (1977), have already established Proposition 7. Although they have shown it in a different context, and thus in a different way, some elements of the proof of Lemma 3 are adopted from these works.

One of the most prominent two-stage allocation rules in Network Theory is the often-studied Position Value (e.g., Borm *et al.*, 1992). The main idea is to determine the connections' values according to the Myerson Value and then, in the second step, the members of each connection apply the Equal Division Solution in order to share the corresponding value equally. Of course, this motivation can be extended to the more general setting considered here. Given an allocation problem (N, M), the *Position Value* (for coalitional networks) $PV \in \mathcal{Y}$ is given explicitly by

$$v_{ic}^{PV}(h) = (PV_{ic}(v))(h) := \begin{cases} \frac{1}{|h(c)|} \cdot v_c^{MV}(h^*), & \text{if } i \in h(c) \\ 0, & \text{if } i \notin h(c) \end{cases}$$

for all $i \in N$ and $c \in M$. Note that each player i receives

$$v_i^{PV}(h) = \sum_{c: i \in h(c)} \frac{1}{|h(c)|} \cdot v_c^{MV}(h^*)$$

and the value of each connection is indeed given by the Myerson Value: $v_c^{PV} = v_c^{MV}$ for all $c \in M$. Taking into account the characterizations established in Propositions 5 and 6, these considerations imply that the Position Value needs to satisfy (ADD), (SYM), together with (NULL) at the first stage and (ADD), (IPOS), (cwSYM), and (cwNULI) with respect to each connection. In fact:

Proposition 8. Suppose $Y \in \mathcal{Y}$ satisfies (ADD), (IPOS), (cwSYM), (cwNULI), and assume $Z^Y \in \mathcal{X}$ satisfies (NULL) and (SYM). Then $Y \in \mathcal{Y}^{2stg}$ and, furthermore, $Y = PV$.[8]

Again, the proof can be found in the appendix. Showing $Y \in \mathcal{Y}^{2stg}$ is based on exploiting (cwNULI) and (ADD). By applying the latter axiom and Proposition 4, it is sufficient to show that whenever a connection receives a value of zero in all networks, the positions within this connection receive a payoff of zero, too. This is given, obviously, by (cwNULI). Moreover, since the first-stage allocation rule is the Myerson Value, component-balancedness also carries over to the Position Value.

Lemma 4. If $Y \in \mathcal{Y}$ is component-balanced both induced player-based allocation rules $X^Y \in \mathcal{X}$ and $Z^Y \in \mathcal{X}$ have this property, too.

Proof. Let (N, M) be an allocation problem and $v \in \mathcal{CA}$ be component-additive. Furthermore, let $h \in \mathcal{H}$ and $A \in \mathcal{A}(h)$. Then:

$$\sum_{i \in A} v_i^Y(h) = \sum_{i \in A} \sum_{c \in M} \underbrace{v_{ic}^Y(h)}_{=0, \text{ if } i \notin h(c)} = \sum_{i \in A,\ c: h(c) \subseteq A} v_{ic}^Y(h) = v(h|_A).$$

Analogously for $Z_c^Y = \sum_{i \in N} Y_{ci}$. □

Lemma 5. Let $Y \in \mathcal{Y}$ satisfy (IPOS). Then $Z^Y \in \mathcal{X}$ is component-balanced if and only if $X^Y \in \mathcal{X}$ is component-balanced.

[8] Borm *et al.* (1992) introduce a characterization of the Position Value for a certain class of Myerson's communication situations (cf. Myerson, 1977). Interestingly, they also interpret their "arc game" as a "dual game": "[The value] of an arc is measured by means of the Shapley Value of a kind of 'dual' game [. . .]" (Borm *et al.*, 1992, p. 306). Later, van den Nouweland and Slikker (2012) extended this characterization to bilateral networks. However, it is not possible to extend it further to coalitional networks in a straightforward way, because they use the "superfluous link property", which is not as restrictive here as in the bilateral setting.

Proof. For the proof just note that if Y satisfies the inactive positions axiom, the proof of Lemma 4 works in both directions. $\square$

Proposition 9. Let $Y \in \mathcal{Y}$ satisfy (IPOS) and (ADD). If, furthermore, $Z^Y \in \mathcal{X}$ satisfies (NULL), then $X^Y \in \mathcal{X}$ is component-balanced.

Proof. This is a direct implication of Lemmas 3 and 5. $\square$

Of course, Proposition 9 is also satisfied if X^Y and Z^Y interchange roles.

Proposition 10. The Position Value is component-balanced.

For bilateral networks, van den Nouweland and Slikker (2012) already established Proposition 10. However, to show it, the authors used "component decomposability" which is not needed here. In the context of coalitional networks, the result is a direct implication of the characterization of the Position Value.

Although the Position Value possesses interesting and desirable features, in some situations it might be inadequate. For example, using the Myerson Value first and the Equal Division Solution in the second step violates consistency. Following Hart and Kurz (1983), a two-stage allocation rule is consistent if the way of distributing the value within each connection is the same as the one used for determining the values in the dual game. Of course, there are several ways to construct consistent allocation rules, but in light of the preceding discussion, a canonical candidate is the iterated application of the Myerson Value. Therefore, consider again a two-stage procedure where the first-stage allocation rule is $MV \in \mathcal{X}$. Given an allocation problem (N, M), for each $h \in \mathcal{H}$ and $v \in \mathcal{V}$, the value function of the reduced game within $c \in M$ is then given by $w^{c,h}_{v^{MV}}(S) = v^{MV}_c \left(h|^*_{c,S}\right)$ for all $S \subseteq N$. In contrast to the Position Value, suppose now that the second-stage allocation rules are chosen in a consistent way: that is, the solutions of the reduced games are also determined according to the Myerson Value $MV \in \mathcal{X}$ for all $c \in M$. This iterated application of the Myerson Value will be denoted by $IM \in \mathcal{Y}$ and it is given by $v^{IM}_{ic}(h) \coloneqq \big(MV_i\big(w^{c,h}_{v^{MV}}\big)\big)(h(c))$.
As already discussed in Section 3, the characterizations of the Equal Division Solution and the Myerson Value differ by only one axiom. Consequently, the same is also true for the Position Value and the iterated application of the Myerson Value. In fact, instead of the nullifying player property, IM satisfies connection-wise the null player property.

Proposition 11. Let $Y \in Y_{M,2^N}$ satisfy (ADD), (cwSYM), (cwNULL), and assume $Z^Y \in \mathcal{X}$ satisfies (SYM). Then $Y \in \mathcal{Y}^{2stg}$, Z^Y satisfies (NULL), and, furthermore, $Y = IM$.

Moreover, since in the first step the worth is distributed according to the Myerson Value, Lemma 5 can be applied again.

Proposition 12. IM is component-balanced.

Applying the Myerson Value iteratively is in spirit with a prominent solution concept from Cooperative Game Theory, namely with Owen's Value (Owen, 1977). In his seminal contribution, the author focused on cooperative games with a given coalition structure, that is, the players are partitioned exogenously into coalitions which may not overlap. Given this setting, Owen applied the Shapley Value iteratively, first among the coalitions and then within each coalition. Loosely speaking, this approach does not take the actual form of cooperation $h \in \mathcal{H}$ into account but instead it focuses only on the induced associations $\mathcal{A}(h)$. To fix ideas, let (N, M) be an allocation problem and let Ψ denote Owen's Value. Moreover, for given $v \in \mathcal{V}$, let $w_v^h(S) := v(h|_S)$ be the worth of $h \in \mathcal{H}$ restricted to $S \subseteq N$. Then Owen's Value can be extended analogously to the Myerson Value via $v_i^{OV}(h) = \Psi\left(\mathcal{A}(h), w_v^h\right)$. Note that Ψ depends on the coalition structure $\mathcal{A}(h)$ but not on the network h directly. Consider the special case where the network $h \in \mathcal{H}$ actually forms a coalition structure: that is, assume $\bigcup_{c \in M} h(c) = N$ and $h(c) \cap h(c') \neq \varnothing$ only if $c = c'$ for all $c, c' \in M$. Phrased differently, each player is a member of exactly one connection. Here, the only difference between iteratively applying the Myerson Value IM and Owen's Value OV is that the latter one focuses less on single positions. More specifically, it is possible to show that OV satisfies the null player property only in a player-based way but not connection-wise. However, if the total productivity is determined by means of a component-additive value function, each player's contribution to the whole network is exactly what she contributes to her connection and therefore both axioms are equivalent.

Proposition 13. Assume $h \in \mathcal{H}$ forms a coalition structure and $v \in \mathcal{CA}$ is a component-additive value function. Then $v_i^{IM}(h) = v_i^{OV}(h)$ for all $i \in N$.[9]

The main idea of the proof is to translate the axioms from the characterization given in Owen (1977) to coalitional networks and to show that both allocation rules have the same properties. The previous result immediately implies that the Position Value and Owen's Value differ by just one axiom under certain circumstances (i.e., if $h \in \mathcal{H}$ forms a coalition structure and $v \in \mathcal{V}$ is component-additive). Moreover, given these requirements, it is straightforward to show that component-balancedness is equivalent to what Aumann and Drèze (1974) call "relative efficiency": that is, the aggregated payoff of players within each connection equals exactly the connection's value. For this special case it is also possible to extend the Aumann-Drèze Value to coalitional networks and it coincides with iteratively applying the Myerson Value and with Owen's Value.

5 Conclusions

The structures of many economical and social organizations are too complex to analyze them by means of Cooperative Game Theory or Bilateral Network Theory. The model introduced here uses mathematical hypergraphs (which are called coalitional networks in this study) to extend both approaches to a more general and richer framework. This enlarges not only the field of possible applications but it also allows to model them more realistically. In particular, the extended setting allows to formalize and analyze two-stage allocation procedures in a convenient way. Although these procedures already have been attempted in literature by some studies (e.g., Albizuri *et al.*, 2005; Aumann and Drèze, 1974; Borm *et al.*, 1992; Hart and Kurz, 1983; Owen, 1977) there has been no general framework for analyzing them in a structured way. This study not only provides such a framework but it also extends some particular examples of two-stage allocation rules well-known from Cooperative Game Theory and Network Theory to coalitional networks. Moreover, characterizations of these allocation rules are discussed. Interestingly, under certain requirements the Position Value and Owen's Value satisfy almost the same axioms.

[9] The author is grateful to André Casajus for pointing out a mistake in a previous version of this result.

The analysis of coalitional networks could be extended in several ways. One of the most interesting ones would be to investigate dynamic models of network formation. In the study presented here, the coalitional network was always given exogenously. Therefore, it seems to be crucial to know under which circumstances which structures will arise. To this end, suitable stability concepts and deviation rules would need to be introduced. It would furthermore be reasonable to allow N and M to be infinite and to work with finite carriers in order to have no exogenously given limitation in the number of networks. Another natural extension is to apply the extended model to applications not covered so far. Coalitional networks are richer than coalition structures or bilateral networks, and they allow to study settings where the players cooperate in quite complex structures. Examples of this are large organizations or social clubs, to name but a few.

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A Proofs

Some of the proofs are based on only slightly varied but similar technical arguments. For example, the characterizations of the allocation rules discussed in this paper are all based on basic value functions and they all proceed in a similar way. In order to not repeat the same arguments again and again, the following two lemmas will be helpful.

Lemma 6. Let an allocation problem (N, M) be given. Assume the allocation rule $X \in \mathcal{X}$ satisfies (ADD), (IPLY), and (SYM). Furthermore, let $\alpha \in \mathbb{R}$ be a constant and $1^h \in \mathcal{V}$ a basic value function with $h \in \mathcal{H} \setminus \{h^\varnothing\}$. Then:

$$\left(X_i\left(\alpha 1^h\right)\right)(h') = \begin{cases} 0, & \text{if } i \notin N(h') \\ 0, & \text{if } i \in N(h') \text{ and } h \nsubseteq h' \\ \frac{\alpha}{|N(h')|}, & \text{if } i \in N(h') \text{ and } h = h' \end{cases}$$

Proof. Let $X \in \mathcal{X}$ satisfy (ADD), (IPLY), and (SYM). Furthermore, let $\alpha \in \mathbb{R}$, $1^h \in \mathcal{V}$ for $h \in \mathcal{H} \setminus \{h^\varnothing\}$, and $h' \in \mathcal{H}$. First note that the inactive players axiom implies $\left(X_i\left(\alpha 1^h\right)\right)(h') = 0$ for all $i \notin N(h')$.

Case 1: $h \nsubseteq h'$.

This implies $1^h(h') = 0$. Furthermore, $\alpha 1^h(h'|_{S+i}) = \alpha 1^h(h'|_{S+j}) = 0$ for all $i, j \in N(h')$ and $S \subseteq N(h') \setminus \{i, j\}$. Thus, by symmetry $\left(X_i\left(\alpha 1^h\right)\right)(h') = \left(X_j\left(\alpha 1^H\right)\right)(h')$ for all $i, j \in N(h')$ and, because X is balanced by definition, this implies that all players receive a payoff of 0.

Case 2: $h = h'$.

This case proceeds analogously to the previous one. Again $\alpha 1^h(h'|_{S+i}) = \alpha 1^h(h'|_{S+j}) = 0$ for all $i, j \in N(h')$ and $S \subseteq N(h') \setminus \{i,j\}$. From this follows $\left(X_i\left(\alpha 1^h\right)\right)(h') = \left(X_j\left(\alpha 1^h\right)\right)(h')$ for all $i, j \in N(h')$ by symmetry and, thus, $\left(X_i\left(\alpha 1^h\right)\right)(h') = \frac{\alpha}{|N(h')|}$ by balancedness. □

Note that Lemma 6 makes no statement about $h \subsetneq h'$. For this case, further requirements are needed.

Lemma 7. Assume that the position-based allocation rule $Y \in \mathcal{Y}$ is additive. If it additionally satisfies (cwNULI) or (cwNULL), then $Y \in \mathcal{Y}^{2stg}$.

Proof. Let (N, M) be an arbitrary allocation problem. If $Y \in \mathcal{Y}$ is additive, for applying Proposition 3 it is sufficient to show that $v_c^Y = 0$ for $c \in M$ and $v \in \mathcal{V}$ always implies $v_{ic}^Y = 0$ for all $i \in N$. Phrased differently, if the connection's worth is zero in all networks the corresponding positions receive this worth as well. Therefore, suppose there is given a value function $v \in \mathcal{V}$ with $v_c^Y(h^*) = 0$ for all $h^* \in \mathcal{H}^*$. By applying either (cwNULI) or (cwNULL) this immediately implies $v_{ic}^Y(h) = 0$ for all $i \in N$. Thus, $Y \in \mathcal{Y}^{2stg}$. □

Proof of Proposition 2

This proposition is a direct implication of balancedness. Given an allocation problem (N, M) and a player-based allocation rule $Z \in \mathcal{X}$, each value function $v \in \mathcal{V}$ can be decomposed via $v = \sum_{c \in M} v_c^Z$. Thus, an equivalent formulation of invariance under redistribution for $X \in \mathcal{X}$ is

$$X_i\left(\sum_{c \in M} v_c^Z\right) = \sum_{c \in M} X_i\left(v_c^Z\right)$$

for all $i \in N$, $h \in \mathcal{H}_{M,2^M}$ and each sequence of payoff functions $\left\{v_c^Z\right\}_{c \in M}$, and this is obviously equivalent to additivity. □

Proof of Proposition 3

Let an allocation problem (N, M) be given. First assume $Y_{ic} = X_i \circ Z_c$ for two player-based allocation rules $X \in \mathcal{X}$ and $Z \in \mathcal{X}$. If $v_c^Y = \bar{v}_{\bar{c}}^Y$ for $c, \bar{c} \in M$ and $v, \bar{v} \in \mathcal{V}$, Lemma 1 implies

$$v_{ic}^Y = X_i\left(v_c^Z\right) = X_i\left(v_c^Y\right) = X_i\left(\bar{v}_{\bar{c}}^Y\right) = X_i\left(\bar{v}_{\bar{c}}^Z\right) = \bar{v}_{i\bar{c}}^Y \quad \text{for all } i \in N.$$

For the other direction let $Y \in \mathcal{Y}$ be given. According to Lemma 1 it is feasible to choose $Z_c := Z_c^Y$ for all $c \in M$. Exploiting the condition given in the proposition yields that

$$X_i(v') := \begin{cases} Y_{ic}(v), & \text{if there exist } c \in M \text{ and } v \in \mathcal{V} \text{ with } v' = Z_c(v) \\ X_i^Y(v'), & \text{otherwise} \end{cases}$$

is well-defined for all $v' \in \mathcal{V}$. Of course, it would also be possible to choose any other $X' \in \mathcal{X}$ in the second case. Then balancedness is indeed satisfied because

$$\begin{aligned} \sum_{i \in N} X_i(v') &= \begin{cases} \sum_{i \in N} Y_{ic}(v), & \text{if there exist } c \in M \text{ and } v \in \mathcal{V} \text{ with } v' = Z_c(v) \\ \sum_{i \in N} X_i^Y(v'), & \text{otherwise} \end{cases} \\ &= \begin{cases} Z_c^Y(v) = Z_c(v) = v', & \text{if there exist } c \in M \text{ and } v \in \mathcal{V} \text{ with } v' = Z_c(v) \\ \sum_{i \in N, c \in M} Y_{ic}(v') = v', & \text{otherwise.} \end{cases} \end{aligned}$$

Furthermore, by construction also $Y_{ic} = X_i \circ Z_c$ for all $i \in N$ and $c \in M$. □

Proof of Proposition 4

The proof proceeds analogously to the one of Proposition 3. □

Proof of Proposition 5

Let an allocation problem (N, M) be given. It is easy to verify that ED satisfies all the axioms. For the other direction let $\alpha \in \mathbb{R}$ and $h \in \mathcal{H}$. Furthermore, assume $X \in \mathcal{X}$ satisfies all the axioms. Let $h' \in \mathcal{H}$ be a further network. Applying Lemma 6 yields for $h \nsubseteq h'$ and $h = h'$ that $\left(X_i\left(\alpha 1^h\right)\right)(h')$ is uniquely determined. Therefore, suppose for the rest of the proof $h \subsetneq h'$. Obviously, $\alpha 1^h = \alpha \sum_{\bar{h}:h\subseteq\bar{h}} 1_{\bar{h}}$, where $1_{\bar{h}}$ is called *Standard Value Function* (cf. van den Brink, 2007) and it is given by

$$1_{\bar{h}}(h') = \begin{cases} 1, & \text{if } \bar{h} = h' \\ 0, & \text{if } \bar{h} \neq h'. \end{cases}$$

If $\bar{h} \neq h'$, player $i \in N(h')$ is a nullifying player with respect to $\alpha 1_{\bar{h}}$ if and only if $i \notin N(\bar{h})$. This implies $(X_i(1_{\bar{h}}))(h') = 0$ for all $i \notin N(\bar{h})$. Furthermore, because of symmetry and balancedness the other players receive a payoff of zero, too. Thus, additivity implies

$$\left(X_i(\alpha 1^h)\right)(h') = \sum_{\bar{h}:h\subseteq\bar{h}} \underbrace{\left(X_i\left(\alpha 1_{\bar{h}}\right)\right)(h')}_{=0,\text{ if } \bar{h}\neq h'} = \left(X_i\left(\alpha 1_{h'}\right)\right)(h').$$

Analogously to case 2 in Lemma 6 it is possible to show that

$$\left(X_i\left(\alpha 1_{h'}\right)\right)(h') = (X_i(\alpha 1^{h'}))(h')$$

and, therefore, the payoff is uniquely determined. □

Proof of Proposition 6

Let an allocation problem (N, M) be given. The proof proceeds in a similar way as the one of Proposition 5. First note that the Myerson Value satisfies the axioms. The easiest way for verifying this is by exploiting the properties of the Shapley Value Φ (see, e.g., Roth, 1988) because $v_i^{MV}(h) = \Phi_i(w_h^v)$, where $w_h^v(S) = v(h|_S)$ for all $S \subseteq N$. The rest of the proof proceeds in standard fashion. Because of additivity it is sufficient to concentrate on basic value functions for showing uniqueness. To this end let $\alpha \in \mathbb{R}$ and $h \in \mathcal{H} \setminus \{h^{\varnothing}\}$. Furthermore, assume $X \in \mathcal{X}$ satisfies the axioms given in the proposition. Because the null player property implies the inactive players axiom, Lemma 6 can be applied. That is, for $h \nsubseteq \bar{h}$ and $h = \bar{h}$ the payoff is uniquely determined. For $h \subsetneq \bar{h}$ each player $i \notin N(h)$ is a null player and therefore receives a payoff of 0. Because all players in $N(h)$ are symmetric in $\bar{h}$ and Y is balanced, $\left(X_i\left(\alpha 1^h\right)\right)(\bar{h}) = \frac{\alpha}{|N(h)|}$ for all $i \in N(h)$. Thus, in this case the payoff is uniquely determined as well. □

Proof of Lemma 3

Let an allocation problem (N, M) be given and suppose $X \in \mathcal{X}$ satisfies additivity and the null player property. Furthermore, let $h \in \mathcal{H}$ be a network and $A_i \in \mathcal{A}(h)$ the association containing player $i \in N$. For each component-additive value function $v \in \mathcal{CA}$ define the auxiliary function $v_{A_i} : \mathcal{H} \longrightarrow \mathbb{R}$ by $v_{A_i}(\bar{h}) = v(\bar{h}|_{A_i})$ for all $\bar{h} \in \mathcal{H}$. Then:

$$\begin{aligned} v(h|_{S+i}) - v_{A_i}(h|_{S+i}) &= \sum_{A \in \mathcal{A}(h)} v\left(h|_{((S+i)\cap A)}\right) - v\left(h|_{((S+i)\cap A_i)}\right) \\ &= \sum_{A \in \mathcal{A}(h)} v\left(h|_{((S)\cap A)}\right) - v\left(h|_{((S)\cap A_i)}\right) = v(h|_S) - v_{A_i}(h|_S) \end{aligned}$$

Therefore, player i is a null player with respect to $v - v_{A_i}$ in the network h. The null player property implies $0 = (X_i(v_{A_i} - v))(h)$ and, thus, by additivity $(X_i(v))(h) = (X_i(v_{A_i}))(h)$ for all $v \in \mathcal{CA}$. Obviously, each player $j \notin A_i$ is a null player with respect to v_{A_i}. Hence,

$$\sum_{j \in A_i} (X_j(v))(h) = \sum_{j \in A_i} (X_j(v_{A_i}))(h) = \sum_{j \in N} (X_j(v_{A_i}))(h) = v_{A_i}(h) = v(h|_{A_i}).$$

□

Proof of Proposition 8

Let an arbitrary allocation problem (N, M) be given and assume $Y \in \mathcal{Y}$ satisfies all of the axioms given in the proposition. In particular, Lemma 7 yields $Y \in \mathcal{Y}^{2stg}$. In the following let $X^c \in \mathcal{X}$ for $c \in M$ and $Z \in \mathcal{X}$ be the player-based allocation rules with $Y_{ic} = X_i^c \circ X_c$ for all $i \in N$ and $c \in M$. Proposition 2 implies $Z_c = Z_c^Y$ for all $c \in M$. Moreover, since Z^Y satisfies additivity, symmetry of active players, and the null player property by assumption, Proposition 6 implies that Z has to be the Myerson Value.

The remainder of the proof proceeds in a similar way as the characterization of the Equal Division Solution. In fact, as the axioms already indicate, it will be shown that within each connection the players distribute their value equally. Because of additivity it is sufficient to concentrate on basic value functions. To this end let $\alpha \in \mathbb{R}$ and $h \in \mathcal{H}$. Note that the inactive positions axiom implies $\left(Y_{ic}\left(\alpha 1^h\right)\right)(h') = 0$ for all $h' \in \mathcal{H}$ with $i \notin h'(c)$.

Case 1: $h \nsubseteq h'$.

Lemma 6 yields

$$\left(Z_c^Y\left(\alpha 1^h\right)\right)\left(h'^*\right) = \left(MV_c\left(\alpha 1^h\right)\right)\left(h'^*\right) = 0$$

for all $c \in M$. As all players $i, j \in h'(c)$ are symmetric within c, all positions in this connection receive the same payoff, i.e., $\left(Y_{ic}\left(\alpha 1^h\right)\right)(h') = \left(Y_{jc}\left(\alpha 1^H\right)\right)(h')$ for all $i, j \in h'(c)$. Consequently, $\left(Z_c^Y\left(\alpha 1^h\right)\right)\left(h'^*\right) = 0$ implies that this payoff has to be zero for all players.

Case 2: $h = h'$.

Here, Lemma 6 can be applied again:

$$\left(Z_c^Y\left(\alpha 1^h\right)\right)\left(h'^*\right) = \left(MV_c\left(\alpha 1^h\right)\right)\left(h'^*\right) = \begin{cases} \frac{\alpha}{|M(h^*)|}, & \text{if } c \in M(h^*) \\ 0, & \text{if } c \notin M(h^*) \end{cases}$$

Similar to case 1, for all $c \in M$ all players within the connection are symmetric and by (cwSYM) all positions in c get the same. Thus, $\left(Y_{ic}\left(\alpha 1^h\right)\right)(h') = \frac{\alpha}{|h(c)||M(h^*)|}$ for all $i \in h(c)$. As already mentioned before, if $i \notin h(c)$ the corresponding position receives a payoff of zero.

Case 3: $h \subsetneq h'$.

Again, the connections' values are given by

$$\left(Z_c^Y(\alpha 1^h)\right)\left(h'^*\right) = \left(MV_c(\alpha 1^h)\right)\left(h'^*\right) = \begin{cases} \frac{\alpha}{|M(h^*)|}, & \text{if } c \in M(h^*) \\ 0, & \text{if } c \notin M(h^*). \end{cases}$$

For $c \in M\left(h'^*\right) \setminus M(h^*)$ all players $i, j \in h'(c)$ are symmetric within c and, similar to case 1, applying (cwSYM) yields $\left(Y_{ic}(1^h)\right)(h') = \left(Y_{jc}(1^h)\right)(h') = 0$. Therefore, suppose $c \in M(h^*)$. Furthermore, let again $\alpha 1^h = \alpha \sum_{\bar{h}: h \subseteq h'} 1_{\bar{h}}$ where the $1_{h'}$ are standard value functions (cf. the proof of Proposition 5). If there exists no $T \subseteq M$ with $\bar{h} = h'|_T$, then player $i \in h'(c)$ is a nullifying player in c with respect to $\alpha 1_{\bar{h}}$ if and only if $i \notin \bar{h}(c)$. Thus, $(Y_{ic}(\alpha 1_{\bar{h}}))(h') = 0$ for all $i \notin h'(c)$. Because each connection $c' \in M$ is a null player in h' with respect to $1_{\bar{h}}$, the aggregated value $\left(Z_c^Y(\alpha 1_{\bar{h}})\right)(h') = \sum_{i \in N} (Y_{ic}(\alpha 1_{\bar{h}}))(h')$ of c also has to be equal to zero and by (cwSYM) this implies $(Y_{ic}(\alpha 1_{\bar{h}}))(h') = 0$ for all $i \in N$. Thus, $(Y_{ic}(\alpha 1_{\bar{h}}))(h') \neq 0$ only if there exists $T \subseteq M$ with $\bar{h} = h'|_T$. Now the value of position $i \in h'(c)$ can be decomposed in the following way:

$$\begin{aligned}\left(Y_{ic}(\alpha 1^h)\right)(h') &= \sum_{\bar{h}: h \subseteq \bar{h}} \underbrace{(Y_{ic}(\alpha 1_{\bar{h}}))(h')}_{=0, \text{ if } \nexists T \subseteq M:\, \bar{h} = h'|_T} \\ &= \sum_{T: M(h^*) \subseteq T \subseteq M(h'^*)} \left(Y_{ic}\left(\alpha 1_{h'|_T}\right)\right)(h') \end{aligned} \tag{2}$$

Note that $c \in M(h^*)$ implies $c \in T$ for all $M(h^*) \subseteq T \subseteq M\left(h'^*\right)$ and, thus, $h'|_T(c) = h'(c)$. Therefore, all players in $h'(c)$ are symmetric in c with respect to $1_{h'|_T}$ and (cwSYM) yields

$$\left(Y_{ic}\left(1_{h'|_T}\right)\right)(h') = \frac{\left(Z_c^Y\left(\alpha 1_{h'|_T}\right)\right)\left(h'^*\right)}{|h'(c)|}$$

for all $i \in h'(c)$. Now Equation (2) can be exploited again:

$$\begin{aligned}\left(Y_{ic}(\alpha 1^h)\right)(h') &= \sum_{T: M(h^*) \subseteq T \subseteq M(h'^*)} \left(Y_{ic}\left(\alpha 1_{h'|_T}\right)\right)(h') \\ &= \frac{1}{|h'(c)|} \sum_{T: M(h^*) \subseteq T \subseteq M\left(h'^*\right)} \left(Z_c^Y\left(\alpha 1_{h'|_T}\right)\right)\left(h'^*\right) \\ &= \frac{1}{|h'(c)|} \sum_{\bar{h}: h \subseteq \bar{h}} \left(Z_c^Y\left(\alpha 1_{h'|_T}\right)\right)\left(h'^*\right) \\ &= \frac{1}{|h'(c)|} \left(Z_c^Y\left(\alpha 1^h\right)\right)\left(h'^*\right) = \frac{\alpha}{|h'(c)| \left|M(h^*)\right|} \end{aligned}$$

From this it follows that within each connection the value is allocated equally. □

Proof of Proposition 11

Let an arbitrary allocation problem (N, M) be given. Proposition 6 implies that iteratively applying the Myerson Value satisfies the axioms. For the other direction let $Y \in \mathcal{Y}$. Moroever, assume Y satisfies all the axioms. By applying Lemma 7 this implies $Y \in \mathcal{Y}^{2stg}$. First note that because of additivity it is again sufficient to concentrate on $\alpha 1^h$, where $\alpha \in \mathbb{R}$ and $h \in \mathcal{H}$.[10] Furthermore, let $h' \in \mathcal{H}$.

Case 1: $h \nsubseteq h'$.

In this case every player is a null player in each connection and, thus, $\big(Y_{ic}(\alpha 1^h)\big)(h') = 0$ for all $i \in N$ and $c \in M$.

Case 2: $h \subseteq h'$.

Here, ic is a null position if and only if $i \notin h(c)$. From this follows

$$\big(Z_c^Y(\alpha 1^h)\big)(h'^*) = \sum_{i \in N} \big(Y_{ic}(\alpha 1^h)\big)(h') = 0$$

for all $c \in M$ with $|h'(c)| = 0$ and, thus, symmetry of Z^Y implies $\big(Z_c^Y(\alpha 1^h)\big)(h'^*) = \frac{\alpha}{|M(h')|}$ for all $c \in M(h')$. Finally, exploiting (cwSYM) yields $\big(Y_{ic}\left(\alpha 1^h\right)\big)(h') = \frac{\alpha}{|h(c)||M(h^*)|}$ for all $i \in N$ and $c \in M$ with $i \in h(c)$.

Therefore, Y equals iteratively applying the Myerson Value. In particular, according to Proposition 6 this implies that the first-stage allocation rule satisfies (NULL). □

Proof of Proposition 13

The main idea of the proof is to show that if a network forms a coalition structure and moreover a component-additive value function is given, IM and OV satisfy the same axioms. To this end, it is necessary to introduce some preliminaries from Cooperative Game Theory. As already mentioned in Remark 1, a TU game is a pair (N, γ) with $\gamma : 2^N \longrightarrow \mathbb{R}$ and $\gamma(\varnothing) = 0$. If the set of players is decomposed into a set of coalitions $B = \{B_1, \ldots, B_k\}$, i.e., $\bigcup_{l=1}^k B_l = N$ and $B_l \neq \varnothing$ for all $1 \leq l \leq k$, this set B is called *partition*. Moreover, a (single-valued) *solution* Γ assigns a payoff vector $\Gamma(\gamma, B) \in \mathbb{R}^n$ to each pair consisting of a TU game (N, γ) and a partition B. Analogously to allocation rules it is possible to characterize solutions by means of several axioms: Let two arbitrary TU games (N, γ) and (N, γ') together with a partition B be given. A solution Γ...

Efficiency ... is *efficient* if $\sum_{i \in N} \Gamma_i(\gamma, B) = \gamma(N)$.

Additivity ... is *additive* if $\Gamma(\gamma + \gamma', B) = \Gamma(\gamma, B) + \Gamma(\gamma', B)$.

Null player property ... satisfies the *null player property* if every null player $i \in N$ (i.e., $\gamma(S \cup \{i\}) - \gamma(S) = 0$ for all $S \subseteq N \setminus \{i\}$) always receives a payoff of zero: $\Gamma_i(v, B) = 0$.

Symmetry within coalitions ... satisfies *symmetry within coalitions* if two symmetric players $i, j \in N$ (i.e., $\gamma(S \cup \{i\}) = \gamma(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$) always receive the same payoff: $\Gamma_i(v, B) = \Gamma_j(v, B)$.

Symmetry across coalitions ... satisfies *symmetry across coalitions* if the aggregated payoff of two symmetric coalitions $B_l, B_{l'} \in B$ (i.e., $v(B' \cup B_l) = v(B' \cup B_{l'})$ for all $B' \subseteq B \setminus \{B_l, B_{l'}\}$) is always the same: $\sum_{i \in B_l} \Gamma_i(v, B) = \sum_{i \in B_l} \Gamma_i(v, B)$.

[10] Although Owen (1977) requires the null player property, the remainder of the proof in principle proceeds completely analogously to the characterization of his value. Of course, (NULL) has to be replaced by (cwNULL) but the line of argumentation is the same.

Owen (1977) has shown the following result:[11]

Theorem. Owen's value Ψ is the unique solution satisfying efficiency, additivity, the null player property, symmetry within coalitions, and symmetry across coalitions.

Now, let an allocation problem (N, M) and a component-additive allocation rule $v \in \mathcal{CA}$ be given. Moreover, suppose the network $h \in \mathcal{H}$ forms a coalition structure. Then, each player-based allocation rule $X \in \mathcal{X}$ can be interpreted as a position-based allocation rule via

$$v_{ic}^{X}(h) := \begin{cases} v_i^X(h), & \text{for } i \in h(c) \\ 0, & \text{for } i \notin h(c). \end{cases}$$

According to Corollary 2.4 from Hart and Kurz (1983), $\left(Z_c^{OV}\right)_{c \in M}$ equals the Myerson Value:

$$v_c^{OV}(h^*) = \sum_{i \in N} v_{ic}^{OV}(h) = \sum_{i \in h(c)} v_i^{OV}(h) = \sum_{i \in h(c)} \Psi_i\left(\mathcal{A}(h), w_v^h\right) \overset{\text{HK}}{=} \Phi_c\left(w_v^{h^*}\right) = v_c^{MV}(h^*),$$

where, again, Φ denotes the Shapley Value and $w_v^{h^*}(T) := v(h|_T^*)$ for all $T \subseteq M$ (cf. Footnote 6). In particular, this implies $\left(Z_c^{OV}\right)_{c \in M}$ satisfies symmetry. By applying the characterization of Ψ it is straightforward to show that OV also satisfies additivity and the null player property (with respect to coalitional networks). Because h is a coalition structure and v is component-additive, player i's contribution to the whole network is exactly what she contributes to her connection $c \in M$:

$$\begin{aligned} v(h|_{S \cup \{i\}}) - v(h|_S) &= v(h|_{h(c) \cap (S \cup \{i\})}) - v(h|_{h(c) \cap S}) \\ &= v_c^{MV}\left(h|_{S \cup \{i\}}^*\right) - v_c^{MV}\left(h|_S^*\right) \\ &= v_c^{MV}\left(h|_{c, S \cup \{i\}}^*\right) - v_c^{MV}\left(h|_{c,S}^*\right) \end{aligned}$$

for all $S \subseteq N$. Thus, OV satisfies the null player property not only in a player-based way but also connection-wise. Nevertheless, it should be mentioned that this is true only under the requirements given here. By further exploiting the construction of the Myerson Value and component additivity, it is possible to show that symmetry within coalitions corresponds to connection-wise symmetry (note that from Lemma 7 it follows that $OV \in \mathcal{Y}^{2stg}$): Let $i, j \in N$ such that there exists $c \in M$ with $i, j \in h(c)$. Then:

$$\begin{aligned} & v(h|_{S \cup \{i\}}) = v(h|_{S \cup \{j\}}) && \text{for all } S \subseteq N \setminus \{i,j\} \\ \Leftrightarrow\quad & \sum_{c' \in M} v(h|_{h(c') \cap (S \cup \{i\})}) = \sum_{c' \in M} v(h|_{h(c') \cap (S \cup \{j\})}) && \text{for all } S \subseteq N \setminus \{i,j\} \\ \Leftrightarrow\quad & v(h|_{c, S' \cup \{i\}}) = v(h|_{c, S' \cup \{j\}}) && \text{for all } S' \subseteq h(c) \setminus \{i,j\} \\ \Leftrightarrow\quad & v(h|_{c, S' \cup \{i\}}^*) = v(h|_{c, S' \cup \{j\}}^*) && \text{for all } S' \subseteq h(c) \setminus \{i,j\} \\ \Leftrightarrow\quad & v_c^{MV}(h|_{c, S' \cup \{i\}}^*) = v_c^{MV}(h|_{c, S' \cup \{j\}}^*) && \text{for all } S' \subseteq h(c) \setminus \{i,j\} \end{aligned}$$

Therefore, OV and IM satisfy the same axioms. □

[11] Actually, Owen (1977) used slightly different but equivalent axioms for proving his result. The more common variation of his axiomatization stated here can be found, for example, in Winter (2002).

References

Albizuri, M. J., Aurrecoechea, J. and Zarzuelo, J. (2005). Configuration values: Extensions of the coalitional Owen value. *Games and Economic Behavior*, **57**, 1–17.

Arenas, A., Cabrales, A., Danon, L., Diaz-Guilera, A., Guimerà, R. and Vega-Redondo, F. (2010). Optimal information transmission in organizations: search and congestion. *Review of Economic Design*, **14**, 75–93.

Aumann, R. J. and Drèze, J. H. (1974). Cooperative games with coalition structures. *International Journal of Game Theory*, **3** (4), 217–237.

Berge, C. (1989). *Hypergraphs: Combinatorics of Finite Sets*. North Holland.

Bolton, P. and Dewatripont, M. (1994). The firm as a communication network. *The Quarterly Journal of Economics*, **109** (4), 809–839.

Borm, P., Owen, G. and Tijs, S. (1992). On the position value for communication situations. *SIAM Journal on Discrete Mathematics*, **5** (3), 305–320.

Caulier, J., Mauleon, A. and Vannetelbosch, V. (2012). Contractually stable networks. *International Journal of Game Theory*, **Forthcoming**.

Chalkiadakis, G., Elkind, E., Markakis, E., Polukarov, M. and Jennings, N. (2010). Cooperative games with overlapping coalitions. *Journal of Artificial Intelligence Research*, **39**, 179–216.

de Clippel, G. and Serrano, R. (2008). Marginal contributions and externalities in the value. *Econometrica*, **76** (6), 1413–1436.

Deegan, J. and Packel, E. W. (1978). A new index of power for simple n-person games. *lnternational Journal of Game Theory*, **7** (2), 113–123.

Fershtman, C. and Persitz, D. (2012). Clubs and social networks. *Mimeo*.

Harsanyi, J. C. (1959). A bargaining model for the cooperative n-person game. In A. Tucker (ed.), *Contributions to the Theory of Games IV*, Princeton: Princeton University Press, pp. 325–355.

Hart, S. and Kurz, M. (1983). Endogenous formation of coalitions. *Econometrica*, **51** (4), 1047–1064.

Herings, P. J.-J., van der Laan, G., Talman, A. and Yang, Z. (2010). The average tree solution for cooperative games with communication structure. *Games and Economic Behavior*, **68**, 626–633.

Jackson, M. O. (2005). Allocation rules for network games. *Games and Economic Behavior*, **51**, 128–154.

— (2008). *Social and Economic Networks*. Princeton University Press.

— and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, **71** (0108), 44–74.

Jorzik, N. (2012). Allocation rules for hypergraph games. *Mimeo.*

McQuillin, B. (2009). The extended and generalized shapley value: Simultaneous consideration of coalitional externalities and coalitional structure. *Journal of Economic Theory*, **144**, 696–721.

Myerson, R. B. (1977). Maximin facility location. *Journal of Research*, **79** (B), 107–115.

— (1980). Conference structures and fair allocation rules. *International Journal of Game Theory*, **9** (3), 169–182.

Navarro, N. (2010). Flexible network rules for identified externalities. *Games and Economic Behavior*, **69**, 401–410.

Owen, G. (1977). Values of games with a priori unions. In R. Hein and O. Moeschlin (eds.), *Essays in Mathematical Economics and Game Theory*, New York: Springer Verlag, pp. 76–88.

Radner, R. (1993). The organization of decentralized information processing. *Econometrica*, **61** (5), 1109–1146.

Roth, A. (ed.) (1988). *The Shapley Value*, Cambridge: Cambridge University Press.

Shapley, L. S. (1953). A value for n-person games. In H. W. Khun and A. W. Tucker (eds.), *Contributions to the Theory of Games*, Princeton: Princeton University Press, pp. 307–314.

Sotomayor, M. (2004). Implementation in the many-to-many matching market. *Games and Economic Behavior*, **46**, 199–212.

van den Brink, R. (2007). Null or nullifying players: The difference between the shapley value and equal division solutions. *Journal of Economic Theory*, **136**, 767–775.

van den Nouweland, A., Borm, P. and Tijs, S. (1992). Allocation rules for hypergraph communication situations. *International Journal of Game Theory*, **20** (3), 255–268.

— and Slikker, M. (2012). An axiomatic characterization of the position value for network situations. *Mathematical Social Sciences*, **Forthcoming**.

Wasserman, S. and Faust, K. (1994). *Social Network Analysis.* Cambridge University Press.

Winter, E. (1989). A value for cooperative games with levels structure of cooperation. *International Journal of Game Theory*, **18**, 227–240.

— (2002). The shapley value. In R. J. Aumann and S. Hart (eds.), *The Handbook of Game Theory with Economic Applications*, Amsterdam: North-Holland, vol. 3, pp. 2025–2054.

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