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Abstract

Standard empirical methods for the identification of rational bubbles in asset markets solely rely on examining explosive time series behavior but do not contain any additional information about the fundamental value and the bubble component. However, obtaining an explicit fundamental solution gives a reasonable starting point for estimating these two components simultaneously. In a decomposition approach on monthly S&P 500 stock data from 1871 to 2023, I highlight the importance of market participant's changing information set over time, leading to estimation results that fit the underlying data much better than in an ex-post analysis. Bubbles become analyzable not only on grounds of explosive time series behavior, but also in terms of their size. I further derive a bubble cycle that depicts periods with autoregressive patterns relatable to bubbles. Moreover, by engaging a growth rate perspective, real price growth rates become attributable to fundamental and non-fundamental bubble factors.

JEL Classification: C14, G12, G14

Keywords: Bubble Cycle, Dividend-Price Ratio, Exuberance, Rational Bubble, Time-Varying Discount Rate

1 Introduction

When the term bubble is used in economics, the cohesive term of a rational bubble is generally meant. Though, rational bubbles differ in some respects from what people colloquially define as a bubble. Rational bubbles are a much stricter concept as they explicitly define the expectation formation being based on rational expectations and the law of iterated expectations while colloquial bubble definitions remain nihilistic in this respect. Nevertheless, both views associate a bubble with drastically asset price increases away from fundamentals followed by a sudden drop (Brunnermeier 2016). The focus on rational bubbles in research is straightforward and practically motivated. In a nutshell, Blanchard and Watson (1982) conclude: "[It] is hard to analyze rational bubbles. It would be much harder to deal with irrational bubbles." Thus, leaving potential irrationality (and information asymmetry) of market participants aside, rational bubbles describe deviations of the price path from its long-run equilibrium given by the efficient markets model.

Notwithstanding, even with these simplifications, rational bubbles remain an opaque field of research. Yet, Hirano and Toda (2024) recently summarize the concept of rational bubbles very stringently. Rational bubbles, if existent, are no short-run deviations from fundamentals that vanish after bubble bursts, but a long-run phenomenon. In fact, rational bubbles are "a permanent overvaluation of assets" (ibid.: 3) given the expectations formed on basis of some fixed information set. The notion of this fixed information set is of crucial importance in theoretically deriving rational bubbles since it ultimately guarantees to easily establish the fundamental component of the asset price via forward induction of expectations.

Though, market participants continuously receive new information in reality. In consequence, they ongoingly revise their rational expectations concerning asset pricing which then ultimately changes the equilibrium outcome, that is the fundamental value of the considered asset. Sticking to Hirano and Toda (2024), one must thus necessarily distinguish between an ex-ante, i.e. real-time, and an ex-post perspective on rational bubbles. "[If] expectations change, a bubble may collapse. From an ex-post perspective, this bubble may appear to be a short-run phenomenon" (ibid.: 3).

The consideration of such changes in information and expectations in an empirical exercise using monthly Standard and Poor's (S&P) 500 stock index data since 1871 (Shiller 2024) is the key issue of this paper. Unlike common existing empirical procedures that solely rely on testing for explosiveness in asset prices (see among others Phillips et al. 2011 and Phillips et al. 2015), I apply an approach that is based on Froot and Obstfeld (1991). In their paper on *intrinsic rational bubbles*, i.e., bubbles that solely rely on the exogenous dividend process alone, they derive a straightforward definition of a fundamental value that is given by a positive multiple of the exogenous dividend process. The main advantage of this specific approach to bubbles is its capability to establish estimates of both, the fundamental component and the bubble component simultaneously.

However, fixing the information set and assuming a constant and exogenous discount rate only poorly fits the data in terms of obtained fundamentals. I therefore argue for a real-time analysis of bubbles in contrast to an ex post one. This means to incorporate the issue of enlarging information known to market participants over time. The real-time approach obtains much better results in terms of decomposing the asset price into a fundamental and a bubble component. An application of the Phillips et al. (2015) test for exuberance shows that the obtained bubble component consists of periods with explosive behavior in well-known time periods commonly attributed to bubbles such as the Roaring 20s, the year 1987 or the second half of the 1990s. The test results received by means of the obtained real-time bubble component validate those of the typically applied price-dividend (PD) ratio.

I thus contrive a holistic empirical analysis of rational bubbles in real time as an examination of (i) estimating the sizes of the fundamental and the bubble component, (ii) detecting explosive time series behavior, and (iii) offering changes in information known to market participants that ultimately affect (i) and (ii) in any point of time due to real-time changes in expectation formation and perceptions of economic fundamentals, the latter also being mirrored in time-varying discount rates.

The remainder of this paper is structured as follows: Section 2 summarizes the major theoretical and empirical issues of rational bubbles. Section 3 then introduces the empirical analysis of S&P 500 data in an ex-post setting. The results of the real-time estimation approach are presented in Section 4. Section 5 further discusses and deepens the central results of the real-time analysis. The obtained bubble component is analyzed using the Phillips et al. (2015) test for exuberance, a bubble cycle bridging to the real economy and the business cycle is derived, a further focus on autoregressive and unit root components in the bubble process is presented, and real stock price growth rates are decomposed into fundamental and bubble factors. Section 6 concludes.

2 An Overview of Bubbles in Theory and Empirical Analysis

2.1 Theoretical Background of Bubbles

Rational bubbles are derived out of the efficient markets model which has its ultimate roots in the definition of an ex-post realized one-period total stock return. This return contains a pure price return component on the one hand, and the dividend-price (DP) ratio on the other hand. Let r_t be the one-period total return, p_t the stock price and d_t the dividends being payed, all referring to period t, then, in absence of any transaction costs, the following condition always holds:

$$r_t = \frac{p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} \tag{1}$$

To proceed, three additional assumptions are commonly placed: (i) expectations on returns and prices are formed rationally, i.e. *rational expectations* hold, (ii) market participants are risk-neutral, and (iii) the expected total return is constant over time resulting in a constant discount rate $r > 0$. Accepting these assumptions, one ends up with the no-arbitrage condition for the *endogenous process* p_t, namely:

$$p_t = \alpha E(p_t + 1|I_t) + x_t \tag{2}$$

where $\alpha = 1/(1+r)$, $x_t = d_t/(1+r)$ is the *exogenous process*, $E(\cdot)$ is the rational expectations operator, and $I_t = \{p_t, p_{t-1}, ..., p_0, x_t, x_{t-1}, ..., x_0\}$ is the information set available to all market participants at time t. By applying a forward solution procedure on (2) where the conditional expectation is successively replaced by expectations in later periods, the *fundamental price* (or *fundamental value*) p_t^f becomes reflected by an infinite sum of discounted future dividend payments x_{t+i}, formally:

$$p_t^f = \sum_{i=0}^{\infty} \alpha E(x_{t+i}|I_t) \tag{3}$$

Dividend payments are thus derived to be the central force that should fundamentally drive asset prices in the stock market. On its own, the fundamental price cannot be obtained without any error since crucial assumptions need to be made about future dividend payments and the discount rate. For instance, Shiller (1981) estimates such an approximate fundamental price by applying a backwards induction procedure on (2) starting from an arbitrary terminal date. A more convenient way (see Froot and Obstfeld 1991) will be introduced in Section 2.3 where assumptions concerning the underlying stochastic process of the dividend process are introduced. As a result of this approach, equation (3) converges into an expression that is easy to estimate.

Though, the fundamental price is only one possible solution for the stock pricing path given by (2). Adding a rational bubble component to (3) also delivers a valid solution for the pricing path in (2) but then naturally contradicts the fundamental solution obtained in (3). The latter only holds if the so-called transversality condition

$$\lim_{T\to\infty} \alpha^{T+1} E(p_{t+T+1}|I_t) = 0 \tag{4}$$

holds where $p_\infty < \infty$ and $abs(\alpha) < 1$. Intuitively, the condition states that the discounted stock price should become zero when approaching an infinite time horizon. That is, the influence of future discounted prices on the current price should become the smaller, the more far one goes into the future. Though, by not imposing (4), a positive expected discounted price is added to the fundamental price in (3) and one ends up in a more general case where stock pricing is represented by

$$p_t = p_t^f + p_t^b \tag{5}$$

with p_t^b denoting the rational bubble component. This component is commonly labeled as rational since it appears within a model setup containing rational expectations. Let M_t be an arbitrary martingale, then it follows that $p_t^b = (1/\alpha^t)M_t$ and

$$E(p_{t+1}^b = (1+r)p_t^b \tag{6}$$

The rational bubble component thus grows in expectations at a rate greater than zero that is equal to the discount rate. Hence, it is explosive in expectations and fulfills the submartingale property.[1] Some important but also very restrictive properties of rational bubbles directly emerge out of equation (6). Diba and Grossman (1988a) argue that (i) negative bubbles cannot exist, (ii) if bubbles exist, they must have existed right from the first day of stock trading, and (iii) once a bubble bursts, it cannot restart. Concerning (i), a negative bubble component results in expectations of stock prices to "decrease without bound and, hence, to become negative at a finite future date" (ibid.: 747).[2] Concerning

[1] Before, it was defined that $r > 0$ (and thus $r \neq 0$) should hold for the discount rate leading to $\alpha < 1$. The reason for this definition is due to the necessity of r to be strictly positive in order to conduct the forward solution procedure on equation (2).

[2] The notion of negative bubbles nevertheless occurs in empirical exercises, see for example Siegel (2003), Sornette and Cauwels (2014), and Goetzmann and Kim (2018). According to them, negative bubbles are the result of feedbacks after a burst that result in visible rebound patterns.

(ii), an initial bubble with a size of p_0^b equal to zero leads to expectations of zero for any future date, thus ruling out the possibility of a bubble if it has not already emerged at the first date of trading. Property (iii) builds upon (ii) relying on the fact that once the bubble becomes zero, expectations and thus expected future bubble values will also equal zero.

Blanchard and Watson (1982) mention one additional property of bubbles. Namely, that (iv) bubbles can only exist in case of infinitely-lived assets (like stocks) but are ruled out for finitely-lived assets (like bonds). For an asset with a finite lifetime, its price must equal the fundamental price at the terminal date. Applying a backwards induction argument, market participants then will never buy the asset for a price higher than the fundamental price one period before the terminal date. The same line of thought then continues for any preceding period which ultimately rules out the existence of a bubble.

2.2 Common Empirical Approaches to Detect Bubbles

Empirical bubble identification methods are commonly of indirect nature. In contrast to direct procedures where p_t^b is modelled according to specific bubble processes on theoretical grounds (these may be of Markovian, intrinsic or extrinsic type, see Chapter 2.4 in Salge 1997 for an extensive summary in this regard), indirect procedures circumvent this issue of defining distinct bubble processes before empirical testing but simply test for explosive (or at least non-stationary) behavior in time series which form necessary conditions for bubbles to exist in different settings.

The two most common and thus standard indirect methods to detect bubbles are probably the (co-) integration approach proposed by Diba and Grossman (1988b) and the Phillips et al. (2011) and Phillips et al. (2015) augmented Dickey-Fuller (ADF) test-based procedures with regards to explosiveness or exuberance.[3]

Integration tests directly build upon the integration orders of both, the price and the dividend processes. If the price process has a higher integration order than the dividend process, the null hypothesis of no bubbles is rejected as the price process deviates from the underlying exogenous dividend process in line with (6) which states that bubbles are expected to be explosive stochastic processes given a constant discount rate $r > 0$.[4]

[3]Other indirect approaches are the variance bound test (LeRoy and Porter 1981, Shiller 1979 and 1981) and the specification test (West 1987). Moreover, two further approaches shall be shortly mentioned at this point. Siegel (2003) considers bubbles as significant deviations in total returns from expected future total returns. Sornette et al. (1996), and Sornette and Cauwels (2014) define bubbles as periods of unsustainable growth in which asset prices grow faster than exponentially. This notion is consistent with exuberance in terms of explosive time series behavior, but additionally includes positive feedback mechanisms bridging to behavioralist bubble approaches.

[4]Similarly, the PD ratio or its inverse, the DP ratio, can be tested for stationarity (see Montrucchio

In contrast, cointegration tests make use of the so-called *spread* s_t introduced by Campbell and Shiller (1987) which is defined as a linear combination of endogenous price p_t and the exogenous process x_t, formally:

$$s_t = p_t - \frac{1}{1-\alpha} x_t \tag{7}$$

Considering both, p_t and x_t to be integrated of order one, the spread s_t is stationary (i.e., integrated of order zero) in case of the null hypothesis of no bubbles. Intuitively, there is a stable relation between prices and dividends in this case which does not allow for explosive bubbly prices that deviate from dividends.

Evans (1991) criticizes the (co-) integration-based bubble identification methods for using right-tailed unit root tests that have little power in detecting Markovian-type *periodically collapsing bubbles* (Blanchard and Watson 1982) which have since then become the predominantly used theoretical bubble process. Because of their real-life notion of virtually bursting and reemerging but at the same time being in line with the conclusions of the efficient markets model, the Phillips et al. (2011) sup ADF (SADF) test uses forward-expanding sample sequences in a recursive manner to accommodate this issue. Yet, the procedure is still prone to the Evans critique that pseudo stationary behavior is identified instead of actual bubbles. This is especially found to take place when long time series with more than one bubble period are considered. Phillips et al. (2015) thus introduce a more advanced rolling window-based approach that reduces the sample sizes upon which the test procedure is applied. Their resulting general SADF (GSADF) test uses the test regression

$$\Delta y_t = \beta_{r_1,r_2} + \delta_{r_1,r_2} \gamma_{t-1} + \sum_{i=1}^{k} \rho^i_{r_1,r_2} \Delta y_{t-i} + \epsilon_t,$$

where y_t is commonly considered to be either the asset price or the PD ratio, r_1 and r_2 (in contrast to the SADF test now both variable) refer to the starting and ending sample fractions, k is the lag order and ϵ_t the error term. Using the GSADF test date-stamping algorithm,[5] the methodology is not only able to detect periodically collapsing bubbles appropriately, but to also date periods of bubbly behavior in an ex-ante, i.e., real-time, setting which is of high importance for market monitoring.

2004). Non-stationarity is interpreted as a hint for bubbles to exist.

[5]Not further discussed here for reasons of compactness, see Phillips et al. (2015) directly.

2.3 Simultaneous Extraction of Fundamentals and Bubbles

While most attention is paid to the bubble component itself, empirical bubble research has only poorly considered the fundamental component so far.[6] The reasoning is straightforward: If bubbles are by definition explosive processes, testing for explosiveness seems appropriate on its own. However, speaking in terms of equation (5), no information on the magnitude of the bubble can be given at all as long as no information on the fundamental value is gathered.

Obtaining so-called *explicit* fundamental solutions can circumvent this issue. The idea here is to *explicitly* assume a stochastic process for the exogenous process x_t. Two broad classes of explicit fundamental solutions have been considered so far in the bubble literature: autoregressive moving average (ARMA) solutions and random walk (RW) solutions. Gourieroux et al. (1982) derive explicit fundamental solutions in the ARMA case of x_t while RW solutions are to a large extent to Froot and Obstfeld (1991) who consider the case of x_t following a geometric RW with drift, i.e., a RW in logarithms containing a drift component μ:

$$x_t = x_{t-1} exp(\mu + \epsilon_t),$$

where ϵ_t refers to normally and independently distributed errors with mean zero and variance σ^2. Due to the trending behavior observed in x_t in real-world data that is typically associated with a stochastic trend, such a RW specification forms a reasonable basis for the further empirical analysis of fundamentals and bubbles.[7] For this case, Froot and Obstfeld (1991) derive the corresponding explicit fundamental solution, i.e., the fundamental value, to be given by

$$p_t^f = \frac{1}{1 - \alpha exp(\mu + \frac{\sigma^2}{2})} x_t, \tag{8}$$

such that the fundamental value is represented by a multiple m of the exogenous process with

[6] Note that all the beforehand mentioned indirect tests procedures also do not give any information on specific bubble processes on theoretical grounds, as well as their parametrization.

[7] For the S&P 500 data used in Section 3, several unit root tests are conducted. It turns out that assuming a stationary autoregressive (AR) or ARMA process for x_t is not justified by the data, see Section 3.1. If suitable, future research could consider cases like x_t following an AR(1) process with drift. In this specific case, the explicit fundamental solution would become $p_t = \frac{1}{1-\varphi\alpha} x_t + \frac{\alpha}{(1-\alpha)(1-\varphi\alpha)} \mu$, with φ being the AR coefficient and μ being the drift (see Salge 1997, Section 2.3.1.2).

$$m = \frac{1}{1 - \alpha exp(\mu + \frac{\sigma^2}{2})}. \tag{9}$$

A criterion for the asset price to converge into the explicit fundamental solution according to (8) is given by

$$\mu + \frac{\sigma^2}{2} < r. \tag{10}$$

Given the explicit fundamental solution in (8), obtaining the bubble component now simplifies to defining the residuals between the observed real price data and the estimated fundamental value, as stated in (5). The following Section 3 considers this aspect in an empirical manner and gives first S&P 500 estimates with regards to fundamentals and bubbles.

3 A First Empirical Analysis in an Ex-Post Setting

3.1 Properties of the S&P 500 Data

The S&P 500 stock market dataset taken from Shiller (2024) covers the period from January 1871 up to December 2023 and consists of 1836 monthly observations. For estimating an explicit fundamental solution, plausible processes considered here for the exogenous process are a stationary AR and a geometric RW, both with a drift component. An evaluation of which hypothesis is more credible is obtained by applying several unit root tests to the data. Table 1 depicts the results of these tests concerning the logarithmic S&P 500 real price and real dividend series. The full sample between 1871 and 2023 is considered next to shorter subsamples lasting from 1955 to 2023 and 1995 to 2023.[8] The applied unit root test procedures are: ADF, Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS).

Logarithmic real dividends clearly entail unit root behavior, and with it the logarithmic exogenous process itself. With regards to ADF and PP, the null hypotheses of a RW with drift cannot be rejected at usual significance levels for all three sample sizes. The KPSS tests consider an AR process with drift and linear trend as null hypothesis in order to shed light on potential trend stationary behavior. With respect to real dividends, the null is rejected in favor of a unit root at a 10 % significance level for the 1955–2023 and 1995–2023 sample sizes. Though, concerning the complete 1871–2023 sample size, the hypothesis of an AR process with drift and linear trend cannot be rejected at the 10 % level. Nonetheless,

[8] The formal argument for this selection of subsamples is given in Section 3.3.

Unit Root Tests for Price and Dividend Data

	Sample	ADF		PP		KPSS	
		H0	P Value	H0	P Value	H0	P Value
Real Price (Logarithm)	1871–2023	RW (Drift)	0.909	RW (Drift)	0.912	AR (Drift, Trend)	≥ 0.100
	1955–2023	RW (Drift)	0.946	RW (Drift)	0.931	AR (Drift, Trend)	≥ 0.100
	1995–2023	RW (Drift)	0.618	RW (Drift)	0.640	AR (Drift, Trend)	0.014
Real Dividend (Logarithm)	1871–2023	RW (Drift)	0.794	RW (Drift)	0.920	AR (Drift, Trend)	≥ 0.100
	1955–2023	RW (Drift)	0.974	RW (Drift)	0.990	AR (Drift, Trend)	0.088
	1995–2023	RW (Drift)	0.871	RW (Drift)	0.967	AR (Drift, Trend)	0.048

Table 1:

ADF, PP, and KPSS test results for logarithmic real price and logarithmic real dividend series. Selected lag orders for ADF: 8 (1871–2023), 7 (1955–2023), 6 (1995–2023). For PP: 8 (1871–2023), 6 (1955–2023), 5 (1995–2023). For KPSS: 9 (1871–2023), 6 (1955–2023), 4 (1995-2023).

common theoretical arguments of informationally efficient markets deny such a deterministic market behavior.

For the logarithmic real price data, the test results resemble those of the real dividend data and strengthen the RW with drift hypothesis. Using the ADF and PP test procedure, unit root behavior is found to be evident in all three sample sizes. The KPSS null hypothesis of an AR process with drift and trend cannot be rejected at usual significance level, but for the 1995–2023 sample, the test speaks in favor of unit root behavior given the rejection of the null at the 5 % significance level.

Thus, for both, real prices and real dividends, it is reasonable to assume – in line with the theory of informationally efficient markets – unit root behavior such that the series entail stochastic trends. For estimating an explicit fundamental solution, equation (8) on grounds of the geometric RW with drift hypothesis is thus a suitable starting point.

3.2 Results of the Ex-Post Approach

Figure 1 gives examples of fundamental values estimated according to equation (8) using different discount rates r and sample sizes. Altering the discount rate *ceteris paribus* leads to horizontal shifts of the estimated fundamental value. That is, increasing the discount rate leads to a downward shift of the estimated fundamental value, decreasing the discount rate leads to an upward shift of the fundamental value. The simple mechanism at work is as follows: The smaller the selected discount rate, the more value is given to future dividend payments which then increase the current fundamental price. Thus, speaking in term of equation (9), increasing the discount rate leads to a lower value of the multiple m with regards to the exogenous process and lowering the discount rate leads to a higher value of the multiple m.

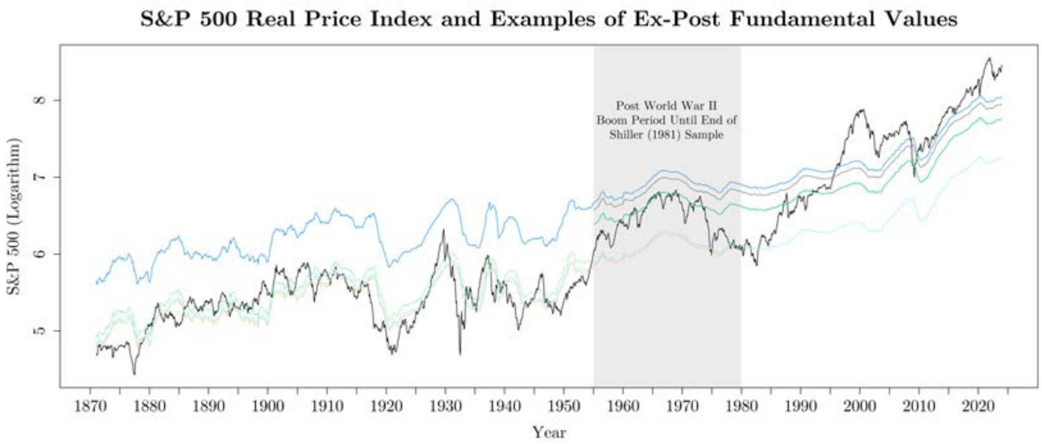

Figure 1:

Logarithmic S&P 500 real price (black solid) and estimated fundamental values according to the ex post approach. Dotted lines refer to discontinued fundamental values. The dark grey one starting in 1955 uses the van Norden and Schaller (1994) method where the fundamental value is obtained as the mean PD ratio multiplied with the dividend process. All others are calculated according to expression (8). Concerning these, the light blue one uses a discount rate that is equal to the average total yearly return of 6.7 % between 1871 and 2023. The remaining ones use constant discount rates that are based on Shiller (1981) and resultant expression (11) – though interpreted ex post –, see Section 4.2. The start of the grey shaded post-WWII boom period is set to the bull market of early 1955 where the pre-WWII high of mid-1937 is exceeded again.

Obviously, according to Figure 1, fundamental values that are given by simple multiples of the exogenous process only poorly fit the actual S&P 500 price data. Taking the example of the fundamental value calculated on basis of a discount rate equal to the average yearly total return of 6.7 % between January 1871 and December 2023 (see the light blue graph in Figure 1), the estimated series might seem reasonable or somehow expectable for the pre-World War II (WWII) period but more and more astonishable for the post-WWII years, especially after the bull market of 1955.[9] The real price series entails patterns that become less explainable by fundamentals according to (8) after WWII such that bubbles become much more pronounced in the post-WWII period if one believes in the conducted decomposition. Obviously, a much lower discount rate compared to annual 6.7 % for the time period between 1871 and 2023 *definitely* results in *unreasonable* fundamentals (see the dark blue graph in

[9]Of course, the occurrence of temporarily negative bubbles (i.e., as soon as the estimated fundamental value is higher than the observed price) before WWII contradicts the rational bubble theory. However, the point made here is the more and more continuously growing bubble component after WWII. Generally, all estimates of fundamentals depicted in Figure 1 are prone to critique as they all contain some poor properties.

Figure 1 depicting ongoingly large negative bubbles before the millennium). Adjusting the discount rate on its own thus cannot give a probable solution to estimating fundamentals for the whole set of data between 1871 and 2023 as just adjusting the discount rate leading to horizontal shifts is not sufficient to account for the changing nonlinear dynamics in observed S&P 500 real prices over the last decades.

Altering the time period under consideration is also crucial for obtaining fundamentals since estimations of (8) are not only dependent on r, but also on the parameters μ and σ^2 that are calculated based on given sample sizes of real prices and real dividends. This issue is also represented in Figure 1 in terms of the two fundamental values depicted in green. The green dotted fundamental value is estimated based on data from January 1871 up to December 1954 while the solid green one uses the remaining time horizon until December 2023. As a matter of fact, a large gap occurs at the turn of the year 1954/1955 in opposite to a continuous fundamental value (like the light- or dark blue ones). Dividing the complete sample into these two parts leads to two different sets of r, μ and σ^2, one for each considered sample, and two alternate horizontally-shifted fundamental estimates for both time periods.

By taking the example of the dotted orange fundamental value up to December 1979 (a sample size of the S&P 500 data already considered by Shiller 1981), it becomes evident that its parametrization with respect to (8) leads to results that are very similar to the light blue fundamental value up to December 2023 but evaluated before 1980. Though, the important point to make here is that, although both are nearly identical before 1980, both make use of two different information sets. That is, while the former one analyzes fundamentals up to December 1979 with only using data up to December 1979, the latter in fact analyzes late 1979 (and the prior past) using information up to future 2023. The choice of the information set being used in the analysis is thus directly connected to either engaging an ex-post or ex-ante view.

3.3 The Disregarded Importance of the Information Set

All estimated fundamental values in Section 3.2 do not seem to fit well the complete course of price data between January 1871 and December 2023. They are all not capable to keep up with nonlinear dynamics in real stock prices, especially after WWII. Estimated fundamentals could be perceived to be either too low after WWII, especially after 1955 concerning the light blue graph, or, for alternate sets of r, μ and σ^2, they are too high for over a whole century up to the 1990s such that prolonged negative bubbles occur that are neglected by rational bubble theory.

Figure 2 shows the S&P 500 DP ratio over the whole set of data since 1871. Clearly,

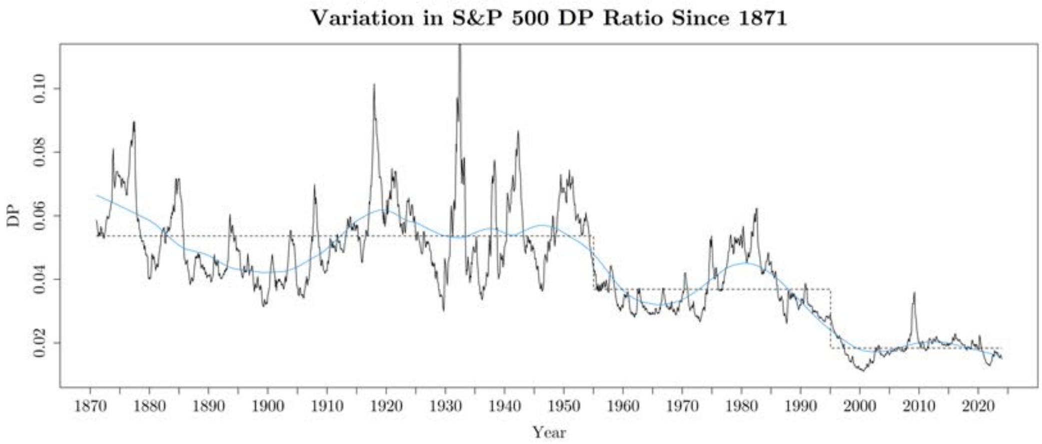

Figure 2:

Identification of breakpoints in the S&P 500 DP ratio. Structural changes in mean and variance are detected in 1955 and 1995. The local mean values of the DP ratio are depicted by the black dotted points. The blue solid line refers to a LLR trend reflecting the average long-run DP ratio. The ratio is measured on a yearly basis.

the DP ratio is not a constant process but a decreasing one. As part of the total return in (1), the total return itself must have decreased over time if the pure price return is regarded as a stationary process. However, assuming a constant discount rate for the whole set of data neglects this issue. Assuming a constant discount rate is equal to assuming constant total returns over the whole sample, which is obviously violated according to Figure 2. By applying a local linear regression (LLR) detrending procedure, the long-run DP ratio in Figure 2 (blue solid) is estimated to decrease from approximately 7 % p.a. in 1871 to below 2 % in 2023. In the ex-post approach, this downward movement is fully disregarded for estimating fundamentals.

The decreasing pattern in the DP ratio is accompanied with two major structural changes in mean and variance.[10]

In accordance with the contemporaneous bull market, one changepoint is detected in 1955. A second one is identified in 1995, which is in line with the buildup of the so-called Dotcom Bubble before the millennium. Obviously, selecting a discount rate (although in an exogenous manner according to the efficient markets model) is dependent on the actual set of information known to the public. The discount rate selected by market participants to estimate expression (8) in December 2023, for example, should be much lower to the one

[10]Different methods are used together with penalties selected according to the Akaike and Bayes information criteria. A potential third breakpoint could be set in early 1928 prior to the 1929 Crash. Of course, the usage of breakpoints is open to criticism since structural changes in fact emerge over periods instead of points in time. The main message here is the clear non-stationarity of the DP ratio.

applied by end-1954 (see Figure 2), as fundamentals themselves have changed drastically in between (see the discontinuous green solid and green dotted graphs in Figure 1 that differ in their discount rates due to underlying interim changes in total returns).

The notion of time-varying discount rates is not in opposition to the efficient markets model (Shiller 2014), but commonly rejected because of reasons of simplicity. In fact, with discount rates being time-dependent, the underlying fundamentals used for discounting themselves are subject to permanent changes, as discussed in Phillips and Yu (2011). Using the notion of an information set, if an arbitrary constant set is assumed to be known to market participants, i.e., a set that in fact rejects any kind of changes in model parameters like r, μ or σ^2 over time, the outcome itself is dependent on the fixation of information to a specific point in time. Then, only an *ex-post* analysis of fundamentals and bubbles *with respect to this fixed information* can be conducted, but by no means an ex-ante analysis in real time. For the latter case, information in fact has to be considered to increase over time and agents are exhorted to ongoingly revise their expectations based on newly gathered information. Thus, for a real-time decomposition of stock prices into fundamentals and bubbles involving immediate market perceptions and prompt stock valuation, the considered information must be changed from being fixed, i.e., constant, to being an increasing sequence.

4 Ongoing Information Changes in Real-Time

4.1 Conceptual Issues

Hirano and Toda (2024) mention the importance of the information set known to market participants at each point in time. Specifically, they note with respect to analyses conducted ex ante, i.e., in real time, and ex post: "What the theory tells us is that *given* the expectations of the agents, permanent bubbles may emerge. Of course, the equilibrium will change if agents revise their expectations. [By applying the efficient markets model], we abstract from expectations formation and study asset pricing implications *given* the expectations" (ibid: 3).

As the ex-post approach of Section 3 fixes information known to the market to December 2023 or to some breakpoint like January 1955, expectations are literally taken as given with respect to these dates. The stock market prior to these dates is then analyzed using information which in fact has not yet been available to market participants, independently of whether December 2023 or January 1955 is considered as terminal date. In short, the ex-post setting identifies past fundamentals and bubbles using future information which, although of potential interest, is inadequate for real-time market monitoring. In fact, and in line with the quotation just above, bubbles become a permanent phenomenon if analyzed ex-post. Estimated fundamentals never closely follow actual prices according to Figure 1, as underlying fundamentals have in fact changed in the interim but remain disregarded.

In order to estimate fundamentals and bubbles in a real-time setting without any usage of future yet unknown information, the decomposition stated in (5) for some point in time $t-u$, $u>0$ cannot be calculated in an ex-post fashion based on an information set $I_t = \{p_t, p_{t-1}, ..., p_0, x_t, x_{t-1}, ..., x_0\}$ containing information up to any later t. Instead, an increasing sequence of information reflecting ongoing changes in information received by the market according to

$$\{I_t, t \in \tau\}$$

must be considered where τ depicts the time horizon (see Salge 1997, Chapter 2.2). Empirical investigation now changes from obtaining static estimates for r, μ and σ^2 that correspond to an ex-post information set I_t to obtaining the respective real-time values for a forward-expanding sequence of information $\{I_t\}$. Thus, all three variables r, μ and σ^2 are prone to ongoing revisions as time proceeds. The idea is to estimate expression (8) iteratively by successively increasing the cardinality of available information. In each (monthly) period, market participants receive new information in terms of real prices and real dividends, and re-estimate parameters r, μ and σ^2. Henceforth, let I_t itself describe forward-expanding $\{I_t\}$ for notation simplicity. Then, in order to obtain real-time estimates for the exogenous process in terms of parameters denoted by $E(\mu|I_t)$ and $E(\sigma^2|I_t)$ in each iteration, all available information up to t is considered. Gathering these estimates over all iterations then gives a long-run picture of immediate market perceptions. With a corresponding to-be-expected real-time discount rate $E(r|I_t)$ that is still assumed to be constant in each single iteration I_t, but variable over a span of iterations, a real-time fundamental value can be obtained that is capable to reflect immediate market perceptions due to revisions in expectations. Bubbles are then obtained, as already in the ex-post approach, by taking the residuals between the actual observed stock price and the fundamental value.

These changes in expectations over time due to a forward-expanding information set known to the market allow the fundamental value (8) to become more adaptable to dynamics in real stock prices over decades. In consequence, fundamental stock pricing is prevented from becoming as sluggish as in terms of the ex-post cases depicted in Figure 1 that all only poorly fit the actual S&P 500 data.

4.2 Derivation of a Real-Time Discount Rate

An open question of the preceding Section 4.1 is how to receive real-time values for the discount rate in order to calculate real-time fundamentals. Figure 2 with its decreasing DP ratio pattern over time is a guiding principle, as the ratio itself should be part of a time-varying discount rate according to the total return equation (1). Shiller (1981) derives an expected value for the discount rate based on the DP ratio. However, as he excludes long-run growth in his derivation, expectations concerning the discount rate have to be modified in sense of a Gordon growth model. For notation simplicity, let $E(r_t)$ denote the expected real-time discount rate based on I_t, i.e., $E(r_t|I_t)$. Further, let expectations $E(d_t)$ and $E(p_t)$ refer to iterations of the respective data series using I_t, i.e., $E(d_t|I_t)$ and $E(p_t|I_t)$. Then, the expected real-time discount rate $E(r_t)$ to be estimated is given by:

$$E(r_t) = \frac{E(d_t)}{E(p_t)} + \frac{E(\Delta p_{t+1})}{E(p_t)}, \tag{11}$$

where $E(\Delta p_{t+1})$ denotes the expected absolute change of the stock price in period $t+1$ (given I_t) and $E(\Delta p_{t+1})/E(p_t)$ is the corresponding growth rate in the stock price. A full derivation of (11) can be found in Appendix B.

Figure 3 gives monthly real-time estimations of (11) for the time period between 1884 and 2023.[11] In the left Figure 3a, the expected real-time discount rate (red solid) is decomposed into the two righthand-side components of equation (11), that is the DP ratio-based component with dividends and prices in expectations (grey dashed), and the expected growth rate in prices (black dotted). The latter can be interpreted in terms of an expected pure price return that is independent of dividends and thus, the exogenous process. The estimated real-time discount rate follows the downward-sloping course of the DP ratio in Figure 2. Starting with approximately 10 % p.a. in early 1884, it decreases to 4 % at the end of 2023. The relationship of the real-time discount rate with the DP ratio in expectations is estimated to be non-linear. Obviously, there are periods where estimated discount rates reflecting expected total returns are either more severe or more loosely backed by expected dividends and thus fundamentals.

To shed more light on this issue, the right graph in Figure 3b depicts the share of the expected pure price return in the expected discount rate. Obviously, this share that is

[11] For conducting the estimations, a geometric RW with drift is also assumed for real price data. The hypothesis is in accordance with the results of Section 3.1. The expected values in (11) are estimated as follows: Let z_t denote either real prices or real dividends in iteration t, then $E(z_t) = z_0(1+\omega_t)^t$ where ω_t is the respective time-varying drift component and z_0 is the first observation of each series fixed to January 1871. The first 157 observations from January 1871 up to December 1883 are used for initializing the iterative and forward-expanding estimation procedure.

independent of fundamentals it is not constant over time. In fact, it decreases in the early 20^{th} century such that total returns are increasingly expected to be justified by dividends and thus market fundamentals. In 1920, this speculative share even decreases to nearly zero crowding out virtually any non-fundamental portion. Then, in the 1920s, the share of expected pure price return increases to about 0.5 prior to the 1929 stock market crash just to fully collapse again afterwards. From the early 1980s up to the millennium, the share sharply increases again from below 0.2 to nearly 0.7 at the peak of the Dotcom Bubble. The 1987 Bubble also becomes visible in terms of a short-term peak in the estimated share. However, after having burst, it does not prohibit the speculative pure price return share to further increase up to the year 2000. The last 25 years are characterized by historically high shares of expected pure price returns above 0.5, exceeding those values of the late 19^{th} century.

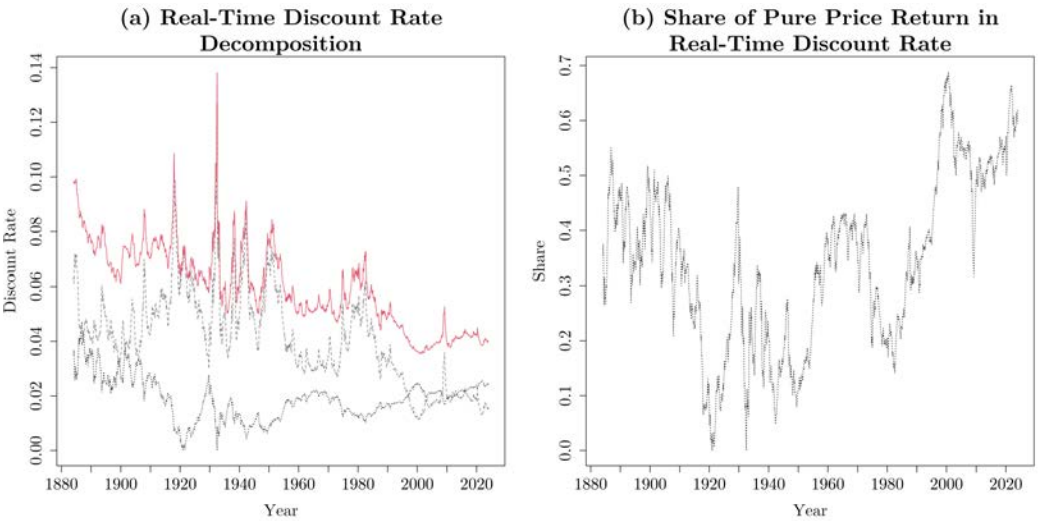

Figure 3:

(a) Real-Time decomposition of the discount rate (red solid) into a DP ratio-based component (grey dashed) and a component of pure price return (black dotted). (b) The share of the black dotted pure price return in (a) relative to the total red solid discount rate in (a).

Actually, without having obtained the real-time fundamental value by now, Figure 3 can already give some first real-time hints on bubbles in terms of speculative return patterns. Expected total market returns that are ultimately used for discounting are far away from being constant and exposed to exuberance. Ongoing changes in market perceptions formed by investors lead to variations in the weighting of fundamental and non-fundamental factors in expected total returns.

4.3 Central Results of the Real-Time Approach

Having obtained expected values for a time-varying real-time discount rate based on forward expanding information sets, it now becomes possible to estimate corresponding real-time fundamental values. Figure 4 depicts estimated real-time market fundamentals for three different sample sizes in accordance with the structural break analysis in Section 3.3 (see also Figure 2). The starting point of each sample is set to a corresponding structural break date, the terminal date is set to December 2023 in all three cases. Figure 4 thus depicts real-time fundamentals for the samples 1871 up to 2023 (red solid), 1955 up to 2023 (dark grey dashed), and 1995 up to 2023 (light grey dotted). Therefore, the three estimated real-time fundamental values differ in terms of the cardinalities they assume with respect to the forward-expanding information sets. Estimates of fundamentals that consider a shortened set of information neglect the importance of some prior data. These fundamentals have less memory and might represent market participants that do only consider the past up to a certain point, as they do not consider long-standing data to be important for current fundamentals.

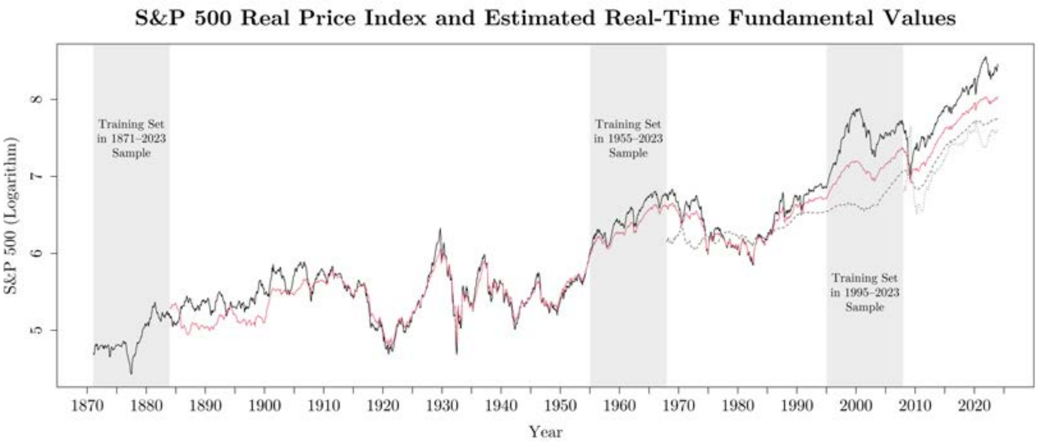

Figure 4:

Logarithmic S&P 500 real price (black solid) and estimated fundamental values according to the real-time approach. Red solid: 1871–2023 sample, dark grey dashed: 1955–2023 sample, light grey dotted: 1995–2023 sample. Initialization period for the 1871–2023 sample: 1871–1883. Initialization period for the 1955–2023 sample: 1955–1967, Initialization period for the 1995–2023 sample: 1995–2007.

For initializing the real-time estimations, sets of at least 150 monthly observations are used in each sample such that depicted fundamentals start in January 1884, 1968, and 2008, respectively.[12] In Figure 4, the initialization periods are depicted by grey areas. Obviously,

[12]See also the preceding footnote in this respect describing the case of estimating expression (11) for the

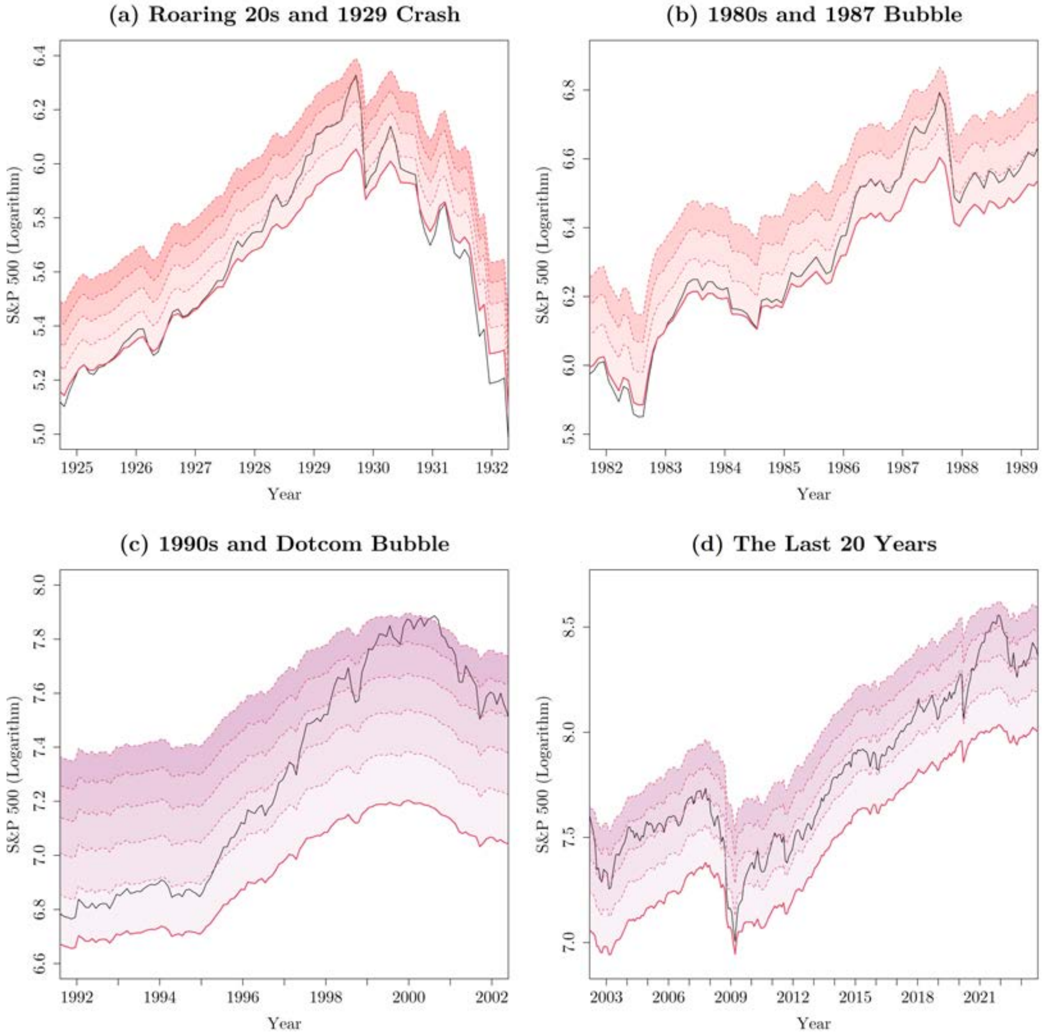

Figure 5:

Selected bubble periods obtained by the real-time approach. The real-time fundamental value using the 1871–2023 sample (red solid) is supplemented by shaded areas showing the sizes of the bubble component. Red shaded areas in (a) and (b) denote +10% cumulative deviations from the fundamental value, purple shaded areas in (c) and (d) denote +20% cumulative deviations from the fundamental value.

compared to the ex-post estimated fundamental values in Figure 1 using constant information sets, the real-time ones seem to fit much better the nonlinear course of actual S&P 500 real prices. Restricting the forward-expanding datasets to start in either January 1955 or January 1995 has a direct impact on estimated fundamentals. In both cases, rejecting some past information leads to larger bubble estimates in recent years as estimated fundamentals turn out to be lower.

complete sample between January 1871 and December 2023.

Figure 5 shows cutouts of four periods usually attributed to bubbles depicting the S&P 500 real price (black solid) and the red solid real-time fundamental value out of Figure 4 regarding the complete 1871–2023 sample: The Roaring 20s and the 1929 Crash (see Figure 5a), the 1980s and the 1987 Bubble in particular (Figure 5b), the 1990s and the simultaneous buildup of the Dotcom Bubble (Figure 5c), and the last 20 years up from 2003 (Figure 5d). Sizes of the real-time bubble component are shown by shaded areas on top of the red solid fundamental component. Red shaded areas each refer to +10 % deviations from fundamentals, purple shaded areas each refer to +20 % deviations from fundamentals.

Even though the 1929 Crash resulting in the Great Depression is often associated with highly non-fundamental pricing, fundamentals are estimated to be not exceeded by more than about 30 % prior to the crash (see Figure 5a) which is rather small, especially in comparison to bubble sizes in the 1990s and after the millennium (see Figures 5c and 5d).[13] Stock prices start to diverge from fundamentals in mid-1927 resulting in a bubble buildup phase of approximately two years. In the 1920s, the estimated real-time discount rate decreases by about one percentage point to slightly below 6 % prior to the crash (see Figure 3a) while the speculative price return share within this rate (that is fully independent from fundamentals) increases up to nearly 0.5 (see Figure 3b). While exuberance is clearly portrayed by the latter rate, the decrease in the real-time discount needs some more investigation as a decline in this rate seems counterintuitive during a bubble buildup phase: In the 1920s, the expected pure price return as part of the real-time discount rate (11) increases to an annual rate of more than 2 % (see the black dotted graph in Figure 3a). Though, at the same time the DP ratio-based component decreases to about 3 % p.a. as sharp increases in expected stock prices outperform increases in expected dividends (see the gray dashed graph in Figure 3a). Two opposite effects thus converge, at which the real-time discount rate is estimated to fall, which is ultimately due to dividend payments not catching up with the development of stock prices.

Yet, this effect of decreasing real-time discount rates *ceteris paribus* spurs contemporaneous market perceptions, increases market fundamentals, and leads to a decrease in the bubble component.[14] What is more, market fundamentals get further pushed in the 1920s as real-time estimates of the drift component regarding the exogenous process in equation

[13] Of course, market participants in 1929 could not judge that 30 % overpricing is small compared to what will follow at the end of the century in terms of the Dotcom Bubble. This argument just holds in an ex-post sense. Certainly, it shows that bubble bursts perceived by the public may occur at any point in time, independently of the degree of estimated overpricing.

[14] An increase in the bubble component due to changing time-varying discount rates is a phenomenon that is also discussed in Phillips and Yu (2011).

(8) increase steadily.[15]

With two these mechanisms at work in the background enhancing fundamentals, the increase in the bubble component observable in Figure 3a becomes more sophisticated to interpret. It becomes traceable that the market was at least perceived to be priced rather fundamentally in this environment even though the bubble component was growing towards 30 % above fundamentals. The conjecture that stock pricing before the 1929 Crash had to be extremely high in the eyes of market participants is difficult to encounter. Yet, with an increase in the drift component of the exogenous process next to a decrease in real-time discount rates, market fundamentals got pushed masking the increase in the bubble size. Albeit, apotential hint for the bubble to burst is probably the large increase in the speculative price return share in the estimated real-time discount rate towards 0.5 according to Figure 3b.

With regards to the 1987 Bubble in Figure 5b, stock prices stick closely to fundamentals up to early-1985. The bubble component then starts to grow afterwards until actual real prices in the S&P 500 exceed the estimated real-time fundamental value by approximately 20 % in October 1987 shortly before the bubble bursts. Again, like prior to the 1929 Crash, decreases in the estimates of real-time discount rates are due to decreases in the DP ratio-based component (see Figure 3a) which shift up fundamentals simultaneously to the buildup of the bubble component. Strikingly, the share of the speculative price return within the discount rate increases in the 1980s like in the 1920s, now towards a value of 0.4 (see Figure 3b), giving again a hint for a potential bubble burst.

Compared to the 1920s and the emergence of the 1987 Bubble, the remaining two phenomena in Figures 5c and 5d depict much larger non-fundamental pricing. In the early 1990s, real prices are already estimated to exceed fundamentals by about 20 %. The Dotcom Bubble then grows continuously until actual stock prices exceed fundamentals by approximately 100 % in early 2000. The bubble component does not vanish in favor of fundamental pricing after the Dotcom Bubble bursts (or deflates). In fact, pricing is estimated to remain highly non-fundamental. This course then continues in the graph of Figure 5d depicting the years since 2003. Prior to the Great Recession with a corresponding through in real prices in early 2009, real prices actually exceed the real-time fundamental value up to 60 %. During the trough, real prices actually fall back to fundamentals, just to steadily increase again afterwards. After 2009, and before early 2020, the bubble component reinforces at more than 20 % up to 40 % above market fundamentals. After a short decline in early-2020 and a

[15] Figure A.1b in Appendix A depicts the drift components of the exogenous process over time starting in 1968. Before, the variation in the drift component is larger, with the pre-1929 period standing out.

runup throughout 2020 to above 60 %, it is estimated to stabilize again later on. Up to the end of the sample in December 2023, the bubble component is reflected by real prices of 40 % above fundamentals, an estimated value that is higher than before both market crashes in 1929 and 1987. Similarly, estimated shares of speculative price return within real-time discount rates increase to historically high values above 0.5 (see Figure 3b).

4.4 Interaction of Variables in Fundamentals

Real-time fundamentals are estimated according expressions (8) and (9). Figure 6 shows how the estimated real-time discount rates for the 1871-2023 (see Figure 6a) and 1955-2023 (see Figure 6b) samples interact with the multiple of the exogenous process denoted by m defined in (9). For the sample between January 1871 and December 2023, a pretty inverse relationship in line with the underlying theory is found: low discount rates *ceteris paribus* correspond to high multiples of the exogenous process and thus high fundamental values. In opposite, high discount rates *ceteris paribus* correspond to low multiples of the exogenous process and thus low fundamental values. This central result also holds for the 1955–2023 sample, but in a lesser degree.

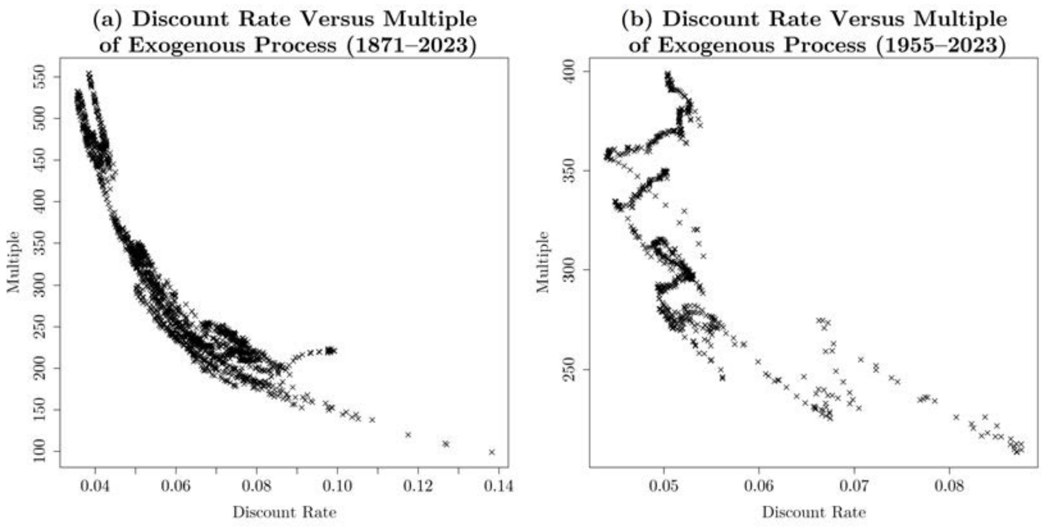

Figure 6:

(a) Correlogram for the real-time discount rate and corresponding multiples of the exogenous process for the 1871–2023 sample. (b) Respective correlogram for the 1955–2023 sample. The correlogram for the 1995–2023 sample is not reported here.

Figure 7 depicts the estimated real-time discount rates (see Figure 7a) and the multiples of the exogenous process (see Figure 7b) for all three considered sample sizes over time. The graphs start in January 1968, i.e., where first real-time estimates for the 1955–2023 sample

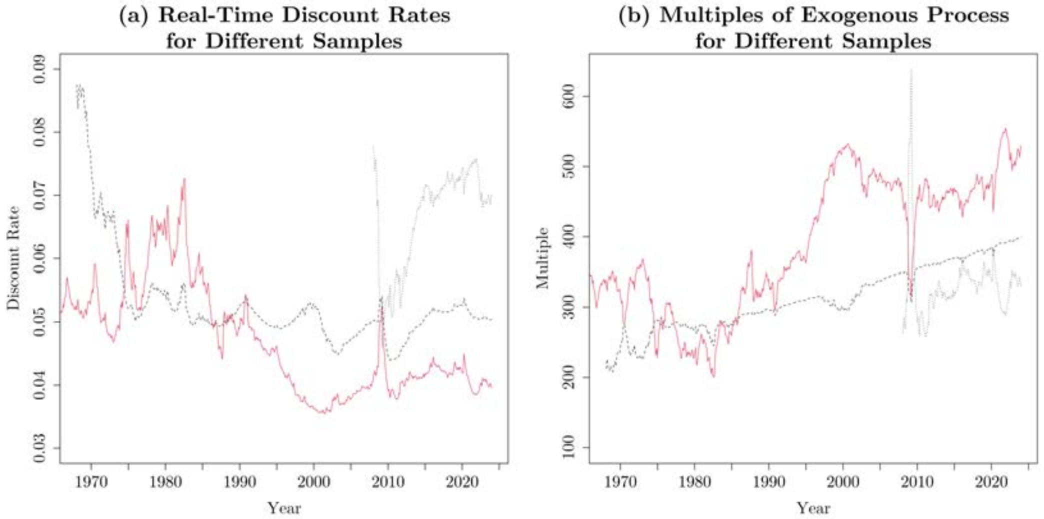

Figure 7:

Cutout for 1968–2023: (a) Estimated discount rates and (b) multiples of the exogenous process in the real-time approach using different sample sizes of data. Red solid: 1871–2023 sample, dark grey dashed: 1955–2023 sample, light grey dotted: 1995–2023 sample.

are obtained. While estimated real-time discount rates for the 1871–2023 sample decrease to below 4 % p.a. in December 2023 (see the red solid line Figure 7a), estimated discount rates are higher as soon as both shorter post-WWII sample sizes are applied (see the dark grey dashed and light grey dotted lines). For the 1955–2023 sample, the estimated real-time discount rate drops to approximately 5 % while for the 1995–2023 sample, estimates remain quite high at 7 %. In consequence, and supported by the findings in Figure 4, those fundamentals that only consider post-WWII data are usually higher in their levels compared to those using the 1871–2023 sample, an exception being the 1955–2023 sample having lower fundamentals in the early 1970s due to a higher contemporaneous discount rate.

4.5 Some Remarks on the Bubble Component

Although the real-time fundamental values in Figure 4 fit actual real price data better than the ex-post calculated ones in Figure 1, negative rational bubbles neglected on theoretical grounds by Diba and Grosmann (1988a) may still occur temporarily. However, in comparison to the ex-post approach in Section 2, the occurrence of such negative bubbles reduces drastically in the real-time approach. Nevertheless, these negative bubble phenomena do always coincide with proper explicit fundamental solutions. Independently of the considered sample, the estimated fundamental solutions always fulfill the convergence criterion (10), that is in each single iteration of the real-time estimation procedure. Thus, even when this criterion is fulfilled and fundamentals are properly obtained, the residual bubble solution may still become negative in some cases. Figure A.1a in Appendix A gives an overview of minimum values for the real-time discount rate based on (10). The occurrence of negative bubbles is thus not the result of non-convergence of fundamentals into an explicit particular solution like (8), but kind of an inevitable byproduct that occurs as soon as the theory of the efficient markets model is transferred into empirical estimations. Blanchard and Watson (1982: 8) argue that the exclusion of negative bubbles might in fact push "rationality a bit too far". In Figure 5a, negative bubbles begin to occur after the 1929 Crash in mid-1930. According to Siegel (2003), this Great Depression period is characterized by sharp decreases in realized total returns leading to significant negative deviations from expected total returns. This is what he defines a negative bubble.

As negative bubbles are a potentially unwished phenomenon, I define a new variable that I call relative bubble size (RBS). It is defined as the ratio between actual observed real price and the estimated real-time fundamental value and circumvents the notion of temporary negative bubbles as it is strictly positive. Figure 8 illustrates some properties of RBS. It depicts relationships that are also derived to hold for intrinsic and Markovian rational bubbles. According to Froot and Obstfeld (1991), a possible but simple intrinsic bubble, i.e., a bubble that solely depends on the exogenous process, is specified by the following expression:

$$b_t = x_t^\theta$$

The parameter $\theta > 0$ depicts the effect of an increasing bubble size due to changes in the exogenous process x_t, that is ultimately dividends. Using a logarithmic transformation can give some information on the parameter θ. Figure 8a plots this relationship for the S&P 500 data between 1871 and 2023 and uses RBS as a proxy for the bubble variable. In line with intrinsic bubble theory, there is a positive relationship between logarithmic RBS and the logarithmic exogenous process. Positive increases in the exogenous process transfer into positive changes in RBS. However, the relationship is clearly non-linear. For some values of the exogenous process, there is a much larger transformation into RBS, highlighting that the real-time approach images large changes in market perceptions with regards to fundamentals over time.

Similarly, a Markovian bubble specification given by

$$b_t = h(b_{t-1})$$

might be considered where h connects b_{t-1} with b_t. In this Markovian case, there is

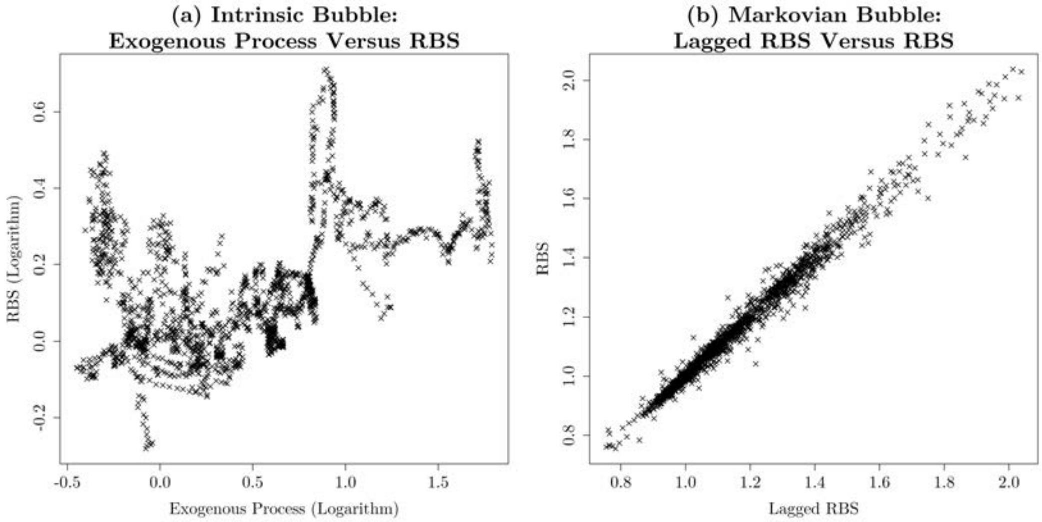

Figure 8:

(a) Correlogram for the logarithmic exogenous process and logarithmic RBS (intrinsic bubble case). (b) Correlogram for RBS and lagged RBS (Markovian bubble case). Both refer to the 1871–2023 sample.

also a clear positive relationship, now between RBS and lagged RBS according to Figure 8b. Albeit the positive relationship, the informational content is rather limited as increases and decreases in terms of lags in RBS outweigh each other to a large extent. This is due to RBS merely building up continuously and its tendency to rather deflate afterwards instead of bursting sharply (see for example the deflating Dotcom Bubble in Figure 5c or Figure 5d for the subsequent years until 2023). Though, some bubble buildups and bursts do become visible in the shape of outliers giving at least a hint for the presence of *periodically collapsing* Markovian bubbles following the notion of Blanchard and Watson (1982).

5 Discussion of the Real-Time Approach

5.1 Testing for Exuberance in the Obtained Bubble Component

The real-time approach with forward-expanding information sets is capable to give a distinct decomposition of real stock prices into their fundamental and bubble comments according to (5) in an ex-ante manner. Now, having obtained the bubble size in terms of RBS, it remains a problem to judge whether the bubble component is in fact becoming somehow dangerous and a burst is immediately to follow. As outlined by Sornette and Cauwels (2014), bubbles can be interpreted as periods of unsustainable growth. To detect this unsustainable growth, the Phillips et al. (2015) GSADF test for exuberance outlined in Section 2.2 has become a standard tool. Therefore, as the essence of the bubble component is practically twofold, that is its size on the one hand, and its explosiveness on the other hand, the RBS measure must withstand empirical test procedures regarding explosive time series behavior. Correspondingly, to show that RBS obtained in the real-time approach is indeed a measure that is capable to give reliable insights on the actual bubble size, the GSADF test procedure is applied on the RBS measure. Commonly periods of exuberance that are attributed to bubbles should then be detectable in RBS. Figure 9 shows the test results with regards to RBS (see Figure 9a) and the PD ratio (see Figure 9b) usually applied in literature.

Astonishingly, the test results are very similar for both, RBS and the PD ratio. Four periods of exuberance stand out in both test applications: (i) 1928 and 1929 before the October 1929 Crash, (ii) 1955 and 1956 corresponding to the structural break in the DP ratio according to Section 3.3 and the contemporaneous S&P 500 bull market, (iii) 1987 containing the 1987 Bubble bursting in October, and (iv) the second half of the 1990s referring to the emerging Dotcom Bubble. All bubble periods detected by the GSADF test are depicted in Table 2.[16]

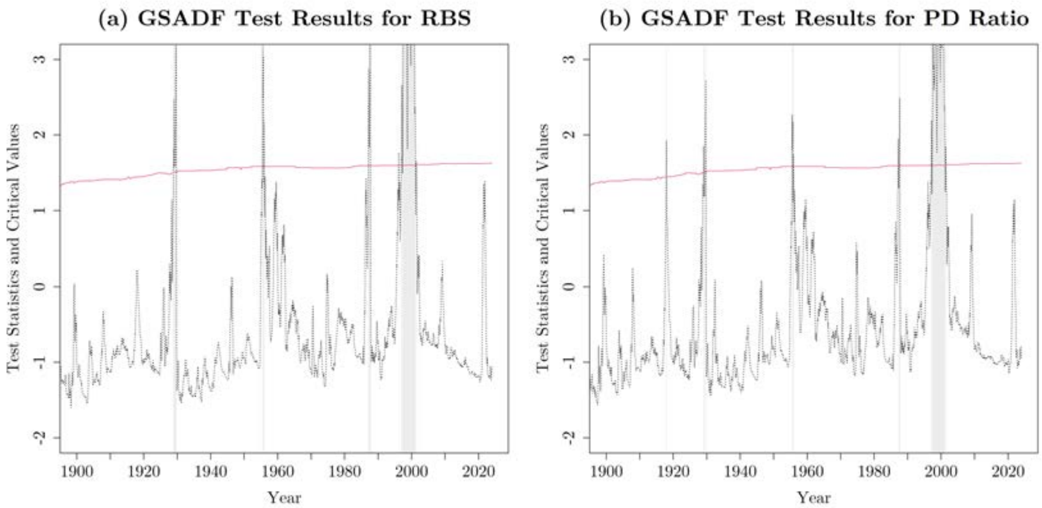

Figure 9:

a) GSADF test results for RBS and (b) the PD ratio. Grey shaded areas denote time periods of identified exuberance. This is the case as soon as the test statistics (black dotted) exceed the critical values (red solid).

The RBS measure is slightly more sensitive with regards to the 1929 Crash and the 1987 Bubble. In both cases, exuberance is detected at least one month earlier than in case of the

[16]The Phillips et al. (2015) procedure is also applied on logarithmic real price and logarithmic real dividends series in Appendix A. Figure A.2 shows the respective graphs, Table A.1 the date stamping.

PD ratio. Moreover, RBS contains exuberance without any interruption between November 1928 and October 1929 whereas the PD becomes insignificant for April and May 1929. A similar observation holds for the 1987 Bubble. While RBS is detected to firstly contain explosiveness in February 1987 up to the burst in October, the PD ratio follows in March, but omits May 1987. Concerning the Dotcom Bubble, RBS firstly contains exuberance in February 1996, and thus earlier than the PD ratio in January 1997. In the aftermath, the results of the tests resemble each other. In both cases, an interruption of exuberance is detected for April 1997, just to enter exuberance again up to March 2001 in both cases. Interestingly, both measures estimate the bubble duration to last for the exact same 46 months. After mid-2001, no further phases of exuberance are detected in both measures, notwithstanding the non-negligible size of the bubble component since then according to Figure 5d.[17]

Periods of Exuberance Detected by the GSADF Test Using RBS and the PD Ratio

RBS				**PD Ratio**			
Bubble Start	**Bubble Peak**	**Bubble End**	**Bubble Duration**	**Bubble Start**	**Bubble Peak**	**Bubble End**	**Bubble Duration**
Nov. 1928	Sep. 1929	Oct. 1929	11	Nov. 1917	Dec. 1917	Jan. 1918	2
Jun. 1955	Jul. 1955	Oct. 1955	4	Jan. 1929	Jan. 1929	Apr. 1929	3
Nov. 1955	Nov. 1955	Jan. 1956	2	Jul. 1929	Sep. 1929	Oct. 1929	3
Feb. 1987	Aug. 1987	Oct. 1987	8	Jun. 1955	Jul. 1955	Oct. 1955	4
Feb. 1996	Feb. 1996	Mar. 1996	1	Nov. 1955	Nov. 1955	Jan. 1956	2
Nov. 1996	Feb. 1997	Apr. 1997	5	Mar. 1987	Mar 1987	Apr. 1987	1
May 1997	Apr. 1998	Mar. 2001	46	Jun. 1987	Aug. 1987	Oct. 1987	4
May 2001	May 2001	Jun. 2001	1	Jan. 1997	Feb. 1997	Apr. 1997	3
				May 1997	Apr. 1998	Mar. 2001	46
				May 2001	May 2001	Jul. 2001	2

Table 2:

Periods of exuberance detected by the GSADF test using RBS and the PD ratio. *Bubble start* is defined as the first point in time where the test statistic exceeds the critical value. *Bubble end* is defined as the first point in time where the test statistic falls below the critical value again. *Bubble Peak* refers to the date where the test statistic exceeds the critical value at most. *Bubble duration* refers to the nmer of monthly observations in a bubble perod with ongoing exuberance.

As far as the plausibility of RBS as a measure for a bubble component can be analyzed, it is not inferior to the standard PD ratio commonly used for detecting exuberance. In doing so, it gives corresponding estimates of the bubble size that cannot be facilitated by

[17] According to the GSADF test results in Table 2, the Dotcom Bubble is estimated to end in June 2001. According to Figure 5c, the bubble component already reaches its maximum size in August 2000 just to deflate afterwards. Commonly, the burst is dated to March 2000 as it is referred to stock price declines in the NASDAQ Composite, and not in the broader S&P 500 index.

the common PD ratio measure. Analyzing bubbles in terms of their sizes, as well as in terms of their explosiveness, is hence attainable.

5.2 Extraction of a Bubble Cycle

RBS can be shown to be trend stationary using nonlinear LLR and Hodrick-Prescott (HP) trends. Table 3 shows the residual analysis for RBS using ADF, PP, and KPSS tests. Concerning the former two test procedures, and for both detrending methods, the unit root null hypotheses of a RW are rejected in favor of stationary residuals at the 1 % significance levels. According to these results, LLR and HP detrend RBS properly in terms of removing non-stationary behavior out of the residuals. In accordance with these results, the KPSS tests cannot reject their null hypotheses of stationary AR processes at the 10 % levels. Hence, the KPSS results support those of the ADF and PP procedures in favor of stationary residuals for both series.

Unit Root Tests for RBS Residuals

	Sample	ADF		PP		KPSS	
		H0	P Value	H0	P Value	H0	P Value
LLR Residuals of RBS	1871–2023	RW	0.01	RW	0.01	Stationary AR	≥ 0.10
HP Residuals of RBS	1871–2023	RW	0.01	RW	0.01	Stationary AR	≥ 0.10

Table 3:

ADF, PP, and KPSS test results for the residual series obtained by LLR and HP detrending of RBS. Concerning the null hypothesis, *RW* refers to a RW without drift and without deterministic trend. *Stationary AR* refers to an AR process without drift and without deterministic trend.

Figure 10 shows RBS together with the estimated LLR and HP trends. Moreover, recessions dates according to the National Bureau of Economic Research (NBER 2025) are added in terms of shaded grey areas. Obviously, there is a cyclical pattern in RBS. Though, the HP trend (solid orange line) is more volatile than the LLR trend (solid blue line) and attributes more variation to the long-run component. Bubbles in the S&P 500 are oscillating which is in accordance with the notion of periodically collapsing bubbles and the lines of thoughts presented in Salge (1997) concerning a bubble cycle (see Section 3.1.2 therein) or Burs (2023) with respect to a limit cycle.

NBER recessions phases are found to correspond with declines in RBS. According to the results depicted in Figure 10, local peaks in RBS often precede recessions. A similar notion is found in Barro and Ursúa (2017) who find general stock market declines in terms of price movements from local peaks to throughs to be associated with recessions or even depressions. However, a general pattern of local peaks in RBS proceeding recessions is not

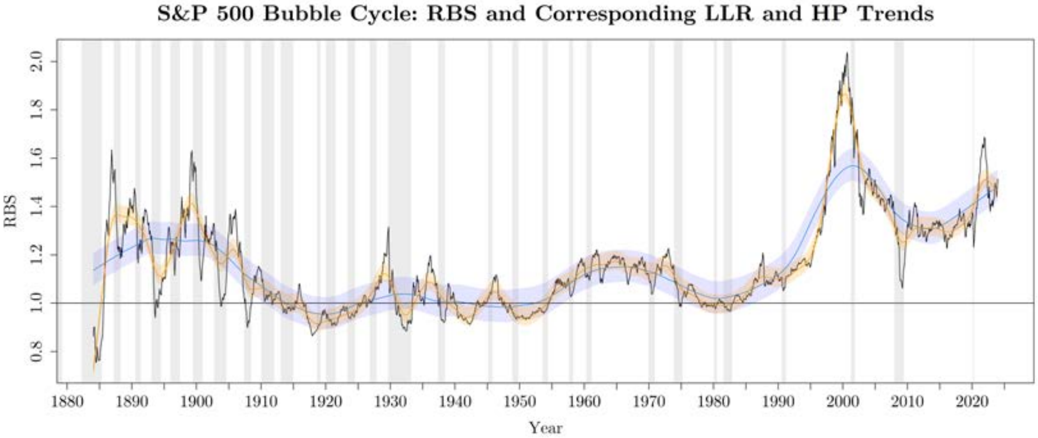

Figure 10:

Derivation of a bubble cycle. The solid black line depicts RBS. NBER recession periods are illustrated by light grey areas. The LLR trend of RBS is depicted in solid blue, the HP trend of RBS in solid orange. The latter trend uses a smoothing parameter of 14400 suitable for monthly data. The corresponding shaded blue and orange areas refer to corridors where RBS is estimated to behave like a unit root process in contrast to an autoregressive process outside of these corridors, see Section 5.3 for more details in this respect.

obtained. For example, with a focus on the crashes in 1929 and 1987, the Great Depression is clearly proceeded by a local high in RBS, but the burst of the 1987 Bubble is not succeeded by a recession period in late-1987.

The trend values of RBS show that bubbles in terms of their sizes are a predominant phenomenon of the late 19^{th} and especially the early 21^{st} century while they are estimated to be rather small or even temporarily negative (see RBS values smaller than one in Figure 10) throughout the 20^{th} century. Prior to the 1929 Crash, RBS is estimated to be approximately 1.3 (see also the 30 % positive deviation in Figure 5a). Prior to the 1987 Bubble burst, it equals approximately 1.2 (see the 20 % positive deviation in Figure 5b). At the same time, the corresponding RBS trend values remain nearly unaffected by these specific periods as they also do in the remaining 20^{th} century. The Dotcom Bubble emerging in the 1990s ends this pattern. As RBS increases sharply up to a value of 2.0 prior to March 2001, RBS trend values increase as well, especially in the HP case. In general, the Great Moderation period starting in the early 1980s has altered the picture concerning bubbles and recessions. While the former increase to levels not observed before since 1871, the latter reduce in their frequency. After the Dotcom Bubble Crash, trend values of the bubble component – with a short exception in 2009 – never downsize to those of the 20^{th} century. Bubbles remain large, while corrections towards market fundamentals fail to appear.

5.3 AR and Unit Root Patterns in the Bubble Component

According to Meese (1986) and Evans (1991), bubbles may resemble mean-reverting autoregressive processes. Since bubbles are by definition explosive processes, they should become observable in terms of regime switching as soon as the market enters an exuberant phase that is not depicted by usual unit root market behavior. In order to find autoregressive behavior in RBS, I estimate several self-exciting threshold autoregressive (SETAR) models (see Tong 1983) for the first differences of RBS residuals, denoted by $\Delta\hat{\xi}_t$, in an ADF unit root test fashion according to:

$$\Delta\hat{\xi}_t = \kappa^{(j)} + \gamma^{(j)}\hat{\xi}_{t-1} + \delta_1^{(j)}\Delta\hat{\xi}_{t-1} + \delta_2^{(j)}\Delta\hat{\xi}_{t-2} + ... + \delta_p^{(j)}\Delta\hat{\xi}_{t-p} + \nu_t^{(j)},$$
$$\text{if } r_{j-1} \leq \hat{\xi}_{t-l} < r_j, j = 1, ..., k \quad (12)$$

First differences of RBS residuals $\Delta\hat{\xi}_t$ are estimated by the level variable $\hat{\xi}_{t-1}$ one period ago and first differences $\Delta\hat{\xi}_{t-1}$ up to $\Delta\hat{\xi}_{t-p}$ where p denotes the respective lag order of those variables in first differences. Corresponding autoregressive coefficients $\kappa^{(j)}$, $\gamma^{(j)}$ and $\delta_1^{(j)}$ up to $\delta_p^{(j)}$ depend on regime j, such that the threshold variable $\hat{\xi}_{t-l}$ (with l depicting the delay parameter) lies in the range of thresholds r_{j-1} up to r_j, dividing the domain of $\hat{\xi}_{t-l}$ into k regimes. The variables $\nu^{(j)}$ refer to white noise error terms with zero mean and constant variance.

Autoregressive patterns in regime j are detected as soon as the respective level parameter $\gamma^{(j)}$ is estimated to be significantly negative (and thus different from zero). Level parameters that do not significantly deviate from zero speak in favor of a unit root regime which is in accordance with the common notion of informationally efficient markets. Considering a three-regime SETAR model according to (12), bubble buildups should become visible by a significant autoregressive high regime (HR). In opposite, rebounds after bubble bursts should become visible by means of a significant autoregressive lower regime (LR). In between, a unit root middle regime (MR) should depict usual, that is any non-autoregressive but unit root, market behavior.

Bec et al. (2004) already use a SETAR model with hypothesized unit root MR. They analyze the purchasing power parity relationship whose unit root behavior is supposed to be bounded by autoregressive behavior at both tails, i.e., at the top and the bottom. Fitting to the purpose presented here, and extending Serletis and Shintani (2003) and Elder and Serletis (2007), Koustas et al. (2008) set up a three-regime SETAR for Dow Jones Industrial Average data. They test for unit root behavior in the MR as well as for autoregressive behavior in the outer two regimes and find evidence for autoregressive mean reversion attributable to bubbles. Similarly, but for US real estate returns, dos Reis and Sobreira (2019) estimate a two-regime SETAR next to RW or autoregressive integrated moving average models. They identify bubbles to occur as soon as returns are depicted by the HR. According to them, nonlinear models like SETAR prove well in capturing exuberant behavior during real estate bubbles.

SETAR Estimation Results for RBS Residuals Obtained by LLR

SETAR: Autoregressive Orders	(1, 1, 1)	(2, 1, 2)	(1, 2, 1)	(3, 1, 3)	(1, 3, 1)	(3, 2, 3)	(2, 3, 2)	(2, 2, 2)	(3, 3, 3)
Lower Threshold	-0.062089	0.001869	-0.062089	-0.006230	0.016560	-0.006230	0.001869	0.001869	-0.006230
Upper Threshold	-0.004727	0.035478	-0.004727	0.056530	0.071040	0.056530	0.035478	0.035478	0.056530
Proportion of Points in LR	17.82%	57.66%	17.77%	52.86%	66.65%	52.86%	57.64%	57.66%	52.86%
Proportion of Points in MR	36.29%	17.77%	36.31%	28.16%	17.72%	28.16%	17.78%	17.77%	28.16%
Proportion of Points in HR	45.89%	24.57%	45.92%	18.97%	15.63%	18.97%	24.58%	24.57%	18.97%
α^{LR}	0.00401078 (0.00282133)	0.00104289 (0.00104201)	0.00392686 (0.00282317)	0.0011864 (0.0011280)	0.00114066 (0.00090724)	0.0011864 (0.0011274)	0.00095978 (0.00104154)	0.00104289 (0.00104227)	0.0011864 (0.0011277)
γ^{LR}	**-0.01304975** **(0.01854699)**	**-0.02980650*** **(0.01182808)**	**-0.01420463** **(0.01861046)**	**-0.032175**** **(0.012373)**	**-0.02942616**** **(0.01104457)**	**-0.032175**** **(0.012366)**	**-0.03238196**** **(0.01186822)**	**-0.02980650*** **(0.01183113)**	**-0.032175**** **(0.012370)**
δ_1^{LR}	0.32534380*** (0.05288974)	0.37234985*** (0.03904946)	0.32872351*** (0.05308082)	0.38318*** 0.039939)	0.30935490*** 0.03395594)	0.38318*** (0.039917)	0.37122273*** (0.03901181)	0.37234985*** (0.03905955)	0.38318*** (0.039929)
δ_2^{LR}		-0.00651197 (0.03942525)		-0.036895 (0.042617)		-0.036895 (0.042594)	-0.00018193 (0.03947935)	-0.00651197 (0.03943543)	-0.036895 (0.042607)
δ_3^{LR}				0.063175 (0.040467)		0.06.3175 (0.040445)			0.063175 (0.040457)
α^{MR}	0.00251627 (0.00209034)	0.00178378 (0.00286585)	0.00245994 (0.00209073)	-0.000069598 (0.0016855)	-0.0027026 (0.00380390)	-0.000078514 (0.0016846)	0.00175234 (0.00286922)	0.00173822 (0.00286921)	-0.000081169 (0.0016854)
γ^{MR}	**0.02116829** **(0.06046696)**	**-0.16618078** **(0.14426019)**	**0.02079778** **(0.06045791)**	**-0.050340** **(0.064815)**	**-0.01116500** **(0.09027354)**	**-0.056514** **(0.064885)**	**-0.16550182** **(0.14446901)**	**-0.16460832** **(0.14435954)**	**-0.056181** **(0.065012)**
δ_1^{MR}	0.41596911*** (0.05465179)	0.02772861 (0.05734915)	0.42831678*** (0.05599025)	0.078019 (0.048960)	0.17822143** (0.06019596)	0.061612 (0.049907)	0.02512961 (0.05805626)	0.02468316 (0.05794688)	0.061757 (0.049948)
δ_2^{MR}			-0.06357937 (0.06287239)		-0.03303113 (0.05569680)	0.085322 (0.051003)	0.02265526 (0.05995459)	0.02224314 (0.05988007)	8.5629e-02 (5.1133e-02)
δ_3^{MR}					-0.07722318 (0.05717208)		-0.00578204 (0.05993522)		-0.0047102 (0.052650)
α^{HR}	-0.00051054 (0.00112180)	0.00017833 (0.00196561)	-0.00051054 (0.00112161)	-0.00036046 (0.0025124)	0.00147036 (0.00305188)	-3.6046e-04 (2.5111e-03)	0.00017833 (0.00196356)	0.00017833 (0.00196612)	-0.00036046 (0.0025118)
γ^{HR}	**-0.03547074***** **(0.00955560)**	**-0.03874284**** **(0.01235723)**	**-0.03547074***** **(0.00955400)**	**-0.037083**** **(0.014032)**	**-0.04523474**** **(0.01550627)**	**-0.037083**** **(0.014024)**	**-0.03874284**** **(0.01234435)**	**-0.03874284**** **(0.01236042)**	**-0.037083**** **(0.014028)**
δ_1^{HR}	0.22994258*** (0.02957773)	0.31468243*** (0.03593530)	0.22994258*** (0.02957276)	0.33305*** (0.037897)	0.29911048*** (0.03821949)	0.33305*** 0.037877)	0.31468243*** (0.03589784)	0.31468243*** (0.03594458)	0.33305*** (0.037888)
δ_2^{HR}		-0.05204767 (0.03591793)		-0.083684* (0.039078)		-0.083684* (0.039056)	-0.05204767 (0.03588049)	-0.05204767 (0.03592721)	-0.083684* (0.039068)
δ_3^{HR}				0.036927 (0.037531)		0.036927 (0.037511)			0.036927 (0.037522)
AIC	-12532	-12545	-12531	-12549	-12531	-12550	-12547	-12543	-12548
BIC	-12472	-12474	-12466	-12468	-12460	-12463	-12465	-12467	-12456

Table 4:

SETAR estimation results for RBS residuals obtained by LLR. The listed autoregressive orders read as follows: For instance, the $SETAR(2, 1, 2)$ refers to a model with two first difference lags in the LR, one in the MR, and two in the HR. Parameter estimates for the level variables are in bold. Negative significan parameters for level variables support the hypothesis of autoregressive behavior in the respective regime, nonsignificant parmaters support the unit root hypothesis. Standard errors in parentheses; corresponding p values $^*P(z) < 0.05,^{**} P(z) < 0.01,^{***} P(z) < 0.001$.

SETAR Estimation Results for RBS Residuals Obtained by HP

SETAR: Autoregressive Orders	**(1, 1, 1)**	**(2, 1, 2)**	**(1, 2, 1)**	**(3, 1, 3)**	**(1, 3, 1)**	**(3, 2, 3)**	**(2, 3, 2)**	**(2, 2, 2)**	**(3, 3, 3)**
Lower Threshold	-0.027809	0.011450	-0.027809	-0.032929	-0.027809	-0.032929	0.011450	0.011450	-0.032929
Upper Threshold	0.004637	0.045740	0.004637	0.009868	0.004637	0.009868	0.045740	0.045740	0.009868
Proportion of Points in LR	23.30%	62.13%	23.32%	19.81%	23.33%	19.81%	62.17%	62.13%	19.81%
Proportion of Points in MR	32.72%	22.66%	32.74%	41.05%	32.76%	41.05%	22.67%	22.66%	41.05%
Proportion of Points in HR	43.98%	15.21%	43.95%	39.14%	43.91%	39.14%	15.16%	15.21%	39.14%
α^{LR}	0.00351543 (0.00220494)	-0.000033286 (0.00089064)	0.00351543 (0.00220368)	0.00437690 (0.00258786)	0.00351543 (0.00220084)	0.00437690 (0.00258234)	-0.000033286 (0.00089011)	-0.000033286 (0.00089072)	0.00437690 (0.00258130)
$\boldsymbol{\gamma^{LR}}$	**-0.07168692**** **(0.02571170)**	**-0.10258***** **(0.017151)**	**-0.07168692**** **(0.02569698)**	**-0.05781688*** **(0.02854654)**	**-0.07168692**** **(0.02566390)**	**-0.05781688*** **(0.02848568)**	**-0.10258***** **(0.017141)**	**-0.10258***** **(0.017153)**	**-0.05781688*** **(0.02847428)**
δ_1^{LR}	0.28856694*** (0.03720687)	0.31249*** (0.031756)	0.28856694*** (0.03718557)	0.30576330*** (0.04002867)	0.28856694*** (0.03713769)	0.30576330*** (0.03994333)	0.31249*** (0.031737)	0.31249*** (0.031759)	0.30576330*** (0.03992734)
δ_2^{LR}		-0.015499 (0.032233)		-0.10649655* (0.04158711)		-0.10649655* (0.04149844)	-0.015499 (0.032213)	-0.015499 (0.032236)	-0.10649655* (0.04148183)
δ_3^{LR}				0.04135692 (0.04394415)		0.04135692 (0.04385046)			0.04135692 (0.04383290)
α^{MR}	0.00248363 (0.00153778)	0.00012060 (0.0034434)	0.00244425 (0.00153707)	0.00057617 (0.00116683)	0.00231995 (0.00153775)	0.00037786 (0.00116642)	0.000093350 (0.0034446)	0.000063897 (0.0034443)	0.00029755 (0.00116714)
$\boldsymbol{\gamma^{MR}}$	**0.16983372** **(0.11160837)**	**-0.045211** **(0.12530)**	**0.16848253** **(0.11154728)**	**-0.00253607** **(0.07601053)**	**0.15270175** **(0.11199302)**	**-0.01453774** **(0.07596516)**	**-0.047725** **(0.12568)**	**-0.045365** **(0.12531)**	**-0.02603439** **(0.07630659)**
δ_1^{MR}	0.28713971*** (0.06339569)	0.098698 (0.064445)	0.25395544*** (0.06627881)	0.34180039*** (0.05243798)	0.26432571*** (0.06662156)	0.31050424*** (0.05346482)	0.085485 (0.066204)	0.086401 (0.066119)	0.31382825*** (0.05348768)
δ_2^{MR}			0.11242234 (0.06590751)		0.11064132 (0.06583539)	0.15606459** (0.05473073)	0.052995 (0.063808)	0.053186 (0.063846)	0.15112344** (0.05480431)
δ_3^{MR}					0.08666748 (0.06301655)		0.012641 (0.057248)		0.07631597 (0.04994504)
α^{HR}	0.00045506 (0.00124938)	0.0019638 (0.0034261)	0.00045083 (0.00124998)	0.00087072 (0.00139576)	0.00036235 (0.00124929)	0.00087072 (0.00139278)	-0.0022226 (0.0034270)	-0.0019638 (0.0034264)	0.00087072 (0.00139222)
$\boldsymbol{\gamma^{HR}}$	**-0.12059450***** **(0.01999427)**	**-0.10612**** **(0.034430)**	**-0.12045506***** **(0.02007177)**	**-0.12316892***** **(0.02203458)**	**-0.11712210***** **(0.02012685)**	**-0.12316892***** **(0.02198760)**	**-0.10126**** **(0.034515)**	**-0.10612**** **(0.034433)**	**-0.12316892***** **(0.02197880)**
δ_1^{HR}	0.25954754*** (0.03429520)	0.29431*** (0.042851)	0.25956116*** (0.03427607)	0.23823937*** (0.03601392)	0.25684814*** (0.03426338)	0.23823937*** (0.03593713)	0.29020*** (0.042887)	0.29431*** (0.042855)	0.23823937*** (0.03592275)
δ_2^{HR}		-0.0091610 (0.046266)		-0.00273965 (0.03750255)		-0.00273965 (0.03742260)	-0.0087116 (0.046239)	-0.0091610 (0.046270)	0.02109235 (0.03618035)
δ_3^{HR}				0.02109235 (0.03627217)		0.02109235 (0.03619484)			0.036927 (0.037522)
AIC	-12663	-12662	-12664	-12665	-12667	-12671	-12662	-12660	-12671
BIC	-12603	-12591	-12599	-12583	-12597	-12584	-12580	-12584	-12579

Table 5:

SETAR estimation results for RBS residuals obtained by HP. The listed autoregressive orders read as follows: For instance, the $SETAR(2,1,2)$ refers to a model with two first difference lags in the LR, one in the MR, and two in the HR. Parameter estimates for the level variables are in bold. Negative significan parameters for level variables support the hypothesis of autoregressive behavior in the respective regime, nonsignificant parmaters support the unit root hypothesis. Standard errors in parentheses; corresponding p values $^{*}P(z) < 0.05,^{**} P(z) < 0.01,^{***} P(z) < 0.001$.

Tables 4 and 5 show the SETAR estimation results according to model (12). Table 4 refers to the SETAR models that use LLR-based RBS residuals, Table 5 refers to those that use the HP-based residuals. In accordance with Meese (1986) and Evans (1991), significant autoregressive behavior in the HR is detected for both RBS residual series. RBS is found to contain segments of data that cannot be attributed to usually-assumed unit root behavior. As autoregressive patterns are also found for many LR in the SETAR models (albeit in a lesser degree), RBS contains segments of buildups and rebounds that are typical for bubbles or similar market anomalies.

To visualize in which time periods RBS is in fact estimated to follow autoregressive processes, Figure 10 depicts a further graphical exercise: The blue shaded corridor enclosing the LLR trend and the orange shaded corridor enclosing the HP trend depict where RBS itself (i.e., not its residuals) is regarded to be driven by unit root behavior. Hence, within the corridors, RBS is estimated to follow a unit root process which is in line with an informationally efficient financial market. Outside the corridors, RBS is estimated to follow an autoregressive process due to bubbles (and their rebounds) or similar phenomena not in line with unit root pricing. To obtain these corridors, I take the estimated SETAR thresholds (that are specified in levels and not in first differences) and add or subtract them from the LLR and HP trend functions for RBS.[18]

As with regards to Table 2 depicting the periods of market exuberance according to the Phillips et al. (2015) GSADF test, the corridors can be used for date stamping bubbly behavior as well. Once an upper threshold is crossed from beneath, RBS shows autoregressive patterns that may be in line with temporary explosive bubble behavior. Table 6 gives an overview of these time periods where RBS residuals are in a HR of a SETAR. This is equivalent to RBS itself being above a depicted corridor in Figure 10. Approximately one third of all 1680 monthly observations lie in the outer two regimes, independently of whether LLR or HP detrended residuals of RBS are used.

The periods where RBS residuals lie in the upper SETAR regimes correlate with well-known bubble periods, as shown in Table 7. Prior to the bubble bursts in 1929, 1987, and 2000, autoregressive patterns are detected. Though, compared to the GSADF results concerning market exuberance listed in Table 2, periods with autoregressive patters are

[18] For constructing the corridors, I take those thresholds that minimize the outer regimes. This approach is due to the argument of keeping the outcomes in terms of autoregressive time series behavior as conservative as possible. Thus, regarding LLR in Table 4, the $SETAR(1,3,1)$ upper threshold is taken, next to the $SETAR(1,1,1)$ lower threshold. Regarding HP in Table 5, the $SETAR(2,1,2)$ upper threshold is taken, next to the $SETAR(3,1,3)$ lower threshold. In cases of identical thresholds for several SETAR parametrizations, I just refer to one single model at this place.

Overview of RBS Residuals Attributed to Different SETAR Regimes

	Sample	Observations in HR	Observations in LR	Observations in Outer Regimes	Total Observations
LLR Residuals of RBS	1871–2023	262	299	561	1680
HP Residuals of RBS	1871–2023	258	331	589	1680

Table 6:

Number of RBS residuals obtained by LLR and HP detrending lying in the outer SETAR regimes. The SETAR thresholds are selected according to Footnote 18.

Periods with RBS Residuals in the Outer SETAR Regimes

Periods with LLR Residuals of RBS in HR				
Nov. 1885 – Sep. 1887	Aug. 1897 – Apr. 1900	Apr. 1928 – Oct. 1929	Mar. 1972 – Apr. 1973	Jan. 2011 – Mai 2011
Sep. 1888 – Sep. 1890	Nov. 1904 – Nov. 1906	Aug. 1935 – Nov. 1936	Feb. 1987 – Sep. 1987	Jan. 2021 – Apr. 2022
Sep. 1891 – May 1892	Oct. 1915 – Jan. 1916	Sep. 1945 – Aug. 1946	Feb. 1998 – Apr. 2002	

Periods with HP Residuals of RBS in HR				
Jan. 1884 – Apr. 1884	Apr. 1902 – Feb. 1903	Jun. 1933 – Sep. 1933	Aug. 1961 – Dec. 1961	Jan. 2007 – Dec. 2007
Nov. 1885 – Jun. 1887	Nov. 1904 – Apr. 1906	Jan. 1934 – Apr. 1934	Nov. 1972 – Mar. 1973	Dec. 2009 – May 2010
Sep. 1889 – Feb. 1893	Aug. 1906 – Nov. 1906	Nov. 1935 – Aug. 1936	Mar. 1987 – Sep. 1987	Oct. 2010 – Jul. 2011
May 1895 – Feb. 1896	Dec. 1908 – Jan. 1910	Nov. 1938 – Mar. 1939	Apr. 1999 – Jul. 1999	Feb. 2021 – Apr. 2022
Aug. 1897 – Oct. 1897	Oct. 1915 – Feb. 1916	Dec. 1945 – Aug. 1946	Nov. 1999 – Jun. 2001	
Jan. 1899 – Dec. 1899	Nov. 1928 – Oct. 1929	May 1959 – Aug. 1959	Jan. 2004 – Apr. 2004	

Table 7:

Time periods where RBS residuals obtained by LLR and HP detrending lie in the HR of estimated SETAR models. The SETAR thresholds are selected according to Footnote 18. Listed periods are restricted to those which have a length of at least three consecutive months. Periods with intermediate breaks up to three months are consolidated to one larger period.

much more frequent. Interestingly, some periods after the burst of the Dotcom Bubble are detected to be of autoregressive nature whereupon the GSADF test in Table 2 gives no analogue results in terms of market exuberance since mid-2001.

5.4 Fundamentals and Bubbles in Real Price Growth

A so far disregarded advantage of RBS is given by engaging a growth rate perspective. As real prices are explicitly decomposed into a fundamental and a bubble component according to (5) in the real-time approach, this decomposition idea can be transferred to growth rates. Using the RBS measure, the real price growth rate, that is g_{p_t}, can be decomposed into the growth rate of the real-time fundamental value, g_{f_t}, and the growth rate of RBS, g_{RBS_t}, such that the following condition holds:

$$g_{p_t} = g_{f_t} + g_{RBS_t} \tag{13}$$

Figure 11 shows this decomposition graphically. The actual observed yearly price growth rate is depicted by black crosses. For each point in time, real-time fundamental value growth rates are depicted by the solid green line and real-time RBS growth rates by the solid red line. Both sum up to the real price growth rate. For further orientation, 5-year periods prior to RBS peaks in 1929, 1987 and 2000 (see Figure 10) are depicted by dark grey areas in Figure 11. Subsequent 5-year periods after those peaks are depicted in light grey.

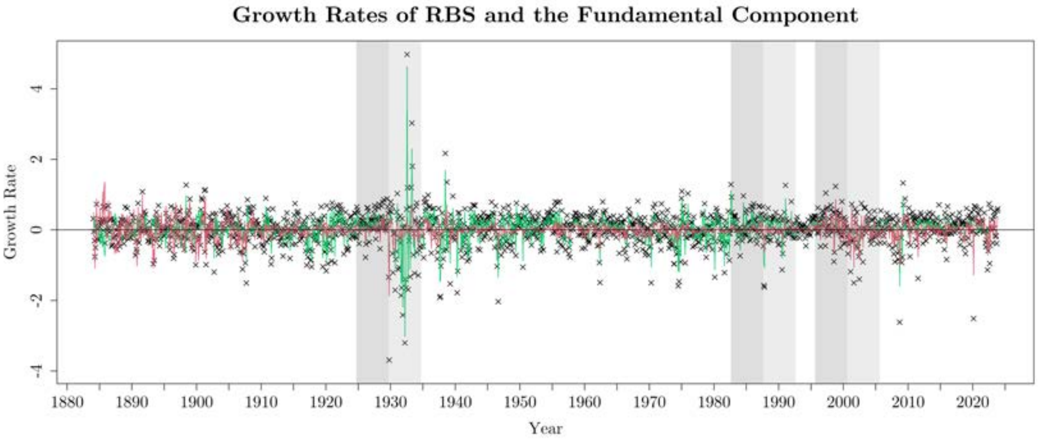

Figure 11:

Decomposition of S&P 500 yearly real price growth rates (black crossed) into a fundamental growth rate (green solid) and a non-fundamental RBS growth rate (red solid). The dark grey areas refer to 5-year phases before the bubble peaks in 1929, 1987, and 2000 (see Figure 5). The light grey areas refer to 5-year phases afterwards.

Generally, the period between 1920 and 1990 is rather explainable by growth in fundamentals. Afterwards, and before 1920, actual real price growth rate is more driven by RBS as displayed by the higher emergence of the red graph over the green one. The 1929 Crash at about 30 % above fundamentals (see Figure 5a) is accompanied by a large reduction in the RBS growth rate of approximately -200 % p.a. in Figure 11, leading to a corresponding monthly decrease of about 17 % (see the red graph at the border of the corresponding dark grey and light grey areas). The recession period after the 1929 Crash, i.e., the Great Depression, also clearly stands out in terms of price growth rates. However, the variation in the real price growth rate is to a large extent explainable by the growth rate in the fundamental value itself, but not in RBS. Due to large reductions in dividend payments, the growth rate in fundamentals first decreases sharply after 1930. Afterwards, there is a positive overshooting where monthly rates exceed 30 %, just to revert back to usual rates before 1935 (see the light grey area depicting the 5-year period after the 1929 Crash).

With a focus on the emergence of bubbles, Figure 11 shows that the 1987 Bubble can be attributed to growth in fundamentals at most, followed by the pre-1929 Crash years where price growth is attributable to growth in fundamentals and RBS in more or less equal parts, vindicating the notion that market participants back then might have still perceived the market to be rather fundamental. Certainly, the emergence of the Dotcom Bubble clearly stands out as corresponding real price growth is clearly attributable to non-fundamental RBS growth with monthly rates up to 8 %.

Equation (13) can be rewritten in order to explicitly contain the two components in fundamentals given by equations (8) and (9), that is exogenous process and its corresponding multiple. Real price growth g_{p_t} is then the sum of growth in the exogenous process, g_{x_t}, growth in the multiple, g_{m_t}, and growth in RBS, g_{RBS_t}. Accordingly, equation (13) becomes:

$$g_{p_t} = g_{x_t} + g_{m_t} + g_{RBS_t} \tag{14}$$

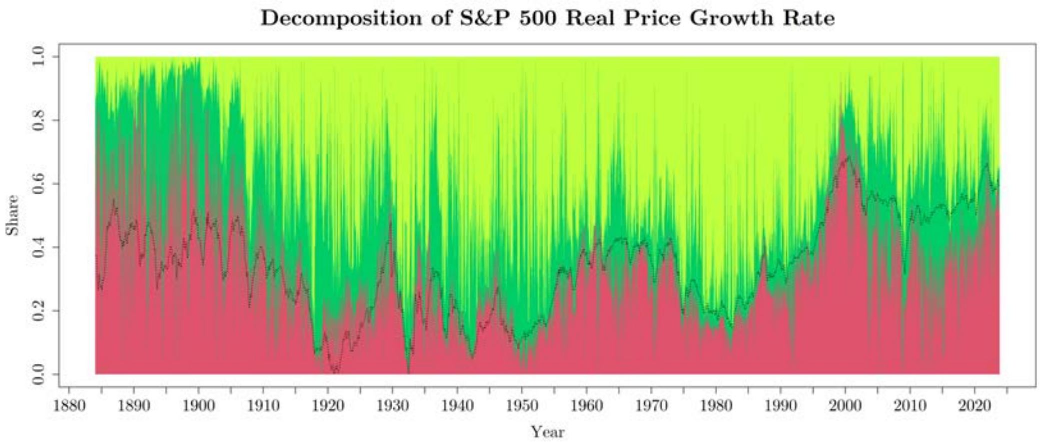

Figure 12:

Decomposition of S&P 500 real price growth rate into normalized fundamental and non-fundamental shares. Fundamental growth shares (dark green for growth in the exogenous process, light green for growth in the multiple) and growth shares for non-fundamental RBS (red). The black dotted line depicts the share of expected pure price return within the real-time discount rate already depicted in Figure 3b following expression (11).

Figure 12 depicts the normalized shares of g_{x_t} (dark green area), g_{m_t} (light green area), and g_{RBS_t} (red area) in total g_{p_t} following equation (14). Moreover, it shows the share of expected pure price return in expected real-time discount rates (dotted black line) that is already covered in expression (11) and plotted in Figure 3b.

As suspected in Section 4.2 and outlined in Section 4.3 with regards to selected bubble periods, this share of expected pure price return (that is independent from the exogenous process and thus dividends) is relatable to bubbles. According to Figure 12, it is positively correlated with the share of non-fundamental RBS growth in real price growth. Bubbles can be attributed to non-fundamental pricing in the data. During the Roaring 20s, as well as in the 1980s or 1990s prior to the bursts of the 1987 or Dotcom Bubbles, the share of expected pure price return increases uniformly with the growth share of RBS. Similarly, it decreases again after those bubbles burst.

While actual real prices in the S&P 500 have diverged from the real-time fundamental value in last decades (see Figures 4 and 5), thus expanding the bubble size, real price growth rates have become more fundamental again after the peak of the Dotcom Bubble in 2000 according to Figure 12. Especially the growth rate of the exogenous process (and thus dividends) represented by the dark green area has increased such that growth in RBS accounts for less than half of the growth in real stock prices by the mid-2010s. Though, a decreasing impact on the size of the bubble component back to fundamentals cannot be observed (see Figures 5d and 10). What is more, the importance of RBS in real price growth rates has increased again in recent years, as the increase in the red area in Figure 12 depicts. Thereupon, RBS itself has increased towards values above 1.4 in 2023, a bubble size that is unusually high in comparison to the 20^{th} century excluding the 1990s and the Dotcom Bubble. Correspondingly, by the end of 2023, about 60 % of the growth in real stock price can be attributed to RBS again, a value that also lastly appeared in the late 19^{th} century.

6 Conclusion

Assuming an explicit fundamental solution gives a reasonable starting point for strictly modelling both the fundamental and the bubble component of an asset. In a real-time estimation approach using monthly S&P 500 stock data from January 1871 to December 2023, I employ such a decomposition in order to detect bubbles at an early stage of their occurrence. To do so, the importance of forward-expanding information sets known to market participants is highlighted. In opposition to simple ex-post decompositions that cannot properly adjust to nonlinear behavior in observed real prices, I allow the discount rate and other parameters needed to calculate market fundamentals to alter over time. Hence, the presented approach introduces ongoing changes to the information set known to market participants such that these are allowed to continuously adjust their expectations to market fundamentals.

The presented real-time approach is founded on explicit fundamental solutions that assume a geometric RW with drift for the exogenous (dividend) process. I obtain bubbles as residuals between estimated fundamentals and the observed real price. In that way, I receive real-time estimates for the size of the bubble component, a factor that is so far fully disregarded in empirical bubble detection. Bubbles in the data are analyzed by combining three aspects that are so-far considered on their own: (i) estimating the sizes of the fundamental and the bubble component, (ii) detecting explosive time series behavior, and (iii) let information sets be forward expanding such that (i) and (ii) may be prone to revisions in market expectations.

To further analyze the obtained bubble component, I define RBS, that is the ratio between the realized real price and the estimated real-time fundamental price. RBS is shown to be a reasonable measure for the size of bubbles as it captures periods commonly attributed to bubbles properly. Using the Phillips et al. (2015) GSADF procedure testing for exuberance by means of explosive time series behavior, I show that the use of RBS is not inferior to the PD ratio that is usually applied for detecting bubbles. In fact, test results are even a little bit more sensitive in case of RBS which is a plus for the early detection of bubbles in a real-time setting.

RBS is moreover shown to be trend stationary using nonlinear LLR and HP detrending approaches. Based on the obtained residual series, three-regime SETAR models are estimated. In an ADF unit root test fashion, these models find significant autoregressive behavior in their outer two regimes. The result is in line with Evans (1991) who observes that bubbles may resemble autoregressive time series processes. The corresponding time-stamped periods also contain those obtained by the GSADF procedure using RBS or the PD ratio.

RBS has the further advantage of allowing the real stock price growth rate to be decomposed into fundamental and non-fundamental shares. I show that real price growth in the S&P 500 is mostly based on fundamental factors between 1920 and 1990, while it is much more based on non-fundamental factors, that is RBS, in the late 19^{th} century and the beginning 21^{st} one. Bubbles have become larger in absolute terms since the 1990s, but the Dotcom Bubble burst directly after the millennium has not deflated stock prices back to real-time fundamentals. Both, in terms of estimated bubble sizes, as well as in terms of the highly non-fundamental real price growth, current stock pricing in the early 21^{st} century seems to resemble those of the late 19^{th} century.

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Appendix

A Figures A.1, A.2 and Table A.1

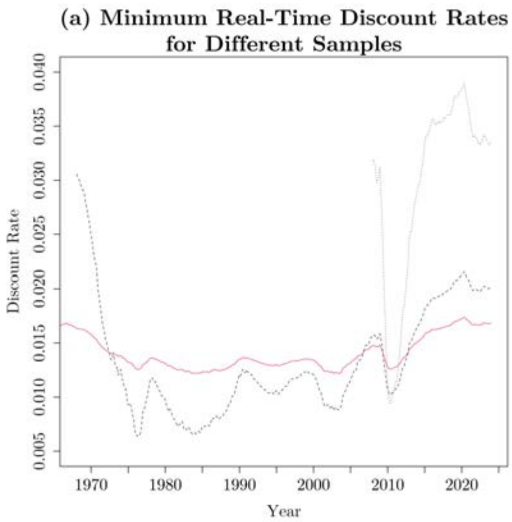

Figure A.1:

Cutout for 1968–2023: (a) Minimum real-time discount rates according to equation (10). (b) Estimates for real-time drift parameters of the exogenous process. Red solid: 1871–2023 sample, dark grey dashed: 1955–2023 sample, light grey dotted: 1995–2023 sample.

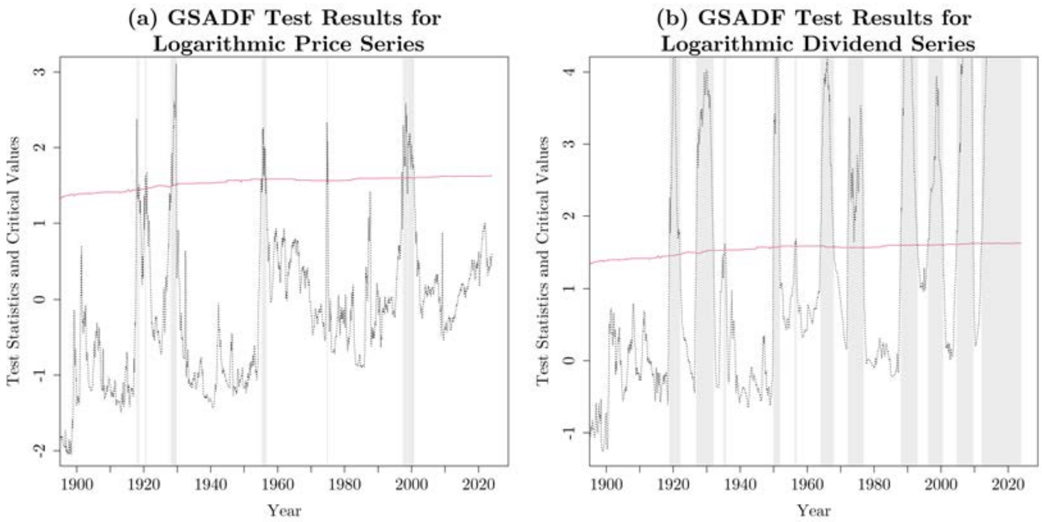

Figure A.2:

(a) GSADF test results for logarithmic real price and (b) real dividend data. Grey shaded areas denote time periods of identified exuberance. This is the case as soon as the test statistics (black dotted) exceed the critical values (red solid).

Periods of Exuberance Detected by the GSADF Test Using Real Price and Real Dividend Data

Real Price (Logarithm)				Real Dividend (Logarithm)			
Bubble Start	Bubble Peak	Bubble End	Bubble Duration	Bubble Start	Bubble Peak	Bubble End	Bubble Duration
Nov. 1917	Dec. 1917	May 1918	6	Aug. 1893	Aug. 1893	Sep. 1893	1
Jul. 1918	Sep. 1918	Oct. 1918	3	Mar. 1894	Mar. 1894	Apr. 1894	1
Feb. 1920	Feb. 1920	Mar. 1920	1	Nov. 1918	Jun. 1920	Feb. 1922	39
May 1920	Jun. 1920	Sep. 1920	4	Nov. 1926	Dec. 1929	Feb. 1932	63
Dec. 1920	Dec. 1920	Jan. 1921	1	Sep. 1934	Sep. 1934	Oct. 1934	1
Sep. 1927	Sep. 1927	Oct. 1927	1	Feb. 1935	May 1935	Oct. 1935	8
Dec. 1927	Dec. 1927	Jan. 1928	1	Nov. 1949	Dec. 1950	Nov. 1951	24
Mar. 1928	Sep. 1929	Nov. 1929	20	Mar. 1956	Sep. 1956	Oct. 1956	7
Feb. 1955	Feb. 1955	Mar. 1955	1	Dec. 1963	Jan. 1966	Dec. 1967	48
Apr. 1955	Apr. 1955	May 1955	1	Feb. 1972	Dec. 1975	Nov. 1976	57
Jun. 1955	Sep. 1955	Jun. 1956	12	Dec. 1987	Dec. 1989	Jan. 1993	61
Jul. 1956	Jul. 1956	Sept. 1956	2	Feb. 1996	Sep. 1998	Aug. 2000	54
Sep. 1974	Sep. 1974	Nov. 1974	2	Aug. 2004	Dec. 2007	Aug. 2009	60
Dec. 1974	Dec. 1974	Jan. 1975	1	Jan. 2012	Apr. 2020	Dec. 2023	144
Feb. 1997	Feb. 1997	Mar. 1997	1				
Jun. 1997	Apr. 1998	Sep. 1998	15				
Nov. 1998	Apr. 1999	Oct. 2000	23				

Table A.1:

Periods of exuberance detected by the GSADF test using logarithmic real price and logarithmic real dividend data. *Bubble start* is defined as the first point in time where the test statistic exceeds the critical value. *Bubble end* is defined as the first point in time where the test statistic falls below the critical value again. *Bubble Peak* refers to the date where the test statistic exceeds the critical value at most. *Bubble duration* refers to the nmer of monthly observations in a bubble perod with ongoing exuberance.

B Deriving Expression (11)

Using the definition of the spread s_t in (7), p_t can be stated as

$$p_t = \frac{1}{1-\alpha} + s_t.$$

Following the notion of Shiller (1981), take unconditional expectations on both sides:

$$E(p_t) = E\left(\frac{1}{1-\alpha}x_t + s_t\right)$$

In a first step, this has to be rearranged to

$$r = \frac{E(d_t)}{E(p_t) - E(s_t)},$$

such that expression (11) can be derived thereupon afterwards. To start, take (7) just expressed in expectations:

$$E(p_t) = E\left(\frac{1}{1-\alpha}x_t + s_t\right)$$

Use $x_t = \frac{d_t}{1+r}$ and $\alpha = \frac{1}{1+r}$, then:

$$\begin{aligned}
E(p_t) &= E\left(\frac{1}{1-\frac{1}{1+r}}\frac{1}{1+r}d_t + s_t\right) \\
E(p_t) &= E\left(\frac{1}{(1+r)-\frac{1}{1+r}(1+r)}d_t + s_t\right) \\
E(p_t) &= E\left(\frac{1}{(1+r)-1}d_t + s_t\right) \\
E(p_t) &= E\left(\frac{1}{r}d_t + s_t\right) \\
E(p_t) &= \frac{1}{r}E(d_t) + E(s_t) \\
E(p_t) - E(s_t) &= \frac{1}{r}E(d_t) \\
\frac{E(p_t) - E(s_t)}{E(d_t)} &= \frac{1}{r} \\
r &= \frac{E(d_t)}{E(p_t) - E(s_t)}
\end{aligned}$$

To conclude in deriving (11), derive $s_t = \frac{\alpha}{a-\alpha}E(\Delta p_{t+1}|I_t)$, which states that the spread s_t is an optimal forecast of the expected change in real price Δp_{t+1}. To do so, take (7) and rearrange for p_t again:

$$p_t = \frac{1}{1-\alpha}x_t + s_t$$

Expand with $1-\alpha$ to obtain:

$$\begin{aligned}
p_t &= \frac{x_t}{1-\alpha} + \frac{s_t(1-\alpha)}{1-\alpha} \\
p_t(1-\alpha) &= x_t + s_t(1-\alpha) \\
p_t(1-\alpha) - x_t &= s_t(1-\alpha) \\
p_t - x_t - \alpha p_t &= s_t(1-\alpha)
\end{aligned}$$

Substitute $p_t - x_t$ with $\alpha E(p_{t+1}|I_t)$ according to (2):

$$\alpha E(p_{t+1}|I_t) - \alpha p_t = s_t(1-\alpha)$$

Take expectations for p_t:

$$\alpha E(p_{t+1}|I_t) - \alpha E(p_t|I_t) = s_t(1-\alpha)$$
$$\alpha E(p_{t+1} - p_t|I_t) = s_t(1-\alpha)$$
$$\alpha E(\Delta p_{t+1}|I_t) = s_t(1-\alpha)$$
$$\frac{\alpha E(\Delta p_{t+1}|I_t)}{1-\alpha} = s_t$$
$$\frac{\alpha}{1-\alpha} E(\Delta p_{t+1}|I_t) = s_t$$

Now, plug this result into $r = \frac{E(d_t)}{E(p_t)-E(s_t)}$ obtained beforehand. Thus:

$$r = \frac{E(d_t)}{E(p_t) - E(\frac{\alpha}{1-\alpha} E(\Delta p_{t+1}|I_t))}$$
$$r = \frac{E(d_t)}{E(p_t) - \frac{\alpha}{1-\alpha} E(\Delta p_{t+1}|I_t)}$$
$$r = \frac{E(d_t)}{E(p_t) - \frac{1}{r} E(\Delta p_{t+1}|I_t)}$$
$$r(E(p_t) - \frac{1}{r} E(\Delta p_{t+1}|I_t)) = E(d_t)$$
$$rE(p_t) - E(\Delta p_{t+1}|I_t) = E(d_t)$$
$$rE(p_t) = E(d_t) + E(\Delta p_{t+1}|I_t)$$
$$r = \frac{E(d_t) + E(\Delta p_{t+1}|I_t)}{E(p_t)}$$

The result is expression (11):

$$r = \frac{E(d_t)}{E(p_t)} + \frac{E(\Delta p_{t+1}|I_t)}{E(p_t)}$$

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