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Fatma Aslan, Papatya Duman and Walter Trockel

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Fatma Aslan [*] **Papatya Duman**[†] **Walter Trockel**[‡]

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Abstract

In this article, we draw attention to some inconsistencies and conceptual chinks in the current literature on coalitional TU-games. We criticize the widespread habit of neglecting the classic problem of coalition-building and defining feasibility, efficiency, and duality for general TU-games with respect to the grand coalition. We redefine these properties using the concept of cohesiveness by versions that are meaningful for all TU-games. Based on conceptual and historical arguments we distinguish between subsets and formed coalitions and between classic TU-games and TU-game extensions. We use the *Duality Theorem of Linear Optimization* to motivate the use of cohesiveness. In an Appendix, we collect some results illustrating similarities and differences between our duality and the currently widely used $*$-duality for TU-games.

Keywords: TU-games, duality, c–Core, cohesive games, super-additivity, Pareto efficiency
JEL Classification: C71

[*]Faculty of Economic and Social Sciences, Budapest University of Technology and Economics, and Quantitative Social and Management Sciences Research Centre, Hungary.

[†]Paderborn University, Germany. Corresponding author. Email: papatya.duman@upb.de

[‡]Bielefeld University, Germany.

1 Introduction

In this article, we will present and justify a new concept of duality for general transferable utility coalitional games (TU-games, for short) resulting from a careful reconsideration of some basic concepts of coalitional TU-games.
The intention of coalitional games is to model situations where a set N of players may cooperate in a precisely specified way to achieve their goals. We presume that any set of players can form a coalition by engaging in a binding agreement. This specifies its members' cooperation in accordance with the rules embodied in the coalition function v. This function determines for each coalition its worth, i.e., its guaranteed monetary payoff. The following two fundamental questions of the theory of TU-games are already formulated in Rapoport (1970):

1. "*Which coalitions are likely to form?*"
2. "*How will the members of a coalition apportion their joint payoff?*"

As mentioned in that paper, most of the literature has been concerned with the second question by presuming that the "grand coalition" will be formed and the apportionment of the grand coalition's worth is negotiated among the players based on criteria agreed upon in the coalitional agreement. The presumption that the grand coalition will be formed is implicit in large parts of the treatments of coalitional games in the literature. Indeed, it is optimal to form the grand coalition as long as the worth of the coalition of all players is at least as large as the maximal aggregate payoff obtained by any partition of the set of players. Games with this property are called cohesive (or complete) (Osborne and Rubinstein, 1994). While for cohesive games the standard notion of efficiency taking the grand coalition's worth $v(N)$ as a benchmark is justified, it is inadequate and even defective for non-cohesive games. This fact is the main reason for our skeptical view of the use of the predominant definition of duality for all TU-games. Therefore, cohesiveness will play a crucial role in our analysis.

In the framework of classic[1] TU-games, we reconsider the concepts of feasibility and efficiency of payoff vectors together with the adequate definition of duality for TU-games. In our analysis, we will choose between three different versions of efficiency for payoff vectors that are defined and discussed, for instance, in Bejan and Gómez (2012). It will be crucial to compare carefully their impacts on those coalition functions they are applied to. In fact, it is instrumental for our definition of a dual game to be applicable and meaningful for all TU-games.

We present notations, concepts, and most of our definitions in Section 2. The most crucial preparatory topic for our expressed goal, *duality for TU-games*, will be the content of Section 3. Section 4 is the central part of our article. There, our duality concept is defined, explained, and consolidated. This section has four subsections. In Subsection 4.1, we deal with the feasibility and efficiency of payoff vectors in TU-games. In Subsection 4.2, we justify for our classic context the use of partitions as opposed to balanced families on which the use of a TU-game extension is based. In Subsection 4.3, we are dealing with the relation of our duality to reduced game properties. Subsection 4.4 discusses the close relation of the duality of TU-games with coalition production economies introduced in Sun et al. (2008), extended and simplified in Inoue (2012). Section 5 concludes with a summary and remarks on future research. Some results indicating differences or similarities of our duality with the $*$-duality in the literature are collected in the Appendix.

2 Notation and Definitions

For a non-empty finite set N, $\mathbb{R}^N$ is an $|N|$-dimensional real vector space. For notational simplicity if $x \in \mathbb{R}^N$ and $S \subseteq N$, we write $x_S = (x_i)_{i \in S}$. We identify a payoff vector x with its induced additive game that by an abuse of

[1]We sometimes use the attribute classic for TU-games in order to distinguish them from TU-games with additional imposed structures as for instance exogenous partitions of the player set N or extensions of the coalition function v allowing simultaneous (partial) membership in more than one coalition.

notation is also denoted x. Thus, $x(S) = \sum_{i \in S} x_i$. The set of all subsets of N is denoted by 2^N. We define $\mathcal{N} := 2^N \setminus \{\emptyset\}$. For any $S \subseteq N$, the indicator function $\mathbb{1}_S : N \to \mathbb{R}$ is defined by $\mathbb{1}_S(i) = 1$ for all $i \in S$, otherwise 0. A *partition* π of $S \subseteq N$ is a set $\pi = \{T_1, T_2, \ldots T_m\}$ of pairwise disjoint subsets $T_i \subseteq S$ covering S. The set of partitions of $S \subseteq N$ is denoted $\Pi(S)$. A collection ß of nonempty subsets S is called a *balanced family* if there is a map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}$ satisfying $\sum_{S \in ß} \lambda(S) \mathbb{1}_S(i) = 1$, where $\lambda(S) > 0$ for all $S \in ß$. We denote the set of balanced families by $\mathcal{B}$.

A cooperative coalitional game with transferable utility, for short TU-game, is an ordered pair (N, v) where $N = \{1, 2, \ldots n\}$ represents the set of players and $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$. The coalition function v is completely described by its restriction to $2^{\mathcal{N}}$. A *subgame* (S, v_S) of (N, v) is defined by $v_S := v|_{2^S}$.

By focusing on non-cohesive games we are leaving the classes of super-additive and balanced games, and loose access to concepts and solutions developed for these classes. However, we need to define those classes of games which are excluded from our analysis when focusing on non-cohesive games, where the grand coalition fails to be efficient.

A TU-game (N, v) is *cohesive* (or *complete*) if $v(N) = \max_{\pi \in \Pi(N)} \sum_{T \in \pi} v(T)$. The *cohesive hull* or *completion* of a TU-game (N, v) is the TU-game (N, v^c) defined by $v^c(S) = v(S)$ for all $S \subset N$ and $v^c(N) = \max_{\pi \in \Pi(N)} \sum_{T \in \pi} v(T)$. The TU-game (N, v) is super-additive if $v(S) + v(T) \leq v(S \cup T)$ for all disjoint $S, T \subseteq N$. The super-additive hull of a TU-game (N, v), denoted by $(N, \tilde{v})$, is the smallest super-additive game (N, w) such that $v(S) \leq w(S)$ for all $S \subseteq N$. Super-additive games are *totally cohesive* in the sense that all its sub-games are cohesive. For all $S \subseteq N$ holds $\tilde{v}(S) = v_S^c(S) = \max_{\pi \in \Pi(S)} \sum_{T \in \pi} v(T)$, and thus $\tilde{v}(N) = v^c(N) = \max_{\pi \in \Pi(N)} \sum_{T \in \pi} v(T)$. A TU-game (N, v) is *balanced* if for each balanced family ß of subsets of N and associated system $\lambda_ß = \{\lambda(S)\}_{S \in ß}$ holds: $v(N) \geq \sum_{S \in ß} v(S) \lambda(S)$. The *balanced hull* of (N, v) is the TU-game (N, v^b) with $v^b(S) = v(S)$ for all $S \subset N$ and $v^b(N) := \max_{ß \in \mathcal{B}} \sum_{T \in ß} \lambda(T) v(T)$.

A TU-game (N, v) is *totally balanced* if all subgames (S, v_S) are balanced. Neither need balanced games to be super-additive nor super-additive games to be balanced. Both are cohesive, however. For an arbitrary TU-game (N, v) holds $v(N) \leq v^c(N) = \tilde{v}(N) \leq v^b(N)$.
The concept of the core is introduced by Gillies (1959) via a domination relation for any arbitrary set P of payoff vectors for a game (N, v). Taking P as either $\{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$ or as $\{x \in \mathbb{R}^N \mid x(N) \leq v^c(N)\}$ one receives $Core(N, v)$ and $c - Core(N, v)$, respectively. The core is defined as $Core(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}$, and the *c-core* of a TU-game (N, v) is the core of its cohesive hull, i.e., $c - Core(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v^c(N),\ x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A TU-game (N, v) is called *c-balanced* game if its cohesive hull (N, v^c) is a balanced game. It is very well-known that the (c-)core of a TU-game (N, v) is non-empty if and only if (N, v) is (c-)balanced.

3 Duality for TU-games

The first definition of a dual game we are aware of is given in Rapoport (1970) for the very restricted domain of simple games. That duality is consistent with the current standard definition as for all S and $N \setminus S$ the worth there can only be 0 or 1 and $v(S) = 0$ if and only if $v(N \setminus S) = 1$. However, transfers of payoffs or adding up are not meaningful operations in simple games as payoffs are just non-cardinal descriptions of not transferable winning or losing. Aumann and Maschler (1985) defines duality for bankruptcy problems which are not TU-games such that given a bankruptcy problem and its dual, both can be in a unique specified way represented by TU-games, and the game representing the dual problem is "the dual game" of the game representing the original bankruptcy problem. Kalai and Samet (1987) is to our knowledge the first to define duals of arbitrary TU-games (N, v) as follows:

Definition 1. *The dual of a TU-game (N, v) is denoted by (N, v^*) where $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.*

Tadenuma (1990) defines the dual of a TU-game (N, v) as $(N, -v)$. Funaki (1998) criticizes that and adopts the definition of the dual game v^* from Kalai and Samet (1987) as in Definition 1.[2] Deviating from the literature, we restrict the domain of this duality definition to the class of super-additive games. Later in this work, we shall give a definition of duality for all TU-games that coincides with the duality in Definition 1 (hereafter $*$-*duality*) on the class of super-additive games.

Consider the interpretations of dual games in the literature: Weber (1994) : "As $v(S)$ represents the payoff that can be "claimed" by the coalition S, so $v'(S)(= v^*(S))$ represents the share of the payoff which can be generated by the coalition N which cannot be claimed by the complement of S." Peleg and Sudhölter (2007): "$v^*(S)$ describes the amount which can be given to S , if the complement $N \setminus S$ receives what it can reach by cooperation. The complement $N \setminus S$ cannot prevent S from [receiving] the amount $v^*(S)$." The interpretations of Oishi and Nakayama (2009) and Oishi et al. (2016) coincide with the ones of Weber (1994) and Peleg and Sudhölter (2007).

According to the interpretations above, $v^*(S)$ describes what remains available as a possible payoff for a coalition S, provided that $N \setminus S$ receives its worth $v(N \setminus S)$. As $v(S)$ is guaranteed to S by the coalitional contract to participate in (N, v), we must have $v^*(S) \geq v(S)$. Hence, $v^*(S) - v(S)$ measures the Pareto optimality gap (opportunity costs) caused by not using $v(N) - v(N \setminus S)$ as the payoff to S. Whenever v is cohesive, such gap is non-negative. Therefore, v^* is compatible with the interpretation of duality reflected in the above quotations.

What we can distill from the explanation and interpretation of the meaning of TU-games and their duals in the current literature is shaped by the following

[2] A different approach to duality of TU-games is proposed by Martínez-Legaz (1996). He considers for any TU-game an informationally equivalent representation by a non-increasing polyhedral convex function, the *indirect function*. That is related to the characteristic function via Fenchel-Moreau generalized conjugation theory. However, here the dual of a TU-game is not a TU-game associated with this game, but rather a dual (non-TU-game) representation of the original TU-game. We are not aware of an attempt in this literature to check for any potential relations between those different duality concepts.

features:

1. The grand coalition is built, and its worth is the benchmark for the efficiency of payoff vectors.
2. The $*$-duality is defined for all TU-games ignoring the conflict with the interpretations in non-cohesive games.

As a consequence of these deficiencies of $*$-duality, we will define a new duality concept in Section 4. The crucial difference will be that we skip the assumption "the grand coalition is built" and define the efficiency of payoffs via maximizing the members' aggregate worths over all partitions of N. This leads to a key role of the notion of cohesiveness.

4 A new definition of duality for TU-games

The perhaps most troublesome deficiency of the current theory of coalitional games is in the extension of the characteristic functions to coalition functions. The characteristic function is defined by von Neumann and Morgenstern (1944) based on a non-cooperative game, and the construction of the concept leads the characteristic function to be super-additive. That is due to the fact that in strategic games the coordinated choice of strategies in groups of players is "super-additive". Hence, for von Neumann-Morgenstern characteristic functions the natural benchmark for feasibility and efficiency is $v(N)$. The importance of permitting all coalition structures rather than just $\{N\}$ had been recognized and emphasized by several game theorists, for instance, Neuefeind (1974), Böhm (1974), and Guesnerie and Oddou (1979), yet without a strong impact on the literature. We shall focus on the consequences of skipping super-additivity for the current $*$-duality which is defined for any TU-game (N, v), but is only compatible with the interpretation prevalent in the literature if v is super-additive.[3]

[3]Peleg and Sudhölter (2007) considers games with coalition structure and abandons thereby the benchmark feature of the "grand coalition". But its coalition structure is

Before giving our definition of duality for all classic TU-games, we first will be confronted with terminological vaguenesses about coalitions. Although the term coalition is the central notion in defining coalitional games, a large fraction of the literature just uses it as a synonym for a subset of the players in a TU-game (N, v). A coalition as just a subset makes "forming a coalition" meaningless. We need to consider players' pre-play negotiations anteceding their decisions about which subsets of N commit themselves to participate as formed coalitions in (N, v). Hence, we need a coherent terminology for denoting player sets. This problem is discussed in a detailed and transparent manner in Rapoport (1970), based on Harsanyi (1959). The merit of Rapoport (1970) is the explicit distinction of mere subsets of players, i.e., syndicates, from subsets of players loaded with certain contractual commitments of its members, i.e., coalitions. We shall adopt this terminology in the rest of this article.[4] We call a syndicate S a *cartel* if its members, in order to maximize their aggregate payoffs, cooperate with each other via partitioning into subsets that become coalitions by entering the game. Such a cartel would receive an aggregate payoff $\tilde{v}(S) > v(S)$.

In a non-super-additive game (N, v), there must be a part of legal or physical constraints embodied in v that prevents some coalitions S from splitting into subcoalitions and thereby prevents a higher total payoff $\tilde{v}(S)$. However, if a syndicate S as a coalition would have a worth $v(S) < \tilde{v}(S)$, it may as a cartel partition into subsets, entering the game as coalitions committed to share their aggregate worth $\tilde{v}(S)$ in a specific way. This example already gives an indication of why even for non-super-additive games, the super-additive hull will enter into our definition of duality. With $S = N$ in the example above, the syndicate N could not achieve more than $v(N)$ as the grand coalition, but it could achieve $v^c(N) = \tilde{v}(N)$ as an efficiently partitioned cartel.

As in TU-games cohesiveness and hence super-additivity are not parts of the definition, one may have $v(S) + v(N \setminus S) > v(N)$ for some coalition S.

exogenously given, and its efficiency is not an issue. Moreover, duality in this modified framework is not discussed or only defined.

[4]Our concern with the concept of a coalition disregards TU-game extensions and TU-games with an exogenously given coalition structure.

In this case, the grand coalition is unable to afford the total payment, in contradiction to the generally accepted interpretation of the worth $v(S)$ as the guaranteed worth. So who is providing the guaranteed amounts $v(S)$ and $v(N \setminus S)$? The logical consequence is either to restrict the class of TU-games to cohesive ones or to abandon the benchmark role of the grand coalition's worth $v(N)$. We will follow the second way.

Definition 2. *Let (N, v) be a TU-game. Its dual game (N, v^d) is defined by $v^d(S) = \tilde{v}(N) - \tilde{v}(N \setminus S)$ for all subsets S of N.*

Remark 1. *Our definition of duality reflects the idea that the players of any syndicate S that is planning to cooperate in the game (N, v) do so only if $v(S) = (v_S)^c(S)(= \tilde{v}(S))$. The worth $v(S)$ of a syndicate is an assured payoff contingent on its members having formed a coalition.*

The following two subsections will be concerned with explaining and motivating our definition of duality for all classic TU-games.

4.1 Feasibility and efficiency

The standard version of *feasibility* and *efficiency* for TU-games is based on the realization of the grand coalition. Depending on which of feasibility and efficiency concepts is used different types of payoff vectors are considered in the literature. Bejan and Gómez (2012) discusses three versions of feasibility defined by the following sets:

$$\begin{aligned} X(N, v) &= \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\} \\ X_\Pi(N, v) &= \{x \in \mathbb{R}^N \mid x(N) \leq v^c(N)\} \\ X_\Lambda(N, v) &= \{x \in \mathbb{R}^N \mid x(N) \leq v^b(N)\} \end{aligned}$$

The respective subsets of *efficient* payoff vectors x are defined by replacing the inequality by equality.

As we want to get rid of the restriction by the worth of the grand coalition, we refute the first version. We refute the third version because it admits partial memberships of players in various "coalitions" and thus, goes beyond the scope of our classic framework. Hence, we will work with the second one.

Definition 3. *A payoff vector x for a TU-game (N, v) is called feasible if x is an element of $X_{\Pi}(N, v)$ and efficient if x is an element of $X^*_{\Pi}(N, v) :=$ $\underset{x \in X_{\Pi}(N,v)}{\operatorname{argmax}}\ x(N)$.*

We shall prove later in Subsection 4.2 that the worth $v^c(N)$ results via maximization in a game (N, v) without assuming that this game is cohesive.

As a first step, we generalize feasibility by basing it on partitions of the player set N maximizing their members' aggregate worth rather than on the worth of the coalition N. Our modified ("first-best") efficiency degrades the current standard version to "second-best" efficiency, which is first-best only for cohesive games with $v(N) = v^c(N) = \tilde{v}(N)$.

The second step is to mimic the definition of $*-$duality for this more general framework as far as the grand coalition is concerned. That would result in a duality where $v(N) - v(N \setminus S)$ is replaced by $v^c(N) - v(N \setminus S)$.

The third and last step is very subtle and is based on the rationality and efficient behavior of players in their preplay coalition-building negotiations. Every syndicate S forms a coalition only if the guaranteed worth $v(S)$ equals $(v_S)^c(S) = \tilde{v}(S)$. Moreover, every coalition S expects the same rational behavior of its opponents in the syndicate $N \setminus S$. Therefore, $N \setminus S$ will be optimally partitioned and thus, legally receive $\tilde{v}(N \setminus S)$ in the aggregate.

These three steps convert for every TU-game (N, v) the term $v^*(S) = v(N) - v(N \setminus S)$ into $v^c(N) - (v_{N \setminus S})^c(N \setminus S) = \tilde{v}(N) - \tilde{v}(N \setminus S) = v^d(S)$.

It is important to understand that we **do not** assume that the game (N, v) is super-additive or even cohesive. These properties arise from the efficiency of pre-play negotiations prompting players in a syndicate S to form the coalition S only if $v(S) = (v_S)^c(S) = \tilde{v}(S)$. Not every coalition T in (N, v) needs to satisfy $v(T) = \tilde{v}(T)$. Yet, whenever a game (N, v) is super-additive, we get $v^d = (\tilde{v})^* = v^*$.

Lemma 1. *For every super-additive TU-game (N, v), one has $v^d = v^*$.*

Therefore, we have three related concepts of efficiency: as a property of syndicates qualifying them for forming coalitions, as a property of vectors maximizing total payoff and as a property of partitions to be effective with their aggregate worth for an efficient payoff vector.

4.2 Why partitions rather than balanced families?

Within the classic framework of TU-games, the membership is an indivisible feature of each agent. In the context of balanced families, the players are allowed to be members of several coalitions simultaneously, and a coalition's worth is a function mapping degrees of membership to shares of a coalition's worth in its classical meaning. Though it is a coherent concept, it is not a part of the classic problem. In order to formalize it, one needs to extend the notion of a game from (N, v) to (N, w) with $w : 2^N \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $w(S, t) := tv(S)$ for any $(S, t) \in 2^N \times \mathbb{R}_+$. Depending on the socio-economic scenario represented by (N, v), this may or may not be a coherent way of defining an extended TU-game.[5] On the other hand, as membership is indivisible and the worth of a coalition is a real number, partitions are the best fit in our classic framework.

Partial memberships in various coalitions can be financed by the players of N when $x(N) = v^b(N)$. If this is the case, then the secured payoff to be allocated to each syndicate T is at least as large as its entitled worth $v(T)$. This fact is known via the *Duality Theorem of Linear Optimization* based on the following optimization problems which are taken from Myerson (1991).

$$(9.2)\ \min_{x \in \mathbb{R}^N} x(N) \text{ subject to for all } S \in \mathcal{N},\ x(S) \geq v(S)$$

$$(9.3)\ \max_{\mu \in \mathbb{R}_+^{\mathcal{N}}} \sum_{S \subseteq N} \mu(S) v(S) \text{ subject to for all } i \in N,\ \sum_{S \subseteq N : i \in S} \mu(S) = 1$$

But this requirement $x(N) = v^b(N)$ is too restrictive for non-c-balanced games as it leads to the aggregate payoff $v^b(N)$ which cannot be produced by a non-c-balanced game. In order to get a smaller "budget" available for the payoffs, we guarantee only the worth of those syndicates who take part in the game (N, v) as coalitions. In other words, in order to reach a smaller value as the optimal value, we want first to weaken the restrictions in (9.2) that for all syndicates S holds $x(S) \geq v(S)$. Hence, we use a restriction

[5] Shapley and Shubik (1975) explains why balancedness of a TU-game (N, v) requires a continuous, concave, positive linear-homogeneous extension of v.

for our primal linear program (P) to the effect that the feasible set becomes larger and the minimum becomes smaller.

We denote the set of partitions containing a syndicate S as a component by $\Pi_S(N) = \{\pi \in \Pi(N) \mid S \in \pi\}$ whose elements are denoted by π_S. Define $v(\pi_S) := \sum_{T \in \pi_S} v(T)$. For any syndicate S, we define $\Pi_S^*(N) = \operatorname{argmax}_{\pi_S \in \Pi_S(N)} v(\pi_S)$ and denote its elements by π_S^*.

Take any $\pi_S^* \in \Pi_S^*(N)$ for a coalition S with $v(\pi_S^*) = v^c(N)$. For those efficient partitions, there should be enough payoffs to be allocated only to the coalitions generating those partitions. To do so (P) is designed as an analogue of (9.2) to guarantee the payoffs to the coalitions, but not to all syndicates. (D) is the dual maximization problem of (P).

$$\text{(P)} \min_{x \in \mathbb{R}^N} x(N) \text{ subject to for all } S \in \mathcal{N},\ x(N) \geq v(\pi_S^*)$$

$$\text{(D)} \max_{y \in \mathbb{R}_+^{\mathcal{N}}} \sum_{S \subseteq N} y(S) v(\pi_S^*) \text{ subject to } \sum_{S \subseteq N} y(S) = 1$$

Let x^* and y^* be optimizers of these dual problems, respectively. By the Duality Theorem of Linear Optimization, the joint optimal value for both programs is $x^*(N) = v^c(N)$. The optimizers y^* in problem (D) are those probabilities y^* that put positive mass only on S with $v(\pi_S^*) = v^c(N)$. Among them are all Dirac measures $\delta_S = \mathbb{1}_{\{S\}}$ with $y^*(S) = \delta_S(S) = \mathbb{1}_{\{S\}}(S) = 1$. Thus, all partitions $\pi_S^* \in \Pi_S^*(N)$ satisfy $v(\pi_S^*) = \tilde{v}(N) = v^c(N) = x^*(N)$. This completes the proof of the following analogue of the relation between the dual programs (9.2), (9.3) and the balanced hull v^b.

Theorem 1. *The joint optimal value of the dual linear programs (P) and (D) is the worth of the grand coalition N in the cohesive hull v^c.*

While switching from the grand coalition efficieny to the Pareto efficiency, choosing partitions as the coalition structure was not a preference, but it is inevitable.

Obviously, any optimal x^* is Pareto efficient in (N, v). We call all optimal partitions of N resulting from (P) and (D) *efficient partition* in (N, v), i.e., a partition of N is *efficient* if its aggregate worth is $v^c(N)$. A Pareto efficient vector x is *partition efficient* if $x(S) = v(S) = \tilde{v}(S)$ for all components of some efficient partition of N. Although the partition efficient vectors x of (N, v) are not aspirations[6], they are distinguished anticipations.

Definition 4. *For a TU-game (N, v), we define the set $PEA(N, v)$ is the set of anticipations which are partition efficient. Formally,*

$$PEA(N, v) = \{x \in \mathbb{R}^N \mid \textit{there exists } \pi^* \textit{ s.t. } x(S) = \tilde{v}(S) \textit{ for all } S \in \pi^*\}$$

Remark 2. *It is a consequence of Theorem 1 that for any TU-game (N, v), the set $PEA(N, v)$ is non-empty.*

Being a vector in $PEA(N, v)$ does not mean that every subset T of any component of any efficient partition receives $x(T) \geq v(T)$. To be able to achieve that via an internal reallocation of $x(T)$, we would need $v^c(N) = v^b(N)$, i.e., (N, v) is c-balanced. This implies that for any game, $PEA(N, v)$ is a non-empty super set of $c - Core(N, v)$.
Sacrificing the concept of balanced families means that we need to find some kind of substitute for the core and the c-core for the large class of non-c-balanced TU-games, on which the (c-)core, one of the most important concepts in cooperative game theory, is empty. Such a core surrogate deserves investigations, which we leave to future research.

4.3 The relation between our duality and reduced games

In this section, we introduce some versions of reduced games, discuss their interpretations, and explain their relation to duality. This will, in particular, underline appropriateness of this duality in situations where $*-$duality fails.

[6]An *aspiration* is a special kind of *anticipation*, which is a payoff vector x such that for all $i \in N$, there exists some $S \subseteq N$ containing i with $x(S) \leq v(S)$. An *aspiration* $x \in \mathbb{R}^N$ for a TU-game (N, v) is an anticipation such that $x(S) \geq v(S)$ for each syndicate $S \subseteq N$ [cf. Bennett (1983)].

Definition 5. *Let (N, v) be a TU-game, $S \subseteq N$ and $x \in X_\Pi(N, v)$. The reduced game with respect to S and x is the game (S, v_S^x) with*

$$v_S^x(T) = \begin{cases} v^c(N) - x(N \setminus S) & \text{if} \quad T = S, \\ \max\limits_{Q \subseteq N \setminus S} (v^c(T \cup Q) - x(Q)) & \text{if} \quad T \subset S. \end{cases}$$

If (N, v) is cohesive (or even super-additive or balanced), then $v^c(N) = v(N)$ and $X_\Pi(N, v) = X(N, v)$. Hence, our definition of v_S^x coincides with that of v_S, x defined by Davis and Maschler (1965) for arbitrary (N, v) and $x \in X(N, v)$. This means that our definition extends the concept of a reduced game to the class of non-cohesive games while remaining that of Davis and Maschler (1965) on cohesive games.

Peleg and Sudhölter (2007) offers an interpretation of a reduced game as defined by Davis and Maschler (1965):

"*Assume that all members of N agree that members of $N \setminus S$ will get $x(N \setminus S)$. Then, the members of S may get $v(N) - x(N \setminus S)$. Furthermore, suppose that the members of $N \setminus S$ continue to cooperate with the members of S (subject to the foregoing agreement). Then, for every non-empty $T \subset S$, the amount $v_{S,x}(T)$ is the (maximal) total payoff that the coalition T expects to get.*"

The terminology used in this quoted passage would allow to distinguish coalitions from syndicates, but it is not clear how this interpretation, and therefore the above definition fits in our framework.

We already argued that in the simultaneous pre-play negotiations by all players, rational players will form coalitions efficiently. The assumption that "all members of N agree that members of $N \setminus S$ will get $x(N \setminus S)$" is stated without any explanation under which circumstances it is adequate. This assumption is compatible with the rationality and the efficiency of players' decisions, hence makes sense, only if $x(N \setminus S) \geq \tilde{v}(N \setminus S)$. The coalition $N \setminus S$ would not be able to enforce more and not accept less than $\tilde{v}(N \setminus S)$. So only $x(N \setminus S) = \tilde{v}(N \setminus S)$ is an adequate candidate for the above general agreement on $x(N \setminus S)$. As the same holds for $x(S) \geq \tilde{v}(S)$, we get $x(N) \geq \tilde{v}(S) + \tilde{v}(N \setminus S)$ implying that (N, v) must be a cohesive game.

Lemma 2. *On cohesive games, if $x(N \setminus S) = \tilde{v}(N \setminus S)$, we get $v_{S,x}(S) =$*

$v(N) - x(N \setminus S) = \tilde{v}(N) - \tilde{v}(N \setminus S) = v^d(S)$.

Our definition of a dual game on cohesive games is consistent with associating any coalition (as distinguished from syndicate) with the worth $v_{S,x}(S)$, i.e., the (maximal) total payoff that the coalition S expects to get.
Our definition implies that even in non-cohesive games the consistency of duality and the reduced game is obtained. Hence, the (maximal) total payoff that the coalition S expects to get is exactly the amount that the syndicate $N \setminus S$ cannot prevent the coalition S from receiving. This reveals some "dual" relation between the dual and the cohesive hull of a TU-game with two different concepts of efficiency based on maximization and on minimization, respectively: for all S, $v^d(S) = \tilde{v}(N) - \tilde{v}(N \setminus S) \geq \tilde{v}(S)$.
The term $v^d(S) - v(S)$ describes the residual amount that the coalition S might receive if $N \setminus S$ receives its maximal legally guaranteed payoff $\tilde{v}(N \setminus S)$, then the allocation $(v^d(S), \tilde{v}(N \setminus S))$ to S and $N \setminus S$ is efficient. Hence, $v^d(S)$ and $v^d(N \setminus S)$ can be also seen as the efficiency gaps that would need to be paid to S or $N \setminus S$, if S or $N \setminus S$ would be already determined to get $\tilde{v}(S)$ or $\tilde{v}(N \setminus S)$, respectively.
The allocation $(v^d(S), v^d(N \setminus S))$ is not feasible in a TU-game (N, v) unless there exists an optimal partition containing S, so satisfying $\tilde{v}(S) + \tilde{v}(N \setminus S) = \tilde{v}(N)$. For such "efficient" S and $N \setminus S$, we get again $\tilde{v}(N \setminus S) = \tilde{v}(N) - \tilde{v}(S) = v^d(N \setminus S)$ and $\tilde{v}(S) = \tilde{v}(N) - \tilde{v}(N \setminus S) = v^d(S)$. This means that neither S nor $N \setminus S$ is suffering from forgone opportunities, since both of the residual efficiency gaps are zero. On the other hand, we know that for arbitrary S, we have $v^d(S) + v^d(N \setminus S) = \tilde{v}(N) - \tilde{v}(S) + \tilde{v}(N) - \tilde{v}(N \setminus S) = 2\tilde{v}(N) - (\tilde{v}(S) + \tilde{v}(N \setminus S)) \geq \tilde{v}(N)$.
What we have received is the following duality:

Lemma 3.

$$\min_{S \in \mathcal{N}} (v^d(S) + v^d(N \setminus S)) = \max_{S \in \mathcal{N}} (\tilde{v}(S) + \tilde{v}(N \setminus S)) = v^c(N) = \tilde{v}(N).$$

Generally, for all partitions π, we have the relation $v^d(\pi) - v(\pi) \geq 0$. An optimal coalition structure reduces this efficiency gap to zero.

4.4 Duality of TU-markets and market representations

Based on the earlier works of Shapley (1953) and Shubik (1959), *TU-market games* are introduced in Shapley and Shubik (1969) and characterized as being identical with totally balanced TU-games. In their follow-up on market games, Shapley and Shubik (1975) analyses the relation between payoff vectors in the core of a TU-game and their representation of competitive equilibria[7] in the markets inducing this game. Later on, the class of TU-market games is extended to include other games induced by or representing various market models or economies. Garratt and Qin (1996, 1997, 2000) introduce market models that can be represented by super-additive games [see also Bejan and Gómez (2017)]. Sun et al. (2008) defines a coalition production economy that is induced by an arbitrary TU-game, for which the equivalence between the c-core and the utility allocation of competitive equilibria of the induced economy is established. Inoue (2012) simplifies this model by reducing the number of output commodities to just one ("money") allowing two versions, one with divisible and one with indivisible labor input of agents. These two models allow or exclude, respectively, multiple jobbing of agents, i.e., allocating their available labor time to several coalitions. This gives players the opportunity to organize themselves in balanced families of syndicates rather than in partitions into various coalitions. Therefore, only the "single jobbing" model of Inoue (2012) is adequate for representing the TU-games in our framework.

While the explanation of coalition-building via competitive equilibria of a represented coalition production economy is only possible if the game is at least c-balanced [cf. Sun et al. (2008)], it is still possible for arbitrary TU-games via efficiency properties. The efficient self-organization of the players into coalitions that are active and efficient in the production process is compatible with our framework that distinguishes syndicates from coalitions, efficient coalitions from inefficient ones, and efficient from inefficient coalition

[7] A *competitive equilibrium* (or *Walrasian equilibrium*) is a pair (p, X), where p is a vector of commodity prices and X a matrix whose columns represent commodity bundles in the agents' excess demand sets at p such that markets clear, i.e., $pX = 0$.

structures. We will focus now on the single jobbing model. For an elaborate interpretation and technical details, we refer to the short article by Inoue (2012).

Like in Shapley-Shubik markets, each player of the representing TU-game is endowed with one indivisible unit of an idiosyncratic good, her "labor time". Each coalition S has a technology by which a coalition S can produce $tv(S)$ for $t \geq 0$ if each member of S provides t of its input good labor. Indivisibility implies for each player, t to be either 0 or 1. So, the coalition is built if its production set is activated by all members of S providing their full endowments.

This is also the case if the considered game is a market game. As Shapley and Shubik (1975) remarks, the competitive payoff vectors of (N, v) are exactly those maximizing the players' aggregate utility that equals $v^c(N) = \tilde{v}(N)$. If the "right" coalitions build, they form a partition π of N that produces in total the amount $\sum_{T \in \pi} v(T) = v^c(N) = \tilde{v}(N)$. For any TU-game (N, v), there exists such an economy $\mathcal{E}_v$[8].[9]

By Inoue (2012), we know that any TU-game can be generated by $\mathcal{E}_v$, but we do not know what we can say beyond the efficiency if the game is not a c-balanced game. Proposition 1 gives a clue about efficiency in non-c-balanced games, whose straightforward proof is left to the reader.

Proposition 1. *The maximal output of $\mathcal{E}_v$ can be produced only by any vector $x \in PEA(N, v)$ and equals $x(N) = v^c(N) = \tilde{v}(N)$. All active firms are components of the same efficient partition of N underlying that $x \in PEA(N, v)$.*

As Inoue's representation of v by $\mathcal{E}_v$ holds for any TU-game (N, v), it holds in particular for (N, v^d). One may suspect therefore that the economy $\mathcal{E}_{v^d}$ would admit an interpretation as an in some sense dual economy $\mathcal{E}_v^d$. But

[8] A *coalition production economy* $\mathcal{E}$ with n agents is described by a commodity space, the agents' characteristics and the coalitions' production sets satisfying some desired properties. We denote the economy generating the TU-game (N, v) by $\mathcal{E}_v$. For the details please see Inoue (2012).

[9] Notice that several coalition structures $\pi \in \Pi(N)$ may be able to produce the aggregate amount $v^c(N)$.

that is not the case. We know that $v^d(S) = \tilde{v}(N) - \tilde{v}(N \setminus S) \geq \tilde{v}(S)$ for all S with equality for $S = N$ implying that efficiency for (N, v^d) means minimal rather than maximal. But the Inoue economy $\mathcal{E}_{v^d}$ would consider coalition structures in (N, v^d) that maximize the worth of partitions of N.

Even without employing some specific economic model different from Inoue (2012), we may interpret the dual game (N, v^d) under the aspect of economic systems. Two seminal articles Debreu (1951, 1954) introduce and apply the concept of a coefficient of resource utilization ρ that measures "*the dead loss associated with a nonoptimal situation (in the Pareto sense) of an economic system.*" There the fact is underlined that ρ "takes into account three kinds of loss: i) underemployment of physical resources, ii) inefficiency in production, iii) imperfection of economic organization". If we apply that to TU-games as a highly aggregate and simplified economic systems, then when any partition of N where all components "produce efficiently" in the sense that they build coalitions with $v(S) = \tilde{v}(S)$ and where by assumption physical resources (available labor time) are fully employed, there remains only imperfection of organization. This organization is the pre-play selection of all players of a coalition structure they are going to create by their choices of which coalition they will be founding members of. So, the coefficient ρ may be applied to the games (N, v) and (N, v^d). The coefficients ρ_v and ρ_{v^d} are then defined as $\rho_v(N, v) = v(\pi)/v(\pi^*) \leq 1$ and $\rho_{v^d}(N, v^d) = v^d(\pi^*)/v^d(\pi) \leq 1$ with equality for $\pi = \pi^*$, the case of efficiency in both games (N, v) and (N, v^d) and $v(\pi^*) = v^d(\pi^*) = v^d(N) = \tilde{v}(N)$.

5 Concluding Remarks

We have introduced a new concept of duality for general TU-games that is not based on the *second-best efficiency* defined via the grand coalition as the reference point, but rather on Pareto efficiency with the most productive coalition structure(s) as a benchmark. These two concepts coincide on the class of super-additive games, where the grand coalition builds a most productive (singleton) partition. The prevalent interpretation of the duality, which is meaningfully for super-additive games, sometimes leads to logical inconsistency in non-cohesive games but can be maintained with our duality concept. We have presented various different strands of arguments for the use of our new concept and provide results paralleling or deviating from the ones found in the duality literature, mainly due to Oishi and Nakayama (2009) and Oishi et al. (2016). The cohesive hull of a TU-game, the key concept of our definition of duality, admits to cover also balanced and totally balanced games, although in a strict sense these are linear homogeneous extensions of coalitional TU-games.

Still, our much more general concept does not satisfactorily cover **all** games in **all** of its interpretations. Like $*-$duality, our duality also implicitly interprets worth as utility, gain, revenue, profit, or saving rather than as dis-utility, loss, cost, or debt. Both duality concepts formalize kinds of *upper duality* by basing the efficiency concept on *maximizing worth*, where *worth* suggests something positive and desired like *utility* as opposed to *disutility*. There would not be any problem if TU-games would be functions mapping all non-empty sets of players always only into the non-negative or only into the non-positive real numbers. Each game could then be interpreted and analyzed either as a *gain game* or a *loss game*. The analogy to maximization problems which always may be expressed as minimization problems would be obvious. This might have been behind the duality concept of Tadenuma (1990). Also, the definition of the anti-core defined, for instance, in Oishi et al. (2016) by just reverting the inequalities in the definition of the core seems to hint in this direction. But at the same time, it demonstrates the confusion in the TU-literature when it comes to signs or interpretations of games. Maschler et al. (2013) denote both the core and the anti-core by the same name "core" despite the reversal of inequalities. After all, the overwhelming majority of TU-games is neither inherently classified as profit games nor as cost games and does not give a clear indication of whether efficiency has to be approached via maximization or minimization. Even if these approaches are opposite, both may be meaningful based on some upper or lower duality. It is neither the sign ($-$ versus $+$) nor the notation (c versus v) that characterizes TU-games as cost games. As any game is strategically equivalent to positive

games and to negative games, any cost game can be represented as positive and any "ordinary" game as negative. It is just the interpretation of a game, i.e., of its payoffs as utilities or revenues as opposed to dis-utilities or costs that determines, what efficient payoff vectors are.
A sound analysis of options for syndicates in pre-play negotiations presumes their agreement on the interpretation of efficiency, hence the choice between maximizing and minimizing the aggregate worth. A common understanding among players of what efficiency in a game means is necessary for potential cooperation. The focus on players' agreement on efficiency suggests naturally the idea of defining and analyzing also a "lower duality". Based on efficiency rather than on maximization of gains or minimization of costs, one may get such a universally applicable duality notion for all TU-games, embodying upper and lower duality. In this context, we would like to hint at the commendable article by Derks et al. (2014). Although they are not directly concerned with duality and the grand coalition still serves as a benchmark, their analysis may be seminal for further work on duality as their solution concepts cover the core as well as the anti-core. It may be worthwhile to try to build on their ideas but by substituting $v^c(N)$ for $v(N)$, partitions for balanced families, and Pareto efficiency for the currently still predominant second-best efficiency.

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Appendix

Some Results on Duality

A TU-game (N, v) is called *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$, and is called *concave* if the inequalities are reversed.

Proposition 2. *If (N,v) is convex, then (N,v^d) is concave.*

Proof. As convexity implies super-additivity, $v^d(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$ by Lemma 1.
Let $S, T \subseteq N$ be arbitrary. Then, $v^d(S)+v^d(T) = v(N)-v(N\setminus S)+v(N)-v(N \setminus T) = 2v(N) - (v(N \setminus S) + v(N \setminus T)) \geq 2v(N) - (v(N \setminus (S \cup T)) + v(N \setminus (S \cap T))) = v^d(S \cup T) + v^d(S \cap T)$ where the inequality is obtained by the convexity of (N,v).

Oishi and Nakayama (2009) introduces *anti-dual* of a TU-game, and then Oishi et al. (2016) uses it to define an operator that associates with each solution on a given domain a possibly different solution on the same domain. To do the same, we adopt the definitions to our duality concept: the *anti-dual* (or *negative-dual*) game of a TU-game (N,v) as the game (N,v^{ad}) with $v^{ad} = -v^d$. given a solution ϕ on a class V of games, the *anti-dual* of ϕ, denoted ϕ^{ad}, is defined by setting for all $v \in V$, $\phi^{ad}(v) = -\phi(-v^d)$. Additionally, a solution ϕ on V is *self-anti-dual* if for all $v \in V$, $\phi(v) = \phi^{ad}(v)$. Below, you may find some results on anti-duality.

Proposition 3. *If (N,v) is convex, then so is (N,v^{ad}).*

Proof. By Proposotion 2, (N,v^d) is concave. This clearly implies the convexity of (N,v^{ad}) as $v^{ad} = -v^d$.

Proposition 4. *If (N,v) is balanced, then (N,v^{ad}) is balanced.*

Proof. Take an arbitrary $x \in Core(N,v)$. Thus, $x(S) \geq v(S)$ for all $S \subset N$, and $x(N) = v(N) = \tilde{v}(N)$ by the fact that balancedness implies its cohesiveness.
Consider $-x$. Clearly, $-x(N) = -\tilde{v}(N) = v^{ad}(N)$. Take any $S \subset N$.
$v^{ad}(S) = -v^d(S) = -\tilde{v}(N) + \tilde{v}(N \setminus S) = -v(N) + \max\limits_{P \in \Pi(N \setminus S)} \sum\limits_{C \in P} v(C) \leq -v(N) + \max\limits_{P \in \Pi(N \setminus S)} \sum\limits_{C \in P} x(C) = -v(N) + x(N \setminus S) = -x(N) + x(N \setminus S) = -x(S)$.
Thus, $-x \in Core(N,v^{ad})$.

By Proposition 4, we have the following results: the set of balanced games is invariant under the anti-dual operator, and $Core(N, v^{ad})$ is non-empty if (N, v) is balanced. Moreover, there is a equivalence between the core of a balanced TU-game and the core of its anti-dual game.

Proposition 5. *If (N, v) is balanced, then $Core(N, v) = -Core(N, v^{ad})$.*

Proof. By Proposition 4, $Core(N, v) \subseteq -Core(N, v^{ad})$. Let $x \in Core(N, v^{ad})$. Then, $x(N) = v^{ad}(N) = -v^d(N) = -\tilde{v}(N) = -v(N)$ and $x(S) \geq v^{ad}(S)$ for all $S \subset N$.
Consider $-x$. As $x \in Core(N, v^{ad})$, we get $-x(N \setminus S) \leq -v^{ad}(N \setminus S) = \tilde{v}(N) - \tilde{v}(S) = v(N) - \tilde{v}(S) \leq v(N) - v(S) = -x(N) - v(S)$. Thus, $-x(N \setminus S) \leq -x(N) - v(S)$, *i. e.*, $v(S) \leq -x(S)$. Since $-x(S) \geq v(S)$ for all $S \subset N$ and $-x(N) = v(N)$, we have $-x \in Core(N, v)$.

By Proposition 5, we conclude that on the domain of balanced games, the core is *self-anti-dual*.
The natural question at that point is whether these results hold also for the c-balanced games and the c-core.

Proposition 6. *If (N, v) is c-balanced, so is (N, v^{ad}).*

Proof. (N, v^c) is balanced and so is its anti-dual $(N, (v^c)^{ad})$. By the cohesiveness of (N, v^c), we get $(v^c)^{ad} = -(v^c)^d = -v^d = v^{ad}$, i.e., (N, v^{ad}) is balanced, hence c-balanced.

Proposition 7. *On the set of c-balanced games the core is not self-anti-dual, while the c-core is self-anti-dual.*

Proof. Let (N, v) be c-balanced but not balanced. So, $Core(N, v) = \emptyset$ and (N, v^c) is balanced. By Proposition 5 and 6, $Core(N, v^c) = -Core(N, (v^c)^{ad}) = -Core(N, v^{ad}) = Core(N, v^c) \neq \emptyset$. Since $-Core(N, v^{ad}) \neq Core(N, v)$, the core is not self-anti-dual on c-balanced games.

Combining the results above, one can conclude that the c-core is self-anti-dual on the set of c-balanced games: $c\text{–}Core(N, v) = Core(N, v^c) =$

$-Core(N,(v^c)^{ad}) = -Core(N,v^{ad})$, and (N,v^{ad}) is balanced, which implies $Core(N,v^{ad}) = c\text{–}Core(N,v^{ad})$. Thus, we get $c\text{–}Core(N,v) = -\ c\text{-}Core(N,v^{ad})$.

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