UNIVERSITÄT PADERBORN
Die Universität der Informationsgesellschaft

Center For International Economics

Working Paper Series

Working Paper No. 2026-07

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März 2026

Center For International Economics

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March 31, 2026

Abstract

Many economic and financial series exhibit non-stationarity as well as long-memory behavior in both the first and second moments. To capture both non-stationarity and long-memory characteristics simultaneously, a general dual-trend and dual long-memory framework is proposed. In this framework, the error term of the semi-parametric FARIMA model is assumed to exhibit a slowly changing scale and long-memory heteroskedasticity. A four-step estimation procedure is proposed, including a trend and a FARIMA model estimation for the first moment, followed by a scaling function and a long-memory volatility model estimation for the second moment. Three long-memory EGARCH-type models and the FIGARCH model are employed in the final stage. Our results indicate that the proposed approach can effectively model the selected economic series exhibiting dual-trend and dual long-memory features.

Keywords: dual-trend, dual long-memory, semi-strong FARIMA, modulus FILog-GARCH, modified FIEGARCH, FIEGARCH and FIGARCH

JEL Codes: C22, C14, C58

1 Introduction

Many studies have been working on modelling long-term dependencies in the conditional mean and conditional variance of economic and financial time series. Baillie et al. (2002), Conrad and Karanasos (2005a), Conrad and Karanasos (2005b), and Kasman et al. (2009) found out that inflation series and daily returns have dual long-memory features. In this context, the commonly employed approach is the so-called hybrid of a FARIMA model and a FIGARCH model, which together are capable of capturing the long-memory properties in both the first and second conditional moments. The FARIMA model captures the long-memory behavior in the conditional mean, while the FIGARCH model analyzes the squared residuals from the FARIMA model to capture the long-memory property in the conditional variance. Similar approaches like FARIMA-FIEGARCH and FARIMA-FIAPARCH have also been investigated for this purpose.

Recently a dual long-memory FARIMA-FILog-GARCH was proposed by Peitz et al. (2023) and evidence for trend-stationary dual long-memory time series was discovered. Feng et al. (2023) proposed a trend-stationary dual long-memory FARIMA with a novel modulus asymmetric FILog-GARCH (MFILog-GARCH) model. The modulus transformation addresses the near zero observation problem encountered in the Log-GARCH models. Moreover, the authors observed a slowly changing scale in the FARIMA error term and therefore, the errors are no longer i.i.d. Modeling the conditional variance of those non-stationary processes via GARCH-type models can lead to spurious results. A remedy for this issue is to use semiparametric volatility models. Letmathe et al. (2021) proposed different classes of semiparametric long-memory GARCH models to model conditional heteroskedasticity and a slowly changing scale simultaneously. Building on these studies, we propose a dual-trend and dual long-memory framework that accounts for both long-memory features and stationarity issues in the first and second moments. Therefore, this paper is an extension of Peitz et al. (2023) and Feng et al. (2023). We introduce a four-step procedure for this dual-trend and dual long-memory approach. First, the trend in the mean level is estimated using a data-driven algorithm. Second, a FARIMA model is applied for the trend-adjusted series to capture the long-term dependence in the conditional mean. Third, the slowly changing scale is estimated from the squared residuals of the FARIMA model by using the same data-driven algorithm as in the mean level. A long-memory GARCH model is then applied to the stationary process to capture the long-term dependence in the second moment. The stationary-adjusted FIGARCH model (Caporin, 2002; Robinson, 1991; Karanasos et al., 2004) is used as the benchmark at the final estimation stage. In addition, three long-memory GARCH models from the EGARCH family (Peitz et al., upcoming) are considered, namely the fractionally integrated EGARCH (FIEGARCH), the modified FIEGARCH (FIMEGARCH) and the modulus fractionally integrated Log-GARCH (FIMLog-GARCH). The FIMLog-GARCH overcomes the near-zero observation problem of the traditional Log-GARCH models, while the FIMEGARCH combines the features of the FIEGARCH and FIMLog-GARCH. Altogether, these specifications form four types of dual-trend and dual long-memory models, which are, Semi-FARIMA combined with Semi-FIGARCH, Semi-FIEGARCH, Semi-FIMEGARCH, and Semi-FIMLog-GARCH.

The structure of the remaining paper is planned as follows. Trend-stationary FARIMA model is introduced in Section 2. Section 3 explains the semiparametric extensions of long-memory GARCH models. In Section 4, the empirical implementation is discussed, including the the nonparametric estimation algorithm, Maximum Likelihood Estimation (MLE) for the parametric models and their properties. Section 5 presents the application of the proposed dual-trend and dual long-memory models to different data sets, and Section 6 concludes the paper.

2 Semi-strong FARIMA with persistent volatility

Following the definition of Feng et al. (2023), a trend stationary dual long-memory model can be defined as follows,

$$Y_t = m(x_t) + Z_t, \tag{1}$$

$$(1-B)^d \phi(B) Z_t = \psi(B) r_t, \tag{2}$$

$$r_t = \sigma_t \eta_t, \tag{3}$$

where Y_t is a non-stationary process for $t = 1, ..., n$, which is decomposed into a deterministic and a stochastic component. Let $x_t = t/n$ be the rescaled time, which transforms the time index into the fixed domain $(0, 1]$. This ensures x_t lies in a bounded interval, regardless of how large n becomes. $m(x_t)$ stands for the local mean of Y_t, while the trend stationary series Z_t is assumed to follow a semi-strong FARIMA process. The FARIMA (Hosking, 1981) model generalizes the standard ARIMA(p, d, q) model by allowing the differencing parameter d to take non-integer values. B is the backshift operator, $0 < d < 0.5$ is the fractional differencing parameter, $\phi(B) = 1 - \beta_1 B - ... - \beta_p B^p$ and $\psi(B) = 1 + \alpha_1 B + ... + \alpha_q B^q$ are AR and MA characteristic polynomials, respectively. Under the assumption that $\phi(B)$ and $\psi(B)$ have no common factors and all roots lie outside of the unit circle, Z_t admits a linear filter representation in terms of r_t,

$$Z_t = \sum_{k=0}^{\infty} b_k r_{t-k}, \tag{4}$$

where b_k are real coefficients with $\sum_{k=0}^{\infty} |b_k| = \infty$, but $\sum_{k=0}^{\infty} b_k^2 < \infty$. The residuals r_t follow semi-strong white noise and are modeled by a long-memory GARCH model. The classification of strong, semi-strong and weak FARIMA models depends on the assumptions about the innovations r_t. In the sense of strong FARIMA, the error term r_t is assumed to be an i.i.d. sequence with zero mean and variance σ_r^2. In a weak FARIMA, the errors are only assumed to be uncorrelated with finite variance, but no assumptions are made about independence and the conditional mean may be nonzero, that is, $E[r_t|\mathcal{F}_{t-1}]$ could differ from zero, where $\mathcal{F}_{t-1}$ denotes the information set available at time $t-1$. A semi-strong FARIMA builds a bridge between strong and weak FARIMA process. It assumes that r_t is a martingale difference sequence with $E(r_t|\mathcal{F}_{t-1}) = 0$ and $E(r_t^2) < \infty$. This setting is more flexible than strong FARIMA, but still requires a zero conditional mean. The variance of Z_t is finite, if and only if the residuals r_t have finite variance. We also assume that the fourth moment of Z_t exists, $E(Z_t^4) < \infty$, which ensures that the ACF of Z_t^2 is well defined. This implies that the fourth moment of r_t also exist, $E(r_t^4) < \infty$. Such a condition is required for the existing data-driven algorithm used to select the window size in nonparametric regression.

However, the existence of finite fourth moments depends on the subsequent volatility models. For instance, the considered members of the EGARCH family, such as FIEGARCH, FIMLog-GARCH, FIMEGARCH models satisfy these conditions under common regularity assumptions. Since the existence of finite moments in FIEGARCH and FIMEGARCH models require the moment generating function (mgf) of the innovations distribution to exist in a neighborhood of zero, we exclude the conditional t-distribution and assume normally distributed innovations for all considered volatility models. The unconditional variance of the standard FIGARCH model (Baillie et al., 1996) can be infinite. Therefore, we adopt the adjusted FIGARCH model proposed by (Karanasos et al., 2004), which can exhibit a finite fourth moment.

3 Semiparametric long-memory volatility models

GARCH-type models require stationary returns. However, in this study, the squared error term of the semi-strong FARIMA model is assumed to exhibit slowly evolving scale (long-run heteroskedasticity). Ignoring this slow-changing scale feature and relying solely on a long-memory GARCH model will lead to inconsistent estimates. Feng (2004) introduced a non-negative smooth deterministic function $\sqrt{\nu(x_t)}$ into the GARCH model, and proposed the Semi-GARCH model in order to deal with the slowly changing scale. The nonparametric estimator $\nu(x_t)$ typically requires the higher order moment condition $E(\zeta_t^8) < \infty$, where ζ_t is the stationary return. However, researchers have shown that the weaker condition $E(\zeta_t^4) < \infty$ is sufficient for deriving the asymptotic results (Feng, 2004). In this context, σ_t is defined as the conditional standard deviations of ζ_t given $\mathcal{F}_{t-1}$. Consequently, the total standard deviation at time x_t can be obtained by $\sqrt{\nu(x_t)}\sigma_t$. The Semi-GARCH model is given by,

$$r_t = \sqrt{\nu(x_t)}\zeta_t \quad \text{with} \quad \zeta_t = \sigma_t \eta_t, \tag{5}$$

where η_t are i.i.d. with zero mean and unit variance. σ_t follows a GARCH process. It is assumed that ζ_t has unit variance and $\zeta_t \neq 0$ is almost surely. Eq. (5) can be extended to a semiparametric long-memory GARCH model, in which σ_t is estimated using a long-memory GARCH specification. The semiparametric long-memory GARCH models reduce to the corresponding parametric model if $\nu(x_t) \equiv \sigma_r^2$. Taking logarithms on the squared r_t in Eq. (5), we obtain $\ln r_t^2 = \ln \nu(x_t) + \ln \zeta_t^2$. Let $C = E[\ln \zeta_t^2]$, $m(x_t) = \ln \nu(x_t) + C$, $\xi_t = \ln(\zeta_t^2) - C$ and $y_t = \ln r_t^2$. Then Eq. (5) can be transformed into the following additive form,

$$y_t = m(x_t) + \xi_t. \tag{6}$$

the log-transformation of r_t^2 admits the same model form as Eq. (1). Given y_t, $m(x)$ can again be estimated nonparametrically. $\hat{C}$ can be estimated by $-\ln\left[\frac{1}{n}\sum_{t=1}^{n}\exp(\hat{\xi}_t)\right]$, which follows the identity,

$$\ln \hat{\zeta}_t^{\,2} = \hat{\xi}_t + \hat{C}.$$

Taking exponentials and averaging over n gives,

$$\frac{1}{n}\sum_{t=1}^{n}\hat{\zeta}_t^2 = \frac{1}{n}\sum_{t=1}^{n}\exp(\hat{\xi}_t)\exp(\hat{C}).$$

Since $E(\zeta_t^2) = 1$, we obtain,

$$\exp(\hat{C}) = \frac{1}{\frac{1}{n}\sum_{t=1}^{n}\exp(\hat{\xi}_t)},$$

and hence,

$$\hat{C} = -\ln\left[\frac{1}{n}\sum_{t=1}^{n}\exp(\hat{\xi}_t)\right].$$

The estimated scale function $\sqrt{\hat{\nu}(x_t)}$ can be obtained by $\exp\left[(\hat{m}(x_t) - \hat{C})/2\right]$, which follows from,

$$\ln\hat{\nu}(x_t) = \hat{m}(x_t) - \hat{C}$$

Exponentiating both sides gives,

$$\hat{\nu}(x_t) = \exp[\hat{m}(x_t) - \hat{C}],$$

and taking the square root yields,

$$\sqrt{\hat{\nu}(x_t)} = \exp\left[\frac{\hat{m}(x_t) - \hat{C}}{2}\right].$$

Now $\tilde{\zeta}_t = r_t/\sqrt{\hat{\nu}(x_t)}$ can be obtained and then various long-memory GARCH specifications can be applied.

3.1 Semiparametric FIGARCH model

A GARCH(p, q) (Bollerslev, 1986) model is defined as

$$\sigma_t^2 = \alpha_0 + \alpha(B)\zeta_t^2 + \beta(B)\sigma_t^2, \tag{7}$$

where η_t is an i.i.d. sequence. $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, p$, $\beta_j \geq 0$, $j = 1, \ldots q$. $\alpha(B) = \alpha_1 B^1 + \alpha_2 B^2 + \cdots + \alpha_p B^p$ and $\beta(B) = \beta_1 B^1 + \beta_2 B^2 + \cdots + \beta_q B^q$. By defining

$u_t = \zeta_t^2 - \sigma_t^2$ and substituting into Eq. (7), the GARCH(p, q) model can be expressed as an ARMA process for the squared returns, and it is given by,

$$\phi(B)\zeta_t^2 = \alpha_0 + \psi(B)u_t, \tag{8}$$

where $\phi(B) = 1-\alpha(B)-\beta(B) = 1-\sum_{i=1}^{\max(p,q)} \phi_i B^i$ and $\psi(B) = 1-\beta(B) = 1+\sum_{j=1}^{q} \psi_j B^j$, with $\beta_i = -\psi_j$. u_t can be interpreted as the error term of the conditional variance. $E(\zeta_t^2) = E(\sigma_t^2) = \sigma^2$ can be obtained under the assumption that ζ_t is stationary. A stationary adjusted version of Eq. (8) is

$$\phi(B)(\zeta_t^2 - \sigma^2) = \psi(B)u_t. \tag{9}$$

The adjusted FIGARCH is proposed by Karanasos et al. (2004) and defined as,

$$\zeta_t^2 = \sigma^2 + \theta_{fg}(B)u_t, \tag{10}$$

where $\theta_{fg}(B) = \psi(B)\phi^{-1}(B)(1-B)^{-d}$, $0 \leq d < 0.5$. The coefficient of $\theta(B)$ is denoted by $\theta_{fg,i}$. The corresponding ARCH(∞) process of the mean-adjusted FIGARCH model is given by

$$\sigma_t^2 = \omega + \lambda_{fg}(B)\zeta_t^2, \tag{11}$$

where $\lambda_{fg}(B) = 1 - \theta_{fg}^{-1}(B)$ and $\omega = (1 - \sum \lambda_{fg,i})\sigma^2$. For $d = 0$, we have $\sum \lambda_{fg,i} < 1$, which implies $\omega > 0$. In contrast, $\sum_{i=1}^{\infty} \lambda_{fg,i} = 1$ for $d > 0$, implying that $\omega = 0$. Notably, the original FIGARCH model (Baillie et al., 1996) does not exhibit the latter feature. According to Eq. (6) in Karanasos et al. (2004) and Peitz et al. (upcoming), $E(\zeta_t^4) < \infty$ if and only if $\mu_4 = E(\eta_t^4) < \infty$ and $\sum_{i=1}^{\infty} \theta_{fg,i}^2 < [\mathrm{Var}(\eta_t^2)]^{-1} = [E(\eta_t^4) - 1]^{-1}$, which holds for both GARCH and FIGARCH models. This condition implies strong limits on the ARCH and GARCH coefficients, as well as on the long-memory parameter. A larger μ_4 results in stronger restrictions. Peitz et al. (upcoming) discuss the FIGARCH(0, d, 0) under the standard normal distribution with $\mu_4 = 3$, showing that the largest value of d is approximately 0.342. Only the long-memory parameter in a FIGARCH model is limited, however, in long-memory EGARCH models, d is not restricted and can take any value in (0, 0.5) with $\alpha_i, \beta_j \geq 0$. Nevertheless, it is still worth applying the model with d exceeding this value in order to compare empirical results. The Semi-FIGARCH is defined using both Eq. (5) and (11).

3.2 Semiparametric extensions of EGARCH-type models

For the volatility analysis, semiparametric extensions of newly proposed long-memory EGARCH-type models (Peitz et al., upcoming), including the FIMEGARCH and the FIMLog-GARCH, are also investigated, as well as the semiparametric extension of the original FIEGARCH model. Following Eq. (2.1) in Nelson (1991), Peitz et al. (upcoming) defines EGARCH family models as,

$$\ln\sigma_t^2 = \omega + \sum_{k=1}^{\infty} \theta_k g(\eta_{t-k}), \tag{12}$$

where ω is a real number and θ_k are real coefficients with $\sum_{k=1}^{\infty} \theta_k^2 < \infty$. $g(\cdot)$ is a measurable function with mean zero and finite variance, therefore, $\ln \sigma_t^2$ is a stationary process. Eq. (5) and Eq. (12) define the semiparametric extensions of EGARCH family models.

3.2.1 Semiparametric FIEGARCH model

The Semi-FIEGARCH is defined by Eq. (5) together with the following FIEGARCH specification,

$$\begin{aligned} \ln\sigma_t^2 &= \omega + (1-B)^{-d}\alpha(B)\beta^{-1}(B)g_{eg}(\eta_{t-1}) \\ &= \omega + \theta_{fe}(B)g_{eg}(\eta_{t-1}), \end{aligned} \tag{13}$$

where $\alpha_0 = 1$, $\alpha(B) = \sum_{i=0}^{p-1} \alpha_i B^i$, $\beta(B) = 1 - \sum_{j=1}^{q} \beta_j B^j$, $g_{eg}(\eta_t) = \kappa\eta_t + \gamma[|\eta_t| - E(|\eta_t|)]$ is an i.i.d. random sequence with zero mean and finite variance. κ refers to the sign effect or leverage effect, allowing positive and negative shocks of the same size to impact volatility differently, while γ represents the magnitude effect, ensuring large shocks increase volatility regardless of their sign. The two parameters are not simultaneously equal to zero. To ensure that the model is uniquely defined, we set $\theta_{fe,1} = 1$. d is the long-memory parameter. $\ln \sigma_t^2$ is weakly stationary and invertable if d takes values in $[0, 0.5)$. When $0.5 < d < 1$, $\ln \sigma_t^2$ remains strongly stationary but it is no longer weakly stationary, since $\sum_{k=1}^{\infty} \theta_{fe,k}^2 = \infty$. A limitation of the FIEGARCH model is that finite moments of arbitrary orders exist only under the assumption of a conditional normal or a conditional GED (generalized error distribution) with shape parameter greater than or equal to 2. Under a conditional t-distribution with finite degrees of freedom, finite moments of any positive order do not exist.

3.2.2 Semiparametric modulus asymmetric FILog-GARCH model

Feng et al. (2023) proposed to apply the modulus-log $f(x) = \text{sign}(x)\ln(|x| + 1)$ (John and Draper, 1980) to address the issue of zero or near zero observations for Log-GARCH type models. Based on this transformation, they introduced the asymmetric modulus FILog-GARCH model (FIMLog-GARCH). The difference between the FIEGARCH and the FIMLog-GARCH lies in the specification g(·). The Semi-FIMLog-GARCH incorporates both Eq. (5) and the FIMLog-GARCH model, which is defined as

$$\begin{aligned} \ln\sigma_t^2 &= \omega + (1-B)^{-d}\alpha(B)\beta^{-1}(B)g_{ml}(\eta_{t-1}) \\ &= \omega + \theta_{fe}(B)g_{ml}(\eta_{t-1}) \qquad \text{with}, \\ g_{ml}(\eta_t) &= \kappa\{\delta(\eta_t) - E[\delta(\eta_t)]\} + \gamma\{|\delta(\eta_t)| - E[|\delta(\eta_t)|]\}, \end{aligned} \tag{14}$$

where $\delta(\eta_t) = \text{sign}(\eta_t)\ln(|\eta_t| + 1)$. d is the long-memory parameter, which falls within the range of $[0, 0.5)$. κ and γ in $g_{ml}(\cdot)$ are the same as in $g_{eg}(\cdot)$ referring to the asymmetric and magnitude effects. The function $g_{ml}(\cdot)$ is again an i.i.d. sequence with zero mean and finite variance. The centered absolute modulus-log transformation is applied to the magnitude term, while the centered modulus-log transformation is used for the asymmetric term of the $g_{ml}(\cdot)$ function. A key point of the modulus-log transformation is the different limit behaviors of $\ln(1 + |x|)$ at the origin and at infinity. $\ln(1 + |x|) \to |x|$, as $x \to 0$ and $\ln(1 + |x|) \to \ln(|x|)$, as $x \to \pm\infty$. Thus, the tail behavior of modulus Log-GARCH models is comparable to that of standard Log-GARCH models and the conditions for the existence of the fourth moment are similar as well. Hence, the newly proposed model exhibits a finite fourth moment when an appropriate conditional t-distribution is assumed. Additionally, the issue of zero or near-zero returns is now completely solved by the modulus-log transformation. The FIMLog-GARCH model combines the advantageous properties of FIEGARCH and FILog-GARCH models.

3.2.3 Semiparametric FIMEGARCH model

The function $g(\cdot)$ in the recently proposed modified FIEGARCH model (FIMEGARCH) (Peitz et al., upcoming) is defined by combining the absolute magnitude term from the $g_{eg}(\eta_t)$ in the FIEGARCH model and the modulus log-transformed leverage term from the $g_{ml}(\eta_t)$ in the FIMLog-GARCH model. The $g(\cdot)$ from the FIMEGARCH model is denoted as $g_{me}(\eta_t)$, which is given by,

$$g_{me}(\eta_t) = \kappa\{\delta(\eta_t) - E[\delta(\eta_t)]\} + \gamma\{|\eta_t| - E[|\eta_t|]\}, \tag{15}$$

where both the leverage and magnitude term coefficients cannot both be zero simultaneously. The near zero problem of Log-GARCH models is addressed with this specification. However, the existence of finite moments remains similar to that of an EGARCH model as the magnitude term in $g_{me}(\eta_t)$ dominates the tail behavior. The FIMEGARCH model is defined as,

$$\begin{aligned}\ln\sigma_t^2 &= \omega + \beta^{-1}(B)(1-B)^{-d}\alpha(B)g_{me}(\eta_{t-1}) \\ &= \omega + \theta_{fe}(B)g_{me}(\eta_{t-1}).\end{aligned} \tag{16}$$

Both Eq. (5) and Eq. (16) define the semiparametric extension of the FIMEGARCH model.

3.3 Stationary solutions for EGARCH-type models

Following Peitz et al. (upcoming), the strict stationary solutions for all EGARCH family members are similar. Some assumptions are required for the strictly stationary solutions.

- A1: η_t is a continuous random variable with mean zero and unit variance. The PDF of η_t, $f_\eta(x)$ is bounded above and bounded everywhere away from zero for $x \in (-\infty, +\infty)$. This ensures η_t is well behaved.

- A2: $g(\cdot)$ is a measurable function of η_t, with $E[g(\eta_t)] = 0$ and $\text{var}\,[g(\eta_t)] < \infty$ and at least one parameter (γ, κ) is non zero. This ensures $g(\eta_t)$ is also an i.i.d. sequence with zero mean and finite variance.

- A3: $\alpha(z)$ and $\beta(z)$ have no common roots and all roots lie outside of the unite circle. $d < 0.5$ is assumed. This ensures the infinite order coefficients θ_k are squared summable, $\sum_{k=1}^{\infty} \theta_k^2 < \infty$.

Following the theorem 2.1 of Nelson (1991), the strict solutions of σ_t^2 and ζ_t are given by $\exp[g(\cdot)]$ and θ_k, as follows,

$$\sigma_t^2 = e^{\omega} \prod_{k=1}^{\infty} [h(\eta_{t-k})]^{\theta_k} \quad \text{and} \tag{17}$$

$$\zeta_t = \eta_t e^{\omega/2} \prod_{k=1}^{\infty} [h(\eta_{t-k})]^{\theta_k/2}, \tag{18}$$

where $h(\eta_t) = \exp[g(\eta_t)]$. Under the A1, η_t is an i.i.d. sequence. Since $h(\eta_t)$ is a measurable transformation of η_t, hence, $h(\eta_t)$ is also i.i.d. Consequently, both σ_t^2 and ζ_t are strictly stationary. A strict stationary process with finite second moments is weakly stationary. However strict stationarity does not require finite variance, so weak stationarity must be assumed if finite variance is needed.

The stationarity of ζ_t with finite $2m$-th moments or σ_t^2 with finite m-th moments is straightforward in EGARCH-type models. This implies that ζ_t is weakly stationary or that the finite fourth moment exists, if $m = 1$ or 2, respectively. Define $\theta_{\max} = \sup\{\theta_k\}$ and $\theta_{\min} = \inf\{\theta_k\}$, then, following Peitz et al. (upcoming), we have

Proposition 1. *Let $m > 0$. Suppose that A1 and A2 hold, and both $E[h^{m\theta_{min}}(\eta_t)]$ and $E[h^{m\theta_{max}}(\eta_t)]$ exist, then $E[\sigma_t^{2m}] < \infty$. Additionally, if $E[|\eta_t|^{2m}] < \infty$, then $E[\zeta^{2m}] < \infty$. In particular, if these conditions hold for $m = 1$ or 2, ζ_t is weakly stationary or has a finite fourth moment.*

The generic notations $\theta_{\min}$ and $\theta_{\max}$ are used in all cases. The existence of the $2m$-th moment of ζ_t, or equivalently the m-th moment of σ_t^2, can be established from the strictly stationary solution given in Eq. (17). From this, we have

$$E[\sigma_t^{2m}] = e^{m\omega} E[\prod_{k=1}^{\infty} [h(\eta_{t-k})]^{m\theta_k}]. \tag{19}$$

For the EGARCH model, define $h_{eg}(\eta_t) = \exp[g_{eg}(\eta_t)]$ which forms an i.i.d. sequence. The expectation $E\Big[[h_{eg}(\eta_t)]^{m\theta_k}\Big]$ can be obtained as,

$$\begin{aligned} E\Big[[h_{eg}(\eta_t)]^{m\theta_k}\Big] &= E\Big\{\exp\Big[-m\theta_k\gamma E[|\eta_t|]\Big] \cdot \exp\Big[m\theta_k(\kappa\eta_t + \gamma|\eta_t|)\Big]\Big\} \\ &= \exp\Big[-m\theta_k\gamma E[|\eta_t|]\Big] \cdot E\Big\{\exp\Big[m\theta_k(\kappa\eta_t + \gamma|\eta_t|)\Big]\Big\}. \end{aligned} \tag{20}$$

The first term is a constant and the second term requires the mgf of η_t to exist at the argument $\mu = m\theta_k(\kappa \pm \gamma)$, which is the coefficient in the exponent. The mgf, $m_\eta(e^{\mu\eta_t})$, exists for $\mu \in (-\delta, \delta)$. Let the extreme values be, $\delta_1 = \theta_{\max}|\kappa + \gamma|$, $\delta_2 = \theta_{\max}|\kappa - \gamma|$, $\delta_3 = \theta_{\min}|\kappa + \gamma|$ and $\delta_4 = \theta_{\min}|\kappa - \gamma|$. Then Proposition 1 holds for the EGARCH model if $\max\{\delta_1, \delta_2, \delta_3, \delta_4\} < \delta/m$.

For the MLog-GARCH, define the i.i.d. sequence $h_{lg}(\eta_t) = \exp[g_{lg}(\eta_t)]$, we have

$$\begin{aligned}
E\Big[[h_{lg}(\eta_t)]^{m\theta_k}\Big] &= E\Bigg\{\exp\Big\{m\theta_k\Big[\big[\kappa\zeta(\eta_t) + \gamma|\zeta(\eta_t)|\big] - \big[\kappa E[\zeta(\eta_t)] + \gamma E[|\zeta(\eta_t)|]\big]\Big]\Big\}\Bigg\} \\
&= \exp\Big\{-m\theta_k\big[\kappa E[\zeta(\eta_t)] + \gamma E[|\zeta(\eta_t)|]\big]\Big\} \cdot E\Big[\exp\big[m\theta_k[\kappa\zeta(\eta_t) + \gamma|\zeta(\eta_t)|]\big]\Big] \\
&= \exp\Big\{-m\theta_k\big[\kappa E[\zeta(\eta_t)] + \gamma E[|\zeta(\eta_t)|]\big]\Big\} \cdot E\Big[(|\eta_t| + 1)^{m\theta_k[\kappa \mathrm{sign}(\eta_t) + \gamma]}\Big],
\end{aligned}$$

where $\zeta(\eta_t) = \mathrm{sign}(\eta_t)\ln(|\eta_t| + 1)$, the first term is a constant while the second term is random. To ensure the whole expectation is finite, we need the random factor to have finite mean. The worst bound case is $|m\theta_k[\kappa\mathrm{sign}(\eta_t) + \gamma]| \leq m\theta_{\max}(|\kappa| + |\gamma|)$. A sufficient condition for finiteness is K-th moment of $(|\eta_t| + 1)$ exist for $K > m\theta_{\max}(|\kappa| + |\gamma|)$. Since $E(|\eta_t|^K) < \infty$ implies $E(|\eta_t| + 1)^K < \infty$, it is enough to assume $E(|\eta_t|^K) < \infty$. Therefore, Proposition 1 holds for MLog-GARCH, if $E(|\eta_t|^K) < \infty$, with $K > \max(m\theta_{\max}(|\kappa| + |\gamma|), 2m)$. $\theta_{\min}$ is no longer required.

For the MEGARCH, define $h_{me}(\eta_t) = \exp[g_{me}(\eta_t)]$, we can obtain,

$$\begin{aligned}
E\Big[[h_{me}(\eta_t)]^{m\theta_k}\Big] &= E\Bigg\{\exp\Big[m\theta_k\big(\kappa(\zeta(\eta_t) - E[\zeta(\eta_t)]) + \gamma(|\eta_t| - E[|\eta_t|])\big)\Big]\Bigg\} \\
&= E\Bigg\{\exp\Big[-m\theta_k\big(\kappa E[\zeta(\eta_t)] + \gamma E[|\eta_t|]\big)\Big] \cdot \exp\Big[m\theta_k\big(\kappa\zeta(\eta_t) + \gamma|\eta_t|\big)\Big]\Bigg\} \\
&= \exp\Big\{-m\theta_k\big[\kappa E[\zeta(\eta_t)] + \gamma E[|\eta_t|]\big]\Big\} \cdot E\Big[(|\eta_t| + 1)^{m\theta_k\kappa\,\mathrm{sign}(\eta_t)} \cdot \exp(\theta_k m\gamma|\eta_t|)\Big].
\end{aligned}$$

The first term is again a constant. The term, $(|\eta_t| + 1)^{m\theta_k\kappa\mathrm{sign}(\eta_t)}$ is a polynomial factor, its finiteness only requires existence of moments η_t. The term $\exp(\theta_k m\gamma|\eta_t|)$ is more restrictive, it requires the mgf of η_t to exists at the argument $\mu = m\theta_k\gamma$. The mgf of η_t, defined as $m_{\eta_t}(\mu) = E[e^{\mu\eta_t}]$, exist for $\mu \in (-\delta, \delta)$. Define the worse case values $\delta_1 = |\gamma|\theta_{\max}m$, $\delta_2 = |\gamma|\theta_{\min}m$. Therefore, sufficient conditional for Proposition 1 in the MEGARCH model to hold is that δ_1 and δ_2 are smaller than δ.

4 Proposed estimation procedure

Together with Eq. (1), Eq. (2), Eq. (5) and the four long-memory GARCH models, we define the dual-trend and dual long-memory models. The estimation of such models proceeds as follows:

1. The local linear trend $\hat{m}(x_t)$ is estimated using the data-driven algorithm from the `esemifar` package.

2. Either a FARIMA or an ARMA model is then fitted to the trend stationary process $\hat{Z}_t$.

3. The residuals $\hat{r}_t$ obtained in the previous step are used to estimate the slowly changing scale on $\ln \hat{r}_t^2$.

4. Finally, the stationary return, $\hat{\zeta}_t$, can be computed, and a long-memory GARCH model is fitted.

4.1 Nonparametric estimation algorithm

According to Beran and Feng (2002a) and Beran and Feng (2002b), the νth derivative $m(x)$, $m^{(\nu)}(x)$, can be approximated by local polynomial regression with long-memory errors. Given observations $Y_t, t = 1 \ldots n$, the local polynomial estimator of $m^{(\nu)}(x)$ ($\nu < p$) at a specified point x can be obtained by minimizing the locally weighted least squares problem,

$$Q = \sum_{t=1}^{n} \left\{ Y_t - \sum_{j=0}^{p} \beta_j (x_t - x)^j \right\}^2 W\left(\frac{x_t - x}{h}\right), \tag{21}$$

where $h > 0$ is the bandwidth, W is the second order kernel function with a symmetric density on $[-1, 1]$. Let $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, ..., \hat{\beta}_p)^T$, we have $\hat{m}^{(\nu)}(x) = \nu!\hat{\beta}_\nu$. It is assumed that $p - \nu$ is odd, $\hat{m}^{(\nu)}(x)$ is now asymptotically equivalent to some kernel regression with a $k = p + 1$-th order kernel $K(\mu)$ with boundary correction. Under the assumption that the process with finite fourth moments, $m(x)$ can be estimated by an IPI-algorithm, which selects the asymptotically optimal bandwidth by minimizing the AMISE (asymptotic integrated squared error). The AMISE is given by,

$$\text{AMISE}(h) = h^{2(k-\nu)} \frac{I[m^{(k)}]\beta^2}{[k!]^2} + \frac{(nh)^{2d-1}(d_b - c_b)V(1)}{h^{2\nu}}, \tag{22}$$

where $\beta = \int_{-1}^{1} u^k K(u)du$, $I[m^{(k)}] = \int_{c_b}^{d_b} [m^{(k)}(x)]^2 dx$ with $0 \leq c_b \leq d_b \leq 1$ to reduce the boundary effect, which refers to the effect of the estimates at the boundary points. For $d > 0$, $V(1) = 2c_f\Gamma(1-2d)\sin(\pi d) \int_{-1}^{1}\int_{-1}^{1} K(x)K(y)|x-y|^{2d-1}dxdy$. For $d = 0$, $V(1) = 2\pi c_f \int_{-1}^{1} K^2(x)dx$. $c_f = f(0)$ is the spectral density at frequency 0 of the AR and MA components of Z_t in Eq. (1) and of ξ_t in Eq. (6), where ξ_t and Z_t follow a FARIMA process. It is given by,

$$c_f = f(0) = \frac{\sigma^2(1 + \psi_1 + ... + \psi_q)^2}{2\pi(1 - \phi_1 - ... - \phi_p)^2}. \tag{23}$$

By minimizing the AMISE, the asymptotically bandwidth is obtained as follows,

$$h_A = C\, n^{(2d-1)/(2k+1-2d)} \quad \text{with} \tag{24}$$

$$C = \left(\frac{[k!]^2}{2(k-\nu)} \frac{2\nu + 1 - 2d}{\beta^2} \frac{(d_b - c_b)V(1)}{I[m^{(k)}]} \right)^{1/(2k+1-2d)}, \tag{25}$$

where $I[m^{(k)}]$ can be obtained by local polynomial regression and numerical integration. The parameters d and c_f can be obtained via maximum likelihood estimates. The IPI-algorithm for estimating $m(\cdot)$:

i) Set the initial bandwidth h_0 and select the AR- and MA-order (p, q).

ii) Obtain the estimate of $m(x_t)$ from Y_t by minimizing Eq. (21) using h_{j-1}. Then, compute the residuals $\hat{Z}_t = Y_t - \hat{m}(x_t)$. Fit a FARIMA model with the (p, q) orders selected in Step i) to $\hat{Z}_t$ to estimate d and V.

iii) Let α be an inflation factor and set $h_{d,j} = (h_{j-1})^\alpha$, where α is defined as,

$$\begin{aligned} \alpha =& \alpha_{opt} = \frac{2k+1-2d}{2k+3-2d} \quad \text{or} \\ \alpha =& \alpha_{nai} = \frac{2k+1-2d}{2k+5-2d} \quad \text{or} \\ \alpha =& \alpha_{var} = 1/2. \end{aligned} \tag{26}$$

Estimating $I[m^{(k)}]$ via a local polynomial order $p+2$ and $h_{d,j}$. h_{j-1} can be improved by,

$$h_j = \left(\frac{[k!]^2}{2k} \frac{1 - 2\hat{d}}{\beta^2} \frac{(d_b - c_b)\hat{V}(1)}{I[\hat{m}^{(k)}]} \right)^{1/(2k+1-2\hat{d})} \cdot n^{(2\hat{d}-1)/(2k+1-2\hat{d})}. \tag{27}$$

iv) Increase j by 1. Repeat ii) and iii) until convergence or a given number of iterations has been reached. Set $\hat{h}_{opt} = h_j$.

$[c_b, d_b]$ stands for the interior region for the trend estimation. c_b refers to the left cut-off and d_b is the right cut-off. It is well known that high-order derivatives are very sensitive to the boundary bias. In contrast, low-order derivatives are more stable at the boundaries, with bias growing more slowly, which makes the use of the entire sample safer. Following Letmathe et al. (2024), the initial bandwidths are set to $h_0 = 0.1$ for $p = 1$ and $h_0 = 0.2$ for $p = 3$. Additionally, the authors recommend employing $c_b = 1 - d_b > 0$ when $p = 3$. For instance, if $c_b = 0.05$, and we have $d_b = 0.95$. In this case, only 90% of all the sample is used as the interior region, which effectively reduces the boundary effect. For $p = 1$, all data are employed and $c_b = 1 - d_b = 0$, i.e. $c_b = 0$ and $d_b = 1$. The bandwidth $h_{d,j}$ is enlarged by α, which is the exponential inflation factor. There are three types of inflation factors. They satisfy the ordering $\alpha_{var} > \alpha_{nai} > \alpha_{opt}$, and as $d \to 0.5$, we have $\alpha_{nai} \to \alpha_{var}$. α_{opt} minimizes the MSE of $\hat{I}[m^{(k)}]$ and thus yields the optimal convergence rate for the plug-in bandwidth estimator, $\hat{h}_j$. α_{nai} achieves the optimal convergence rate for $\hat{m}^{(k)}$. α_{var} ensures that the bandwidth selection remain stable in practice. The appropriate choice of α depends on the underlying data structure and the estimation goal. For details, see Beran and Feng (2002a).

According to Feng et al. (2023), the asymptotic bias and variance of the estimated $\hat{m}(x)$ are not affected by the nonlinearity in the trend-adjusted series, Z_t. Beran and Feng (2002a) and Beran and Feng (2002b) indicated that the errors in the estimated trend function are asymptotically irrelevant for subsequent parametric estimation. Therefore, when ignoring the volatility structure, the FARIMA MLE estimators are still consistent, but may not be $\sqrt{n}-$consistent.

4.2 MLE for estimating the parametric models

In this study, the volatility models are estimated via quasi maximum likelihood estimator (QMLE), assuming that the innovations are normally distributed with zero mean and unit variance. For long-memory GARCH models, let θ_σ denote the vector of unknown parameter. The QMLE for estimating θ_σ with observations $\zeta_t, t = 1, ..., n$, is obtained by maximizing the following log-likelihood function

$$L(\theta_\sigma) = \frac{1}{n}\sum_{t=1}^{n} l_t, \quad \text{where} \quad l_t = -\frac{1}{2}\left\{\ln(2\pi) + \ln[\sigma_t^2(\mathcal{F}_{t-1};\theta_\sigma)] + \frac{\zeta_t^2}{\sigma_t^2(\mathcal{F}_{t-1};\theta_\sigma)}\right\}. \tag{28}$$

For the $(1, d, 1)$ specifications of the FIMLog-GARCH, FIEGARCH, and FIMEGARCH models, the parameter vector is given by $\theta_\sigma = (\omega, \alpha, \beta, d, \kappa, \gamma)'$, while for the FIGARCH it is $\theta_\sigma = (\phi, \psi, d)'$. In the `fEGarch` package, the `solnp` function is applied as the nonlinear optimization using the augmented Lagrange method. When the assumed distribution is correct, the covariance matrix of the estimated parameters $\hat{\theta}_\sigma$ can be obtained from the inverse of the Fisher information, which is equivalent to the negative inverse of the Hessian evaluated at the maximum of the log-likelihood. Invertibility of models is a required condition for applying the QMLE (for more details, see Straumann and Mikosch, 2006). This means that unobserved conditional variance σ_t^2 can be consistently recovered from the past returns. For GARCH, the proof of invertibility property is straightforward. Peitz et al. (upcoming) conjecture the MLog-GARCH, EGARCH and MEGARCH models are invertible, however, a proof of invertibility for those models remains an open question and is beyond the scope of this study.

For the FARIMA$(1, d, 1)$ model, the log-likelihood function is given by,

$$L(\theta_m) = \frac{1}{n}\sum_{t=1}^{n} l_t, \quad \text{where} \quad l_t = -\frac{1}{2}\left[\ln(2\pi) + \ln(\sigma^2) + \frac{\hat{r}_t^2(\mathcal{F}_{t-1};\theta_m)}{\sigma^2}\right], \tag{29}$$

where $\hat{r}_t$ are residuals under parameter vector $\theta_m = (\phi, \psi, d, \sigma^2)'$ for the FARIMA. The optimization is carried out in two parts. In the first part, Brent's algorithm is used to search for d values, which typically range from 0 to 0.5. In the second part, for each candidate d value from the previous part, the AR and MA parameters are estimated using the nonlinear least squares via Levenberg-Marquardt algorithm.

4.3 Properties of the estimation of parametric models

Boubacar Maïnassara et al. (2021) derived the asymptotic properties of the Least Squares Estimator (LSE) of weak FARIMA model and proved that it is consistent under A0*, A1* and A2*.

A0*: $E[\epsilon_t] = 0$, $\operatorname{var}(\epsilon_t) = \sigma_\epsilon^2$ and $\operatorname{cov}(\epsilon_t, \epsilon_{t+h}) = 0$ for all $t \in \mathbb{Z}$ and all $h \neq 0$.

A1*: The roots of the AR and MA polynomials, ϕ and ψ, lie outside of the unite circle and have no common factors.

A2*: The innovation process $(\epsilon_t)_{t\in\mathbb{Z}}$ is strictly stationary and ergodic.

In the dual-trend and dual long-memory model, the FARIMA innovations are denoted by r_t and are assumed to be a semi-strong white noise or a stationery martingale difference, such that $E(r_t|\mathcal{F}_{t-1}) = 0$ and $E(r_t^2) = \sigma_r^2 < \infty$. The semi-strong FARIMA estimator satisfies A0*, A1* and A2* in Boubacar Maïnassara et al. (2021). For the asymptotic normality of the LSE, additional assumptions are required, including that the parameter lies in the interior of the parameter space, as well as conditions on the moments, strong mixing, and the cumulants (A3, A4 and A4' in Boubacar Maïnassara et al., 2021). However, those assumptions do not hold for the semi-strong FARIMA model with a semiparametric long-memory GARCH error process. Therefore, the FARIMA estimator vector, θ_m, in the dual-trend and dual long-memory model can at least be consistent.

Ling and Li (1997) considered a FARIMA model with the error process following a standard GARCH process. Consistency and asymptotic normality of the approximate maximum likelihood estimator (MLE) are shown. For $|d| < 0.5$, under regular conditions, the sample information matrix of the FARIMA-GARCH model converges almost surely (Theorem 3.1 in Ling and Li, 1997). When Theorem 3.1 holds, the model is consistent and asymptotically normal, with convergence rate of $1/\sqrt{n}$. Since the off-diagonal blocks of the information matrix in Eq. (18) in Ling and Li (1997) are 0, the estimation of FARIMA parameters does not depend on the GARCH parameters and vice versa. Therefore FARIMA and GARCH can be estimated independently without any asymptotic loss of efficiency (see also Theorem 3.2 in Ling and Li, 1997).

Beran and Feng (2001) extend the FARIMA-GARCH model of Ling and Li (1997) to a semiparametric regression with a FARIMA-GARCH error process. The asymptotic normality of the sample mean of the FARIMA-GARCH process is also established in Beran and Feng (2001). In their study, the trend is estimated using local polynomial regression, the detrended series X_t is modeled using a FARIMA process, and its innovations ϵ_t are assumed to follow a GARCH process. Based on Theorem 1 in Beran and Feng (2001), they establish the asymptotic normality of the sample mean $\bar{X} = (1/n)\sum_{t=1}^{n} X_t$ of the FARIMA-GARCH process. The long-memory parameter satisfies $d \in (-0.5, 0.5)$.

Assuming that there is a strictly stationary solution of the GARCH process such that $E(\epsilon_t^4) < \infty$. Additionally, The AR and MA polynomials, ϕ and ψ, have no common factors, and all their roots lie outside of the unit circle. For the sample mean $\bar{X}$ of X_t, they obtained,

$$n^{1/2-d}\bar{X} \xrightarrow{D} N(0, V_d) \quad \text{with}$$
$$V_d = \sigma_\epsilon^2 \frac{|\psi(1)|^2}{|\phi(1)|^2} \frac{\Gamma(1-2d)}{2d+1} \frac{\sin(\pi d)}{\pi d}.$$

The proof can be found in the Appendix in Beran and Feng (2001).

Feng et al. (2007) extended the semiparametric fractional autoregressive model (SEMIFAR) (Beran and Feng, 2002a; Beran and Feng, 2002c) by allowing the error process to follow a GARCH specification. This model is referred to as the SEMIFAR-GARCH model. The semiparametric FARIMA-GARCH in Beran and Feng (2001) is a special case of the SEMIFAR-GARCH when the differencing order is 0. Let X_t be the trend stationary FARIMA-GARCH process as in Beran and Feng (2001), with trend function $m(\cdot)$ at least $(p+1)$-times differentiable, and assume Lemma 1 in Feng et al. (2007) hold, then the asymptotic properties of a local polynomial estimator of $m(\cdot)$ are independent of the unknown GARCH parameter vector θ_σ (see Lemma 2. in Feng et al., 2007). Additionally, the FARIMA innovations satisfy $E(\epsilon_t^2) = \sigma_\epsilon^2$ and $E(\epsilon_t^4) < \infty$. Further assuming that other regularity conditions on the FARIMA model, the bandwidth, the smoothness of $m(\cdot)$ hold. Then the trend $m(\cdot)$ and the FARIMA parameter vector θ_m in the SEMIFAR–GARCH model can be estimated without affecting the asymptotic properties of their estimators. (see Lemma 3. in Feng et al., 2007). The consistency of $\hat{m}(\cdot)$ and the FARIMA estimator vector, θ_m, is established in Feng et al. (2007). Assume that ϵ_t is a GARCH process defined by Eq. (2.2) in Feng et al. (2007) with $E(\epsilon_t^4) < \infty$ and that Theorem 1 in Feng et al. (2007) hold. Then $\sqrt{n}(\hat{\theta}_m - \theta_m) \xrightarrow{D} N(0, \sum)$, if $0 < \alpha < 1/2$ such that $(p+1)\alpha + d > 1/4$, where $\sum = 2D^{-1}$ is defined in Theorem 1 in Beran (1995). The authors show that $\hat{\theta}_m$ is $\sqrt{n}-$consistent under suitable conditions on the bandwidth and other regularity conditions. In this case the effect of the FARIMA estimation on the GARCH estimation is negligible (see Theorem 2 in Feng et al., 2007). If the above condition holds, the error in the FARIMA estimators do not have any effect on the GARCH estimators.

Feng (2004) proposed the Semi-GARCH. They demonstrated that, under the regularity conditions of nonparametric regression and the GARCH model, the GARCH parameter vector, θ, is $\sqrt{n}-$consistent and asymptotically normal up to a bias term $B_\theta = E(\hat{\theta} - \tilde{\theta})$.

For a suitable choice of bandwidth in the nonparametric component, the nonparametric regression does not affect the convergence rate of the GARCH parametric estimator. In particular, if θ is estimated with $h = O(h_a)$, then $B_\theta = O(n^{-2/5})$.

In the dual-trend and dual long-memory model, the trend stationary semi-strong FARIMA errors are assumed to follow a semiparametric long-memory GARCH process. Following the idea of Feng et al. (2023), $\hat{\theta}_m$ is obtained by fitting a FARIMA using the `fracdiff` package in R without further assumptions on the volatility structure of r_t. It is reported in Feng et al. (2023) that $\hat{\theta}_m$ is consistent with a rate of convergence of order $o_p(n^{-1/4})$, for $d < 0.25$. For $d > 0.25$, the convergence rate of θ_m is lower. Regardless of whether $\hat{\theta}_m$ is merely consistent or $\sqrt{n}-$consistent, the obtained covariance matrix $\hat{\theta}_m$ is no longer valid for the FARIMA component in the dual-trend and dual long-memory model, as the FARIMA innovations follow a semiparametric long-memory GARCH process. The standard errors of the estimated coefficients of the FARIMA model are provided for references only. The convergence rate of the parametric estimator might not be affected by the nonparametric part if the bandwidth is selected appropriately (Feng, 2004). However, how the estimation error in the mean level affects the scale function estimation, and how the errors in the first three components affect the final long-memory volatility estimation, remains an open question. A detailed theoretical proof of this issue is beyond the scope of this paper. Nonetheless, it remains worthwhile to explore the model empirically. Simulation studies can be carried out elsewhere to illustrate the consistency of the model estimation.

5 Data and application

Three R packages applied in this study are the `esemifar` (Letmathe et al., 2024), `fracdiff` (Maechler, 2024) and `fEGarch` (Schulz et al., 2025). The `esemifar` package is used to estimate trends in both the conditional mean and variance, while `fracdiff` and `fEGarch` are employed to estimate FARIMA and long-memory GARCH models, respectively. The four mentioned dual-trend and dual long-memory models are applied to real daily wages (WGUK) data, which is measured in pounds sterling. This series consists of yearly observations from 1260 to 1994 and can be accessed in the R package `fma` (Hyndman, 2023). In addition to this, four economic data series from FRED, Federal Reserve Banks of St.

Louis are also analyzed.

1. Asset-backed commercial paper outstanding (ABCOMP) refers to the total amount of asset-backed commercial paper that has been issued but has not reached its maturity date, and it remains in the market. ABCOMP is measured in billions of dollars and contains 1285 weekly observations, spanning from January 3, 2001 to August 13, 2025 (https://fred.stlouisfed.org/series/ABCOMP).

2. 30-year fixed rate mortgage average in the US (MORTGAGE) consists of 2838 weekly observations from April 2, 1971 to August 14, 2025, and it is expressed in percentages (https://fred.stlouisfed.org/series/MORTGAGE30US).

3. Producer price index by industry for petroleum lubricating oil and grease manufacturing is a monthly dataset with 535 observations, covering the period from December 1, 1980 to June 1, 2025 (December 1980 as base period). This index is normalized to December 1980 = 100, values above 100 indicates producer prices are above the December 1980 level. This series is abbreviated as PPIPLOG. (https://fred.stlouisfed.org/series/PCU324191324191).

4. Industrial production: Mining (NAICS = 21) is a monthly dataset spanning from January 01, 1919 to January 01, 2024, and includes a total of 1261 observations. It is short for IPMINE and measures the monthly industrial production specifically for the U.S. mining sector. This index is seasonally adjusted and normalized to 2017 = 100 (https://fred.stlouisfed.org/series/IPMINE).

Firstly, a log transformation is applied to all selected data sets. The logarithmically transformed series and estimated local linear trend are displayed in panel a) of each figure from Figure 1 to Figure 5. A clear trend can be observed in each series. Panel b) in each figure shows the detrended series. The FARIMA(p, d, q) or the ARMA(p, q) models are considered to estimate the conditional mean for the trend stationary series, and the choice between the FARIMA and the ARMA models depends on their BIC value. For simplicity, we only consider p and q taking the values 0 or 1. The estimated bandwidths, coefficients, and their corresponding standard errors of the selected mean models, as well as the $\chi^2_{(30)}$ Ljung-Box (LB) statistics and their respective p-values, are listed in Table 1. It can be observed that, except for the WGUK series, the FARIMA model is selected for all trend-stationary datasets. Notably, significant long-memory coefficients are found in the ABCOMP and PPIPLOG datasets, whereas a moderate long-memory parameter is observed in the MORTGAGE under the FARIMA model. In contrast, the long-memory parameter for the IPMINE is very small and tends toward zero, indicating a lack of long-memory in the conditional mean level for this dataset. For the WGUK series, the ARMA$(1, 1)$ model is selected, and no long-memory is detected for this series as well. The LB statistics are used to test for autocorrelation in the residuals of the FARIMA or ARMA models up to 30 lags. The LB tests results in Table 1 show that the null hypothesis can be rejected, indicating the presence of autocorrelation. Highly dependent FARIMA or ARMA residuals and the estimated scales are displayed in panel c) on each figure. A slowly changing scale can be observed in each series. Therefore, the removal of such a scale is necessary before further analysis of volatility. The estimated bandwidth $\hat{h}$ of the scale function for each data series is reported in Table 2.

Table 1: Estimated bandwidths, coefficients of the selected mean models and the LB test statistics.

Series	Model	$\hat{h}$	$\hat{\phi}$ (se)	$\hat{\psi}$ (se)	d (se)	LB test (p-value)
ABCOMP	FARIMA	0.083	0.883	0.142	0.413	115.360
			0.015	0.010	0.018	6.039e-12
MORTGAGE	FARIMA	0.103	0.980	-	0.150	1143.300
			0.004	-	0.016	$<$ 2.2e-16
PPIPLOG	FARIMA	0.116	0.900	-	0.443	117.090
			0.008	-	0.020	3.119e-12
IPMINE	FARIMA	0.104	0.905	-	4.583e-05	165.940
			0.025	-	0.049	$<$ 2.2e-16
WGUK	ARMA	0.054	0.515	0.261	-	180.53
			0.045	0.049	-	$<$ 2.2e-16

The final step is to estimate the volatility of the stationary series. Four long-memory GARCH-type models, namely FIMLog-GARCH, FIEGARCH, FIMEGARCH and FI-GARCH, are applied for the volatility analysis. The selection of the best volatility model is based on their BIC values. The FIEGARCH model is selected as the best model for the ABCOMP and WGUK series. The FIMEGARCH and FIMLog-GARCH models are the top performers for the PPIPLOG and IPMINE series. The benchmark model FIGARCH delivers the best results for the MORTGAGE series. The estimated parameters of the best volatility model for each data series, along with their standard errors, and the LB-test results can be found in Table 2. $\hat{\phi}$ and $\hat{\psi}$ represent the estimated coefficients of the FIGARCH model, while $\hat{\beta}$, $\hat{\kappa}$ and $\hat{\gamma}$ are the estimated parameters of the FIMLog-GARCH, FIEGARCH, FIMEGARCH models. Both $\hat{\beta}$ and $\hat{\phi}$ are listed in the same column, representing persistent effects, while $\hat{\gamma}$ and $\hat{\psi}$ are stored in the same column, representing magnitude effects. Notably, all coefficients for both persistence and magnitude are positive. As expected, the asymmetry coefficient κ can be either positive or negative. The long-memory parameters of the EGARCH family members lie within the stationary range, however, the estimated long-memory parameter for the FIGARCH model is 0.526, which exceeds the maximum allowed value. From the LB test results, the null hypothesis cannot be rejected, indicating that no significant autocorrelation remains in the errors. Overall, the proposed dual-trend dual long-memory model performs well, effectively capturing both trend and long-memory features in the first and second moments of the series.

Table 2: Estimated bandwidth, coefficients of the selected volatility models and the LB test statistics.

Series	Model	$\hat{h}$	$\hat{\omega}$ (se)	$\hat{\beta}/\hat{\phi}$ (se)	$\hat{\kappa}$ (se)	$\hat{\gamma}/\hat{\psi}$ (se)	$\hat{d}$ (se)	LB test (p-value)
ABCOMP	FIEGARCH	0.097	0.005	0.178	-0.163	0.343	0.180	24.334
			0.101	0.521	0.041	0.055	0.188	0.757
MORTGAGE	FIGARCH	0.176	0.129	0.473	-	0.649	0.526	33.685
			0.023	0.109	-	0.107	0.112	0.294
PPIPLOG	FIMEGARCH	0.128	0.016	0.821	0.154	0.568	0.062	20.720
			0.271	0.050	0.113	0.072	0.120	0.897
IPMINE	FIMLog-GARCH	0.188	0.160	0.625	-0.203	1.000	0.042	14.369
			0.218	0.143	0.064	0.107	0.138	0.993
WGUK	FIEGARCH	0.169	-0.073	0.170	-0.263	0.257	0.316	28.491
			0.129	0.155	0.056	0.084	0.095	0.544

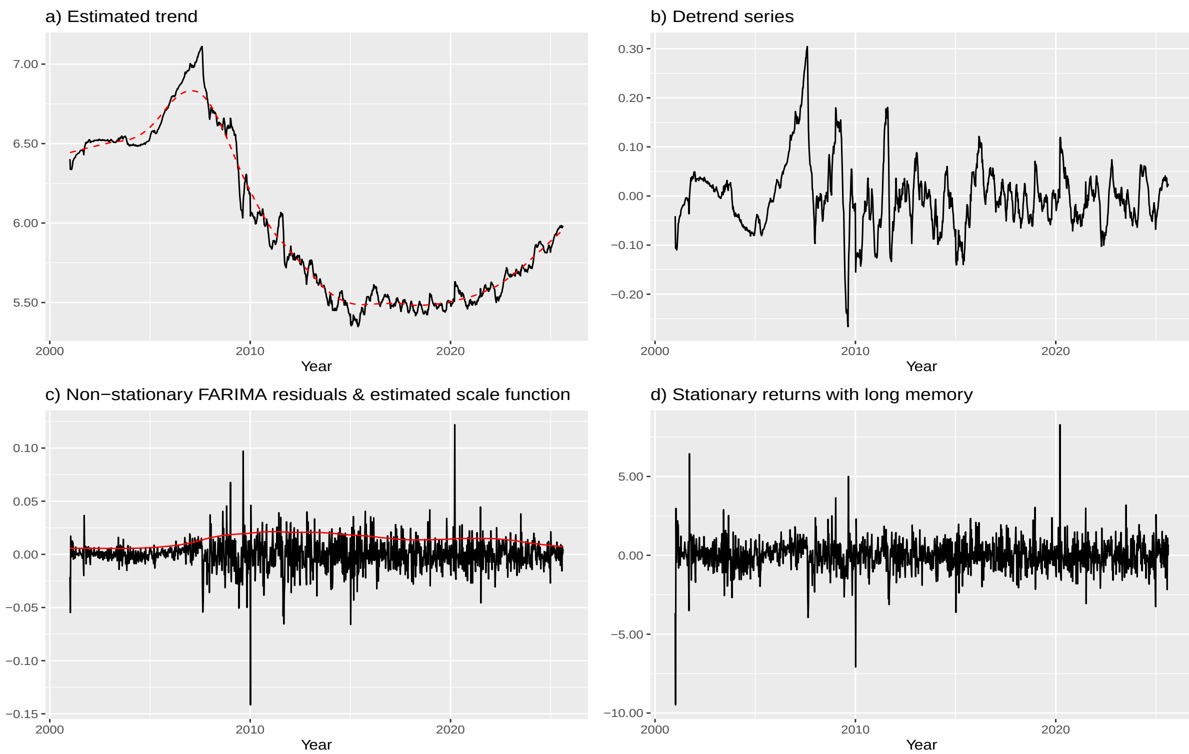

Figure 1: Dual-trend and dual long-memory example - ABCOMP.

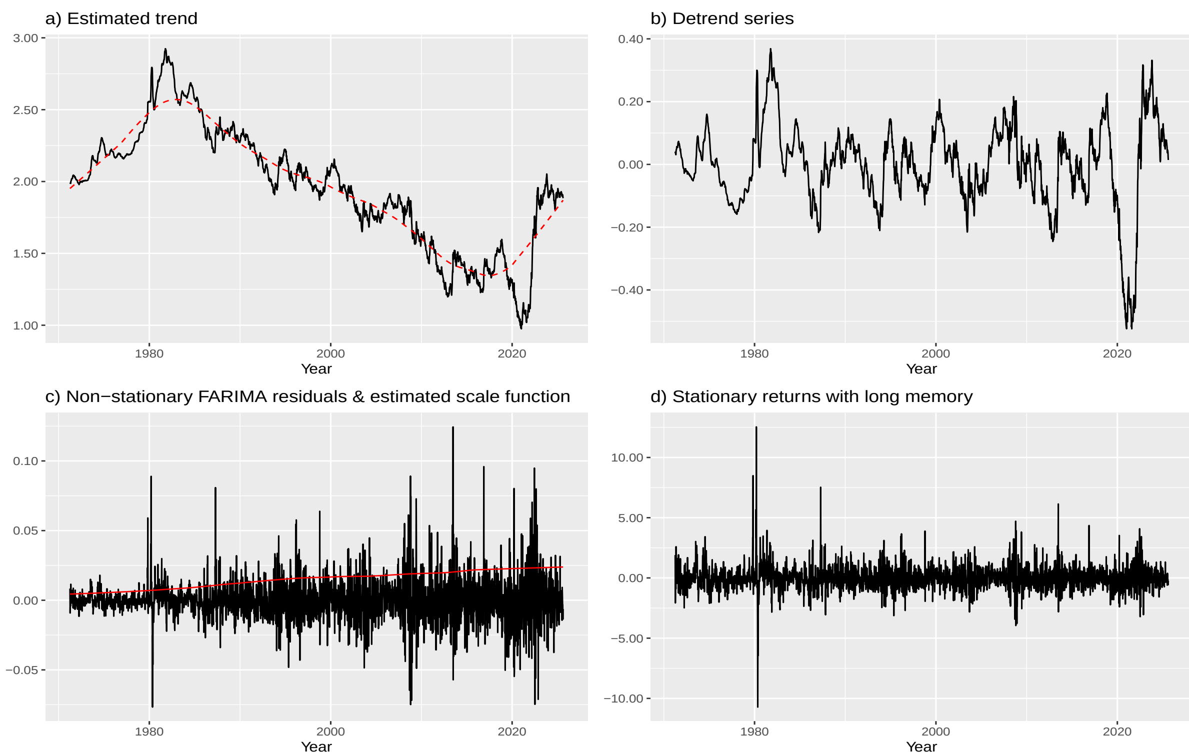

Figure 2: Dual-trend and dual long-memory example - Mortgage.

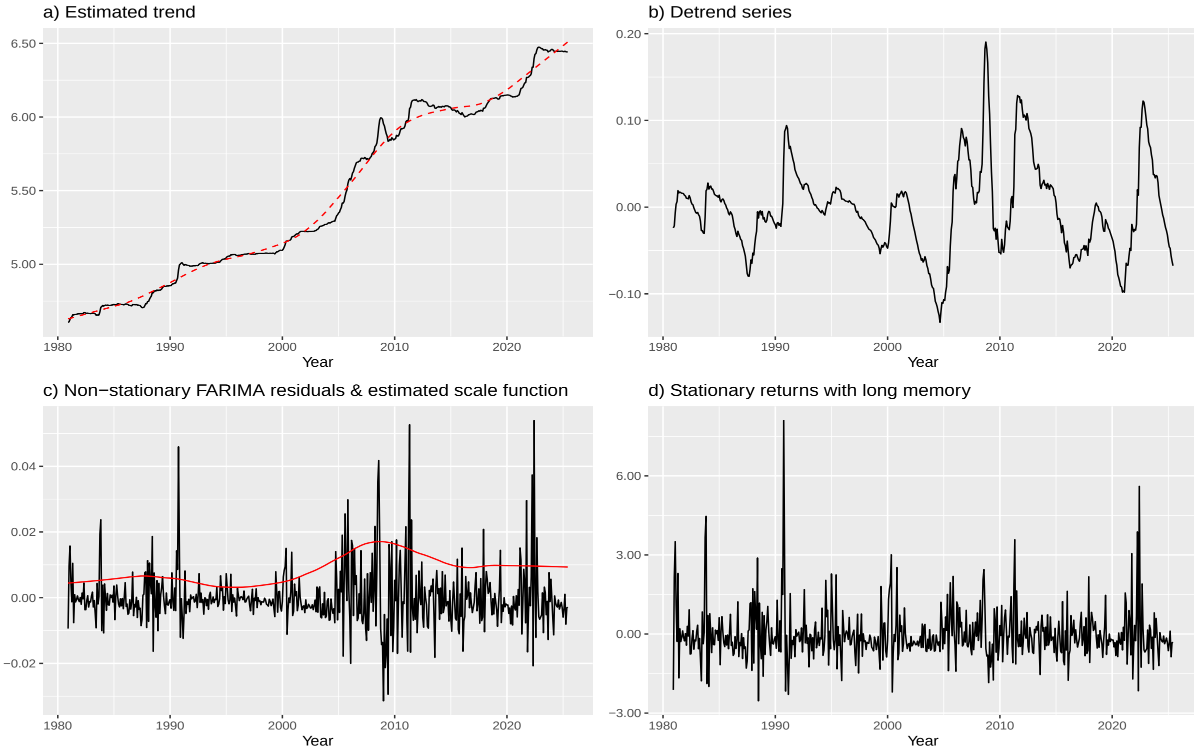

Figure 3: Dual-trend and dual long-memory example - PPIPLOG.

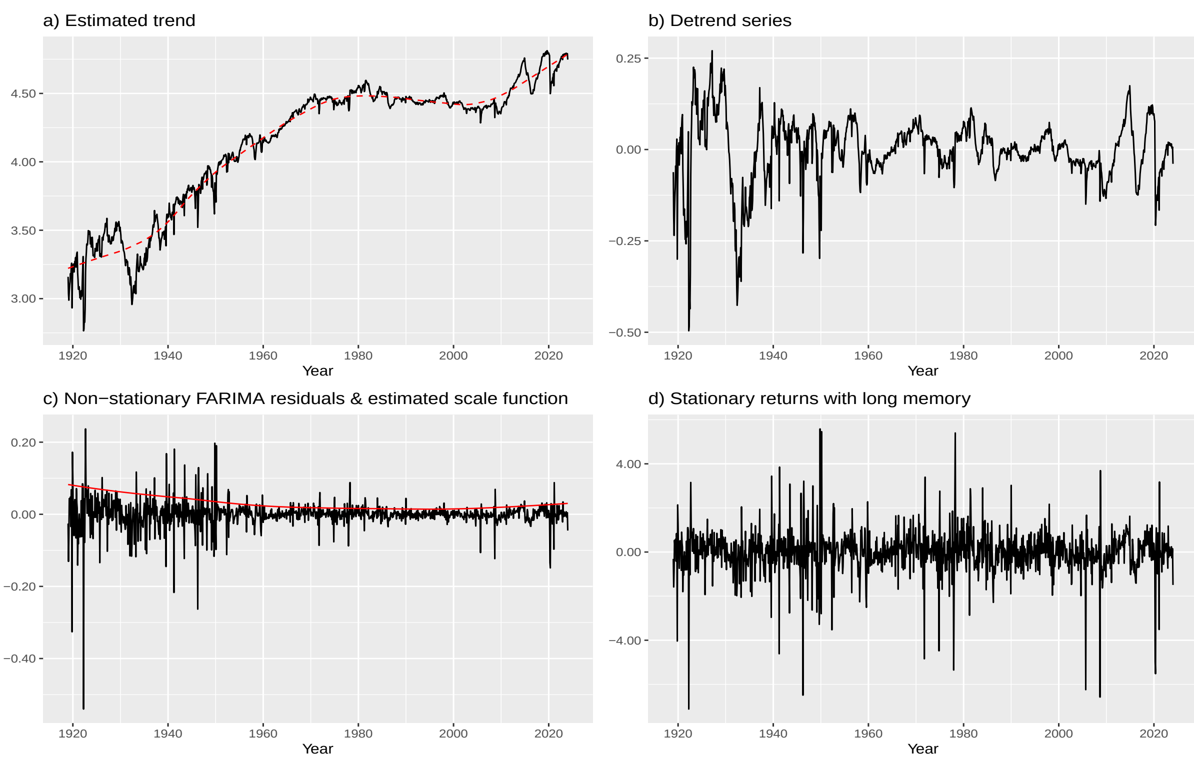

Figure 4: Dual-trend and dual long-memory example - IPMINE.

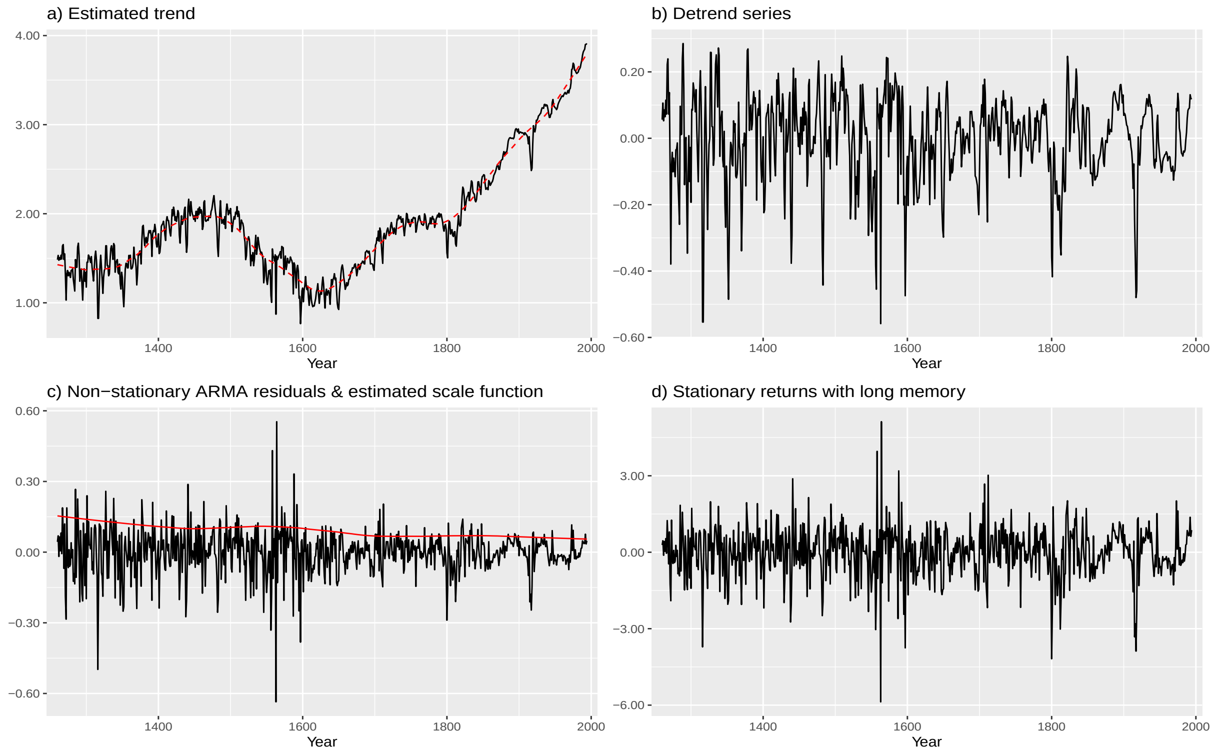

Figure 5: Dual-trend and dual long-memory example - WGUK.

6 Conclusion

This study proposes a dual-trend and dual long-memory model to analyze economic time series. It is defined as the Semi-FARIMA model with a slowly changing scale and long dependency heteroskedastic errors. A four-step estimation procedure is proposed as well. First, the trend in the mean level is estimated via the IPI-algorithm. Second, the conditional mean is modeled using FARIMA or ARMA. Third, the slowly changing scale in the variance is estimated using the same nonparametric approach as for estimating the mean trend. Finally, volatility analysis is implemented using three long-memory GARCH models from the EGARCH family, with the FIGARCH model serving as the benchmark. The results confirm that the proposed models successfully detect trend and long-memory features in both the first and second moments, with no systematic patterns remaining in the residuals. The benchmark model outperforms other GARCH models for the MORTGAGE series, while the EGARCH-type models perform best for the other four series. Simulation studies will be conducted in future research to assess the consistency and examine the approximate normality of the estimators in finite samples.

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