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ON UNIFICATION OF SOLUTIONS TO THE BARGAINING PROBLEM¹

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We establish axioms under which a bargaining solution can be found by the maximization of the CES function and is unique up to specifications of the distribution and elasticity parameters. This solution is referred to as the CES solution which includes the Nash and egalitarian solutions as special cases. Next, we consider a normalization of the CES function and establish axioms, under which a bargaining solution can be found by the maximization of the normalized CES function and is unique up to specifications of the distribution and substitution parameters. We refer to this solution as the normalized CES solution which includes the Nash and Kalai-Smorodinsky solutions as special cases. Our paper contributes to bargaining theory by establishing unified characterizations of existing as well as a great variety of new bargaining solutions.

KEYWORDS: Bargaining problem, CES Function, Normalized CES Function, Nash solution, Kalai-Smorodinsky Solution, Egalitarian Solution.

JEL CLASSIFICATION: C78 · D21

1. INTRODUCTION

The term bargaining refers to a situation in which (i) a group of individuals have the opportunity to reach a mutually beneficial agreement; (ii) there is a conflict of interests about which agreement to reach; (iii) no agreement may be imposed on any individual without his approval. Bargaining is ubiquitous in practice. For example, Rawls (1971, p. 4) observes in his *Theory of Justice* that

... although society is a cooperative venture for mutual advantage, it is typically marked by a conflict as well as an identity of interests. There is an identity of interests since social cooperation makes possible a better life for all ... There is a conflict of interests since persons are not indifferent as to how the greater benefits produced by their collaboration are distributed ...

The bargaining theory pioneered in the seminal papers of Nash (1950, 1953) concerns itself directly with normative principles that players in bargaining situations, such as those described by Rawls, might adopt to decide how to share the benefits from cooperation. According to Nash (1953, p. 129),

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One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely.

The Nash bargaining theory has become one of the most fruitful paradigms in game theory and inspired a large literature on axiomatic solutions and applications. Well-known alternative solutions include the *Kalai-Smorodinsky*, *egalitarian*, *utilitarian*, and *relative utilitarian solutions*.¹ Each of these solutions is characterized by a specific collection of axioms. It is desirable to integrate separate collections of axioms into a unifying class. Such unifications are useful for understanding, characterizing, and comparing various axiomatic solutions.

In this paper, we consider two ways for unifying solutions to the bargaining problem. First, we establish axioms under which a bargaining solution can be found by the maximization of the CES function and is unique up to specifications of the *distribution* and *substitution* parameters.² We refer to this solution as the CES solution, which includes the egalitarian and Nash solutions as special cases. The key unifying axiom is a generalization of the axiom of *scale covariance* (SC), also known as the axiom of *invariance to affine transformations*. An affine transformation of bargaining problems rescales and translates players' payoffs. The unifying axiom is imposed on the collection of *hyperplane problems* (i.e., those whose Pareto frontiers are determined by hyperplanes). Roughly, for each hyperplane problem and for any two players, the axiom requires that the ratio of the players' payoff gains, measured relative to the *disagreement point*, in the solution of the transformed problem have constant elasticity with respect to the ratio of the players' payoff scales resulting from the transformation.³ This scale ratio elasticity is shown to be identical to the elasticity of substitution of the CES function whose maximization gives rise to the solution. The scale ratio elasticity is one for the Nash solution and zero for the egalitarian solution.

As the scale ratio elasticity becomes smaller, the payoff gain ratio tends to be unresponsive to the payoff scale ratio. In comparison, the payoff gain ratio becomes more responsive to the payoff scale ratio the larger the elasticity is. The magnitude of the elasticity can be understood as a control of the balance between more individual (smaller elasticities leading to more equal payoff gains) and collective orientations (larger elasticities leading to bigger total payoff).⁴ In contrast, the distribution parameter captures the asymmetries between the players.

Second, we consider an alternative unification that retains the SC axiom. Specifically, we

¹See Kalai (1977a,b), Kalai and Smorodinsky (1975), Roth (1979a), Moulin (1988), Myerson (1991), Thomson (1981, 1994, 2009), Cao (1982), Segal (2000), Dhillon and Mertens (1999) for references.

²The CES function was first introduced in Arrow et al. (1961) which is characterized by the efficiency, distribution, and substitution parameters. It has many appealing technical properties useful in microeconomic analysis. For instance, with the CES production, the elasticity of substitution specifies how two factors substitute each other as a result of a change in the ratio of their prices. As is well known, one obtains from the CES function three special functional forms: the *Leontief* (zero elasticity), the *Cobb-Douglas*, (unit elasticity), and the *linear* (infinitely large elasticity) forms.

³This axiom along with several other characterizing axioms implies that the solution of a general convex problem coincides with the solution of a supporting hyperplane problem (see Remark 3 on p. 11). The axiom has strong implications to solutions of general convex problems.

⁴The unifying axiom for the CES solution involves interpersonal utility comparison, which was often observed in experiments (see, e.g., Nydegger and Owen, 1974). It is argued by some scholars that interpersonal comparison of utility is necessary for an ethical theory (see, e.g., Binmore, 2010).

consider a normalization of the CES function using both the disagreement and *ideal* points of the bargaining problem. We establish axioms under which a bargaining solution can be found by the maximization of the normalized CES function and is unique up to specifications of the distribution and substitution parameters. We refer to this solution as the normalized CES solution. The key unifying axiom involves payoff truncations of the *divide-the-dollar problem* that lower the ideal point. Roughly, the axiom requires that for any two players, the ratio of shares of the truncated ideal payoff gains, which the players capture in the solution of the truncated divide-the-dollar problem, have constant elasticity with respect to the ratio of their truncated ideal payoff gains.⁵ This truncation ratio elasticity is shown to be identical to the elasticity of substitution of the normalized CES function whose maximization gives rise to the solution. The Nash solution as the normalized CES solution has truncation ratio elasticity equal to one, while for the 2-person case, the Kalai-Smorodinsky solution has truncation ratio elasticity equal to zero.⁶

As the truncation ratio elasticity becomes smaller, the ratio of shares of the truncated ideal payoff gains tends to be less responsive to the truncation ratio, but it tends to be more responsive the larger the elasticity is. As with the non-normalized case, the magnitude of the elasticity can be understood as a control of the balance between more individual (smaller elasticities leading to more equal shares of truncated ideal payoff gains) and collective (larger elasticities leading to bigger total share of truncated ideal payoff gains) orientations. In contrast, the distribution parameter captures the asymmetries between the players.

There have been several papers in the literature trying to unify bargaining solutions. Myerson (1981) investigates social choice rules related to the utilitarian and egalitarian principles in terms of the timing of social welfare analysis. Suppose that bargaining takes place today but the choice set is random which will be realized tomorrow. Myerson shows that the condition calling for the solution to provide the players with incentives to each agreement today makes a unified characterization of the utilitarian and egalitarian solutions possible.⁷

Cao (1982) analyzes a class of bargaining solutions for 2-person problems using *preference functions*. A preference function for a player is a function of both players' payoffs and the choice set of the bargaining problem. It represents the player's evaluation of feasible bargaining outcomes. For each player, Cao considers, as a preference function, the sum of the player's own payoff and the fraction of the ideal payoff given up by the other player weighted by a factor. The weighting factor is taken as a quantification of the trade-off between the player's gain (loss) and the other player's loss (gain). Given the weighting factors of the players' preference

⁵This axiom along with several other characterizing axioms implies that the solution of a convex problem coincides with the solution of a truncated hyperplane problem with the same ideal point (see Remark 5 on p. 18). The axiom has strong implications to solutions of general convex bargaining problems.

⁶As shown in Roth (1979b), the Kalai-Smorodinsky solution cannot be extended to allow for more than two players.

⁷Chun (2005) compares the Nash solution with the egalitarian solution. With three or more players, Chun shows that replacing the SC axiom in the collection of the axioms characterizing the Nash solution with *Converse Consistency* yields the egalitarian solution. In his investigation of the extent to which the Nash solution can be viewed as a compromise between egalitarianism and utilitarianism, Rachmilevitch (2015) shows for 2-person games that the symmetric Nash solution lies between the utilitarian and egalitarian solutions.

functions, a solution for the bargaining problem is determined by maximizing the product of the players' preference functions over the choice set. In this way, a class of solutions is found by varying the weighting factors. Cao shows that the relative utilitarian, Kalai-Smorodinsky, and Nash solutions correspond to values -1, 0, and 1 of the weighting factor for both players.⁸ No attempt is made to integrate the axioms characterizing these solutions into a unified class.

Yu (1973), Freimer and Yu (1976), Yu and Leitmann (1976) propose a parameterized solution for 2-person bargaining problems, known as the *compromise solution*.⁹ This solution is defined as the one that minimizes the p -norm from the ideal point.¹⁰ The authors show that the compromise solution ranges from the utilitarian to the *equal loss* solution as the parameter p changes from 1 to ∞ . The authors verify properties of the solution without aiming at axiomatizing their parameterized solution.

Bertsimas et al. (2012) apply the symmetric CES function as the welfare function for resource allocation problems. The welfare maximizers are the realizations of the CES solution, which we axiomatize in this paper, at different values of the substitution parameter. The welfare function ranges from the *max-min* (egalitarian) to the utilitarian welfare function as the elasticity of substitution changes.¹¹ The authors do not axiomatize their solution. Instead, they provide upper bounds for the prices of fairness and efficiency by relating payoffs in the solution for given elasticity of substitution to payoffs in the solutions at extreme elasticities of substitution (egalitarian and utilitarian solutions).

When solution concepts make predictions on the basis of unobservable information, agents will have incentives to distort their private information. Sobel (2001) analyzes strategic misrepresentation of preference information through a *distortion game*. In the distortion game, there is a fixed quantity of a finite number of commodities. Players' strategies consist of utility functions defined over these commodities. Given the reported utility functions, the social planner provides an allocation that maximizes the weighted sum of reported utility functions according to the relative utilitarian solution. Sobel shows, among other results, that if reports are limited to continuous, increasing, and concave functions, then any constrained *equal-income competitive equilibrium* (EICE) allocation for the true utility functions is an equilibrium allocation for the distortion game. (In an exchange economy, an EICE is a competitive equilibrium of the economy in which agents have equal initial endowments.) Furthermore, the result also holds when the distortion game is induced by the symmetric CES solution.¹² Sobel (2001) contributes to the literature that studies the robustness of axiomatic bargaining models to

⁸Cao (1982) uses the term *modified Thomson rule* instead of *relative utilitarian solution* in reference to an axiomatization of the utilitarian solution in Thomson (1981).

⁹We thank William Thomson for pointing to these references.

¹⁰The p -norm of a vector x in $\mathbb{R}^n$ is $\|x\|_p := (\sum_{i=1}^n (x_i)^p)^{1/p}$.

¹¹The equivalence is explicitly shown in Mas-Colell et al. (1995, p. 829).

¹²Due to the fact that Sobel (2001) considers 0-1-normalized bargaining games, there is no difference between the CES

and normalized CES solutions that we consider in the paper.

strategic misrepresentation of preference information through the distortion game.¹³

The rest of the paper is organized as follows. The next section introduces bargaining problems and well-known axioms for bargaining solutions. Section 3 establishes axioms resulting in the CES solution. Section 4 establishes axioms resulting in the normalized CES solution. Section 5 concludes the paper. Most of the proofs are organized in two appendices.

2. BARGAINING PROBLEM AND AXIOMS

Let $N = \{1, 2, \dots, n\}$ denote the set of players and let $\mathbb{R}^n$ denote the n -dimensional Euclidean space. Given vectors $x, y \in \mathbb{R}^n$, $x \geq y$ and $x \gg y$ if $x_i \geq y_i$ and $x_i > y_i$ for all i , respectively. We use $x * y = (x_1 y_1, x_2 y_2, \dots, x_n y_n)$ to denote the *Hadamard product* of x and y . For $\lambda \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, $\lambda * S = \{\lambda * x | x \in S\}$. Set $\mathbb{R}_+^n = \{x \in \mathbb{R}^n | x \geq 0\}$, $\mathbb{R}_{++}^n = \{x \in \mathbb{R}^n | x \gg 0\}$, $\Delta^{n-1} = \{x \in \mathbb{R}_+^n | x_1 + \dots + x_n = 1\}$, and $\Delta_{++}^{n-1} = \Delta^{n-1} \cap \mathbb{R}_{++}^n$.

A bargaining problem in the sense of Nash (1950, 1953) is composed of a choice set $S \subset \mathbb{R}^n$ in the payoff space and a disagreement point or status quo $d \in S$. Elements in S are payoff allocations that players can achieve with agreements, whereas d specifies payoffs the players end up getting in case of disagreement.¹⁴ A bargaining problem (S, d) is *symmetric* if for any permutation of player, $\sigma(S) = S$ and $\sigma(d) = d$.¹⁵ By a hyperplane problem we mean a bargaining problem with

$$S = \{x \in \mathbb{R}_+^n | p_1 x_1 + \dots + p_n x_n \leq I\}$$

for some vector $p \in \mathbb{R}_{++}^n$ and number $I > 0$. Given a bargaining problem (S, d) , let $\beta(S) \in \mathbb{R}^n$ denote the ideal point of S , which is defined by

$$\beta_i(S) = \max_{x \in S: x \geq d} x_i, \quad i \in N.$$

The ideal point assigns the largest achievable payoff in the choice set lying above the disagreement point to each player. What makes bargaining interesting and challenging is that the ideal point is often infeasible. Set $S^\circ = \{x \in \mathbb{R}_+^n | x_1 + \dots + x_n \leq 1\}$. The bargaining problem $(S^\circ, \mathbf{0})$ is known as the divide-the-dollar problem, where $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$.

Let $\mathcal{B}$ denote the class of bargaining problems (S, d) such that $S \subset \mathbb{R}_+^n$ is compact, convex and there is an element $x \in S$ with $x \gg d$. Due to *translation invariance* and *individual rationality* satisfied by all the solutions considered in this paper, we assume $d = \mathbf{0}$ and restrict

¹³In particular, Sobel (2001) extends earlier results in Sobel (1981) that any EICE allocation for the true utility functions is an equilibrium payoff for the distortion game derived from a class of bargaining solutions, including the Nash and Kalai-Smorodinsky solutions, that satisfy the axioms of Pareto optimality, symmetry, scale covariance, and *midpoint domination*. The last axiom requires that the bargaining solution of the n -person problem give each player a payoff that exceeds his disagreement payoff by at least one n -th of the difference between his ideal and disagreement payoffs. The relative utilitarian solution fails to satisfy the midpoint domination axiom (see Sobel, 2001, p. 205). Results in similar spirit are established in Thomson (1984) (proportional solutions and Nash solution) and Trockel (1999) (Kalai-Smorodinsky solution).

¹⁴In applications, S and d are derived from more primitive data.

¹⁵For $x \in \mathbb{R}^n$, $\sigma(x) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)})$.

attention to $S \subset \mathfrak{R}_+^n$ for what follows without loss of generality and write a bargaining problem as S instead of (S, d) .¹⁶ Accordingly, we only need to consider linear transformations for the axiom of scale covariance throughout the rest of the paper.

REMARK 1 *Under the normalization of the disagreement point, elements in the choice set are allocations of payoff gains relative to the disagreement point. For brevity, we continue to term the elements as payoff allocations and the components of the allocations as payoffs in the rest of the paper.*

A *solution* is a mapping $F : \mathcal{B} \rightarrow \mathfrak{R}^n$ such that $F(S) \in S$ for $S \in \mathcal{B}$. The following axioms for bargaining solutions are standard: For all bargaining problems $S, T \in \mathcal{B}$,¹⁷

IR (Individual Rationality): $F(S) \geq 0$.

SIR (Strict Individual Rationality): $F(S) \gg 0$.

PO (Pareto Optimality): There does not exist any $u \in S$ such that $u \geq F(S)$ and $u \neq F(S)$.

IIA (Independence of Irrelevant Alternatives): $S \subset T$ and $F(T) \in S$ imply $F(S) = F(T)$.

SC (Scale Covariance): $F(\lambda * S) = \lambda * F(S)$ for $\lambda \in \mathfrak{R}_{++}^n$.

SYM (Symmetry): For any symmetric problem $S \in \mathcal{B}$, $F_i(S) = F_j(S)$ for all i, j .

IM (Individual Monotonicity): $F_i(T) \geq F_i(S)$ whenever $S \subseteq T$ and $\beta_j(S) = \beta_j(T)$ for $j \neq i$.

Nash (1950) shows that for the 2-person case,

$$N(S) = \operatorname{argmax}_{u \in S: u \geq 0} u_1 u_2,$$

known as the *symmetric* Nash bargaining solution, is the only solution on $\mathcal{B}$ that satisfies IR, PO, SC, IIA, and SYM. Without the SYM axiom, **Kalai** (1977a) shows that

$$N^\alpha(S) = \operatorname{argmax}_{u \in S: u \geq 0} u_1^\alpha u_2^{1-\alpha}, \quad S \in \mathcal{B},$$

known as the *asymmetric* Nash bargaining solution, is the only solution on $\mathcal{B}$ that satisfies IR, PO, SC, IIA, and $F(S^\circ) = (\alpha, 1 - \alpha)$. That is, without the symmetry axiom, the remaining axioms uniquely characterize the bargaining solution up to specifications of bargaining powers or equivalently, up to specifications of solutions for the divide-the-dollar problem. These results can be extended to the n -person case (see, for example, **Peters**, 1992).

Individual rationality is more primitive than collective rationality. With the property that the disagreement point is strictly Pareto dominated within the choice set, **Roth** (1977) shows

¹⁶Translation invariance means that the solution covaries with translations of the set S and disagreement point d .

¹⁷We refer the reader to **Moulin** (1988), **Peters** (1992), **Myerson** (1991), **Roth** (1979a), and **Thomson** (2009) for discussions of these axioms.

that SIR, IIA, and SC together imply PO. Hence, Nash solution is uniquely characterized by SIR, IIA, SC, SYM. In contrast, the asymmetric Nash solution is uniquely characterized up to specifications of solutions for S° by SIR, IIA, and SC. Assuming that there is no danger of confusion, by Nash solution we mean either the symmetric or asymmetric Nash solution in the rest of this paper.

Kalai and Smorodinsky (1975) replace IIA with IM and show that IR, IM, PO, SC, and SYM jointly characterize a unique solution for the case with $n = 2$. This solution has come to be called the Kalai-Smorodinsky solution. It can be computed via the following maximization problem:

$$KS(S) = \max_{u \in S: u \geq 0} \min \left\{ \frac{u_1}{\beta_1(S)}, \frac{u_2}{\beta_2(S)} \right\}.$$

In the Kalai-Smorodinsky solution, players obtain the same share of their ideal payoffs.

We end this section with an example of a bankruptcy problem that illustrates a key difference between the Nash and the Kalai-Smorodinsky solutions.

EXAMPLE 1 Two creditors, 1 and 2, have valid claims for \$40 million and \$60 million, respectively, against a bankruptcy company. But, the company only has \$60 million. The creditors have to reach an agreement about how to divide the money between them. That is, they must choose u_1 for creditor 1 and u_2 for creditor 2 such that

$$u_1 + u_2 \leq 60, \quad u_1 \leq 40, \quad u_2 \leq 60.$$

In case they fail to reach any agreement, they each receive zero payment. The Nash solution assigns \$30 million dollars to each. In contrast, the Kalai-Smorodinsky solution allocates \$24 million dollars to creditor 1 and \$36 million to creditor 2. Hence, in the Kalai-Smorodinsky solution, each creditor obtains 60% share of his claim which appears to be more reasonable.

The Nash and the Kalai-Smorodinsky solutions in Example 1 differ in their sensitivity to the claims that determine ideal payoffs. If, for example, the claim of creditor 1 falls to \$30 million instead, then the Nash solution remains unaltered, while the Kalai-Smorodinsky solution assigns more to creditor 2. Our unification result in Section 4 provides a menu of solutions of various degrees of sensitivity to ideal payoffs. We will revisit Example 1 later in this paper.

3. UNIFICATION WITH THE CES FUNCTION

The n -variable CES function over $\mathfrak{R}_+^n$ is given by

$$H(x; A, \delta, \rho) = A[\delta_1 x_1^\rho + \delta_2 x_2^\rho + \cdots + \delta_n x_n^\rho]^{\frac{1}{\rho}},$$

where $A \geq 0$, $\delta \in \Delta_{++}^{n-1}$, and $\rho \in (-\infty, 0) \cup (0, 1)$ are the efficiency, distribution, and substitution parameters (see Arrow et al., 1961). While the efficiency parameter affects the value of the CES function, the maximization of the CES function does not depend on it. For this

reason, we assume $A = 1$. Denote by ε the elasticity of substitution, which measures the percentage change in the ratio of any two variables with respect to change in the marginal rate of substitution between these two variables. Then,

$$\varepsilon = \frac{1}{1 - \rho}.$$

Observe that $\varepsilon \in (0, 1) \cup (1, \infty)$ as ρ varies in $(-\infty, 0) \cup (0, 1)$. We can rewrite the CES function as

$$H(x; \delta, \varepsilon) = \left[\delta_1 x_1^{\frac{\varepsilon-1}{\varepsilon}} + \delta_2 x_2^{\frac{\varepsilon-1}{\varepsilon}} + \dots + \delta_n x_n^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

The CES function satisfies the following well-known limiting properties:

$$\lim_{\varepsilon \rightarrow \bar{\varepsilon}} H(x; \delta, \varepsilon) = \begin{cases} \min\{x_1, x_2, \dots, x_n\} & \text{if } \bar{\varepsilon} = 0, \\ x_1^{\delta_1} x_2^{\delta_2} \dots x_n^{\delta_n} & \text{if } \bar{\varepsilon} = 1, \\ \delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n & \text{if } \bar{\varepsilon} = \infty. \end{cases} \quad (3.1)$$

We may let $H(x; \delta, \bar{\varepsilon}) = \lim_{\varepsilon \rightarrow \bar{\varepsilon}} H(x; \delta, \varepsilon)$ for $\bar{\varepsilon} = 0, 1, \infty$.

Given $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon \in [0, \infty)$, define

$$C(S; \delta, \varepsilon) = \operatorname{argmax}_{x \in S: x \geq 0} H(x; \delta, \varepsilon), \quad S \in \mathcal{B}. \quad (3.2)$$

For $\varepsilon \neq 0, \infty$, $H(\cdot; \delta, \varepsilon)$ is strictly quasi-concave; hence, $C(\cdot; \delta, \varepsilon)$ is single-valued and well-defined on $\mathcal{B}$. When $\varepsilon = 0$, uniqueness of the maximizer in (3.2) can only be guaranteed, if there are no flat segments on the Pareto frontier lying above the disagreement point $\mathbf{0}$, which will be referred to as the *IR frontier*. Let $\mathcal{B}^\circ$ be the subclass of $\mathcal{B}$ consisting of such bargaining problems only. We refer to $C(\cdot; \delta, \varepsilon)$ as the CES solution with distribution and substitution parameters δ and ε for $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon \in (0, \infty)$ on $\mathcal{B}$ and $\varepsilon \in [0, \infty)$ on $\mathcal{B}^\circ$. When there is no danger of confusion, we will suppress δ and ε from $C(\cdot; \delta, \varepsilon)$ and refer to it simply as the CES solution. Note that the CES solution coincides with the Nash solution on $\mathcal{B}$ when $\varepsilon = 1$ and with the egalitarian solution on $\mathcal{B}^\circ$ when $\varepsilon = 0$. We exclude the case of $\varepsilon = \infty$ for the reason that our key unifying axiom cannot be formulated in this limit case.

We are now ready to introduce the key unifying axiom for the CES solution.

CSRE (Constant Scale Ratio Elasticity): There exists a constant $\varepsilon \geq 0$ such that

for all hyperplane problems $S \in \mathcal{B}$ and for all $\lambda \in \mathfrak{R}_{++}^n$,

$$\frac{F_i(\lambda * S)}{F_j(\lambda * S)} = \left(\frac{\lambda_i}{\lambda_j} \right)^\varepsilon \frac{F_i(S)}{F_j(S)}, \quad i \neq j.$$

Denote this axiom by **CSRE**- ε when ε is explicitly specified.

Here, the constant ε is the elasticity of the ratio $F_i(\lambda * S)/F_j(\lambda * S)$ of player i 's payoff to that of player j in the solution for the transformed problem $\lambda * S$ with respect to the ratio

λ_i/λ_j of their payoff scales resulting from the transformation. We refer to $F_i(\lambda * S)/F_j(\lambda * S)$ as the payoff ratio and λ_i/λ_j as the payoff scale ratio between player i and player j . We refer to ε as the scale ratio elasticity. The CSRE axiom is a generalization of SC. Indeed, with the help of PO, SC can be replaced by CSRE-1.¹⁸

Let ε be given. To understand CSRE- ε , consider for simplicity the case with $n = 2$ and $\lambda_1 \neq \lambda_2$. As ε approaches 0, the right-hand side becomes closer to $F_1(S)/F_2(S)$, so that the payoff ratio in the solution of $\lambda * S$ is the same as in the solution of S , which is akin to egalitarian solutions. In contrast, as ε grows large, the right-hand side is either very large or very small, depending on whether the ratio λ_1/λ_2 is greater or smaller than 1. Specifically, as ε increases, the solution tends to assign larger payoffs to the player with larger payoff scale, implying that the solution tends to increase total payoff. In this sense, the parameter ε may be understood as a degree of the balance between individually oriented (smaller ε) and collectively oriented (larger ε) solution.

Given δ and ε , the CES solution satisfies CSRE- ε on the whole class $\mathcal{B}$ when $\varepsilon = 1$, but this is no longer the case whenever $\varepsilon \neq 1$ as illustrated in Remark 2 below.

REMARK 2 Given δ and ε , the CES solution does not satisfy the CSRE- ε axiom on the whole class $\mathcal{B}$ whenever $\varepsilon \neq 1$. To see this, let $S = \{y \in \mathbb{R}_+^2 | y_1^2 + y_2^2 \leq 1\}$, $\rho < 1$, and $\rho \neq 0$. Consider

$$\max\left(\frac{1}{2}x_1^\rho + \frac{1}{2}x_2^\rho\right)^{\frac{1}{\rho}} \text{ subject to } x \in (\lambda_1, 1) * S = \{x \in \mathbb{R}_+^2 | (\frac{x_1}{\lambda_1})^2 + x_2^2 \leq 1\}$$

for some $\lambda_1 > 0$. With $\delta = 1/2$ and $\varepsilon = 1/(1 - \rho)$,

$$C(S; \delta, \varepsilon) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad C((\lambda_1, 1) * S; \delta, \varepsilon) = \left(\frac{\lambda_1^{\frac{2}{2-\rho}}}{\sqrt{1 + \lambda_1^{\frac{2\rho}{2-\rho}}}}, \frac{1}{\sqrt{1 + \lambda_1^{\frac{2\rho}{2-\rho}}}}\right).$$

It follows that

$$\frac{C_1((\lambda_1, 1) * S; \delta, \varepsilon)}{C_2((\lambda_1, 1) * S; \delta, \varepsilon)} = \lambda_1^{\frac{2(1-\rho)\varepsilon}{2-\rho}} \frac{C_1(S; \delta, \varepsilon)}{C_2(S; \delta, \varepsilon)}.$$

However,

$$C(S^\circ; \delta, \varepsilon) = \left(\frac{\delta_1^\varepsilon}{\delta_1^\varepsilon + \delta_2^\varepsilon}, \frac{\delta_2^\varepsilon}{\delta_1^\varepsilon + \delta_2^\varepsilon}\right) \quad \text{and} \\ C((\lambda_1, 1) * S^\circ; \delta, \varepsilon) = \left(\frac{\delta_1^\varepsilon \lambda_1^\varepsilon}{\delta_1^\varepsilon \lambda_1^{\varepsilon-1} + \delta_2^\varepsilon}, \frac{\delta_2^\varepsilon}{\delta_1^\varepsilon \lambda_1^{\varepsilon-1} + \delta_2^\varepsilon}\right).$$

¹⁸[Zeuthen \(1930\)](#) proposes a negotiation procedure to resolve labor-management disputes, in which each party, at any point in time, had a proposal on the table (e.g., a point on the payoff possibility frontier). Measuring a party's willingness to risk a conflict by the highest probability of conflict that he would face rather than accept the proposal from the other party, Zeuthen's Principle calls for the party less willing to risk a conflict to make a concession. [Harsanyi \(1956\)](#) discovered that the principle yields an outcome formally identical to the Nash solution (see also [Shapley, 1969](#)). The CSRE axiom can be viewed as a generalization of Zeuthen's Principle.

Hence,

$$\frac{C_1((\lambda_1, 1) * S^\circ; \delta, \varepsilon)}{C_2((\lambda_1, 1) * S^\circ; \delta, \varepsilon)} = \lambda_1^\varepsilon \frac{C_1(S^\circ; \delta, \varepsilon)}{C_2(S^\circ; \delta, \varepsilon)}.$$

It is clear from the preceding payoff ratios that $C(\cdot; \delta, \varepsilon)$ does not satisfy the CSRE- ε axiom on the whole class $\mathcal{B}$ when $\rho \neq 0$ or equivalently $\varepsilon \neq 1$.

There is a fundamental difference between the case with zero and the case with positive scale ratio elasticities. For example, as will be shown in the sequel, SIR, IIA, and positive scale ratio elasticity imply PO. This implication is not guaranteed when the scale ratio elasticity is zero. For this reason, we analyze the two cases separately.

3.1. The Case with Positive Scale Ratio Elasticity

We begin with two properties of the Nash bargaining solution. First, as mentioned earlier, Roth (1977) argues that PO is less primitive than SIR and shows that it can be replaced by the latter for the Nash solution on $\mathcal{B}$. Second, the Nash solution of a bargaining problem coincides with the Nash solution of the hyperplane problem determined by a supporting hyperplane of the original problem at its Nash solution. Both properties are interesting in their own right. We show that they continue to hold when SC is replaced with CSRE- ε for $\varepsilon > 0$.

PROPOSITION 1 *Let $\varepsilon > 0$ be given and let $F : \mathcal{B} \rightarrow \mathbb{R}^n$ be a solution satisfying SIR, IIA, and CSRE- ε . Then, (a) for $S \in \mathcal{B}$, there is a hyperplane problem T whose frontier is part of a supporting hyperplane of S at $F(S)$ such that $F(S) = F(T)$; (b) F satisfies PO.*

PROOF: See appendix A.

With Proposition 1 in place, we now proceed to show that given $\varepsilon > 0$, CSRE- ε along with SIR and IIA implies that the solution can be found by the maximization of the CES function with elasticity of substitution ε and is unique up to specifications of the distribution parameter. First, Lemma 1 below shows that the CES solution satisfies SIR.

LEMMA 1 *Given $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon \in (0, \infty)$, $C(\cdot; \delta, \varepsilon)$ satisfies SIR on $\mathcal{B}$.*

PROOF: See Appendix B.

Our next lemma shows that $C(\cdot; \delta, \varepsilon)$ also satisfies IIA and CSRE- ε .

LEMMA 2 *Given $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon \in (0, \infty)$, $C(\cdot; \delta, \varepsilon)$ satisfies IIA and CSRE- ε .*

PROOF: See Appendix B.

Putting together, Lemma 1 and Lemma 2 establish the existence of a solution on $\mathcal{B}$ satisfying SIR, IIA, and CSRE- ε for given $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon > 0$. The following remark is useful for showing the uniqueness of the solution with explicitly specified scale ratio elasticity.

REMARK 3 Given $\delta \in \Delta_{++}^{n-1}$, $\varepsilon \in (0, \infty)$, and $S \in \mathcal{B}$, let

$$p_i = \frac{\delta_i}{(C_i(S; \delta, \varepsilon))^{\frac{1}{\varepsilon}}}, \quad i = 1, 2, \dots, n.$$

By Lemma 1, p_i is well defined. Next, by Lemma 1, Lemma 2, and Proposition 1, $C(S; \delta, \varepsilon)$ is Pareto optimal in S . Since the CES function $H(\cdot; \delta, \varepsilon)$ is strictly quasi-concave, the unique supporting hyperplane of S that separates S and the upper contour of H at point $x = C(S; \delta, \varepsilon)$ is given by

$$p \cdot x = \sum_{i=1}^n \delta_i (C_i(S; \delta, \varepsilon))^{\frac{\varepsilon-1}{\varepsilon}}.$$

Furthermore, $S \subseteq T$ and $C(S; \delta, \varepsilon) = C(T; \delta, \varepsilon)$ where

$$T = \{x \in \mathfrak{R}_+^n \mid p \cdot x \leq \sum_{i=1}^n \delta_i (C_i(S; \delta, \varepsilon))^{\frac{\varepsilon-1}{\varepsilon}}\}.$$

Remark 3 summarizes for each bargaining problem, an explicit hyperplane problem concluded in Proposition 1(a) under the CES solution.

THEOREM 1 Let $\varepsilon > 0$ be given and let $F : \mathcal{B} \rightarrow \mathfrak{R}^n$ be a solution satisfying SIR, IIA, and CSRE- ε . Then, $F(S) = C(S; \delta, \varepsilon)$ where

$$\delta_i = \frac{(F_i(S^\circ))^{\frac{1}{\varepsilon}}}{\sum_{j=1}^n (F_j(S^\circ))^{\frac{1}{\varepsilon}}}, \quad i = 1, 2, \dots, n. \quad (3.3)$$

PROOF: Choose $\delta \in \Delta_{++}^{n-1}$ such that $C(S^\circ; \delta, \varepsilon) = F(S^\circ)$. Then, since

$$C(S^\circ; \delta, \varepsilon) = \left(\frac{\delta_1^\varepsilon}{\sum_{i=1}^n \delta_i^\varepsilon}, \frac{\delta_2^\varepsilon}{\sum_{i=1}^n \delta_i^\varepsilon}, \dots, \frac{\delta_n^\varepsilon}{\sum_{i=1}^n \delta_i^\varepsilon} \right),$$

the distribution parameter δ must satisfy (3.3). By Lemma 1 and Lemma 2, to complete the proof it only remains to show that with δ in (3.3), $F(S) = C(S; \delta, \varepsilon)$ for all $S \in \mathcal{B}$. To this end, let $S \in \mathcal{B}$ be given. Suppose first $S = \{x \in \mathfrak{R}_+^n \mid p \cdot x \leq I\}$ for some $p \in \mathfrak{R}_{++}^n$ and $I > 0$. Then, $S = \lambda * S^\circ$ with $\lambda = (I/p_1, I/p_2, \dots, I/p_n)$. By CSRE- ε ,

$$\frac{F_i(S)}{F_j(S)} = \left(\frac{\lambda_i}{\lambda_j} \right)^\varepsilon \frac{F_i(S^\circ)}{F_j(S^\circ)}. \quad (3.4)$$

Since $F(S^\circ) = C(S^\circ; \delta, \varepsilon)$, it follows from (3.4), Lemma 1, and Lemma 2 that

$$\frac{F_i(S)}{F_j(S)} = \frac{C_i(S; \delta, \varepsilon)}{C_j(S; \delta, \varepsilon)}. \quad (3.5)$$

By Proposition 1, both $F(\cdot)$ and $C(\cdot; \delta, \varepsilon)$ are Pareto optimal. Consequently, (3.5) implies that $F(S) = C(S; \delta, \varepsilon)$. This shows that $F(\cdot)$ coincides with $C(\cdot; \delta, \varepsilon)$ on the subclass of hyperplane problems.

Now suppose $S \in \mathcal{B}$ is an arbitrary problem. Let T be the hyperplane problem as formulated in Remark 3. Then, $C(T; \delta, \varepsilon) = C(S; \delta, \varepsilon)$. Since $F(T) = C(T; \delta, \varepsilon)$ as shown above, we have $F(T) \in S$. Thus, by IIA, $F(S) = F(T)$ because $S \subseteq T$. Together, $F(S) = F(T)$, $C(T; \delta, \varepsilon) = C(S; \delta, \varepsilon)$, and $F(T) = C(T; \delta, \varepsilon)$ imply that $F(S) = C(S; \delta, \varepsilon)$. *Q.E.D.*

By positive CSRE we mean the CSRE axiom with the corresponding scale ratio elasticity being positive. We are now ready to state and prove our unification of bargaining solutions through the CES function.

THEOREM 2 *Let $F : \mathcal{B} \rightarrow \mathbb{R}^n$ be a solution satisfying SIR, IIA, and positive CSRE. Then, $F(S) = C(S; \delta, \varepsilon)$ with $\delta \in \Delta_{++}^{n-1}$ recovered from F as in (3.3) and ε recovered from F as follows*

$$\varepsilon = \ln \frac{F_2(S^\circ)F_1(\lambda * S^\circ)}{F_1(S^\circ)F_2(\lambda * S^\circ)}, \quad (3.6)$$

where $\lambda = (e, 1, \dots, 1)$ and e is Euler number (the base of the natural logarithms).

PROOF: Let $\varepsilon > 0$ be the constant such that F satisfies CSRE- ε . Then, with $\delta \in \Delta_{++}^{n-1}$ given in (3.3), $C(S^\circ; \delta, \varepsilon) = F(S^\circ)$. Next, for $\lambda \in \mathbb{R}_{++}^n$ with $\lambda_1 = e$ and $\lambda_i = 1$ for $i \neq 1$, CSRE- ε implies

$$\frac{F_1(\lambda * S^\circ)}{F_2(\lambda * S^\circ)} = e^\varepsilon \frac{F_1(S^\circ)}{F_2(S^\circ)}.$$

It follows that ε satisfies (3.6). With δ and ε recovered from F in (3.3) and (3.6), it follows from Theorem 1 that $F(S) = C(S; \delta, \varepsilon)$ for $S \in \mathcal{B}$.

Q.E.D.

By Theorem 1 and Theorem 2, the scale ratio elasticity associated with the CSRE axiom is identical to the elasticity of substitution of the CES function whose maximization gives rise to the solution. Notice that we cannot pin down δ (i.e., the solution $F(S^\circ)$ for S°) and ε from the axioms. It is in this sense that the uniqueness of the solution is up to specifications of the distribution and substitution parameters. Notice also that while the elasticity of substitution measures a degree of balance of individual versus collective orientation, the distribution parameter specifies asymmetries between the players.

When the symmetry axiom SYM is added, the solution must assign $(1/n, \dots, 1/n)$ to S° . Thus, a direct application of Theorem 2 establishes the following corollary.

COROLLARY 1 *Let $F : \mathcal{B} \rightarrow \mathbb{R}^n$ be a solution satisfying SIR, IIA, SYM, and positive CSRE. Then, $F(S) = C(S; \delta, \varepsilon)$ with*

$$\delta_i = \frac{1}{n}, \quad i = 1, 2, \dots, n$$

and

$$\varepsilon = \ln \frac{F_1(\lambda * S^\circ)}{F_2(\lambda * S^\circ)},$$

where $\lambda = (e, 1, \dots, 1)$.

The symmetry axiom provides a specification of the solution for S° that assigns the same payoff to each player. Corollary 1 shows that with the symmetry axiom, the scale ratio elasticity can be recovered from the solution for the problem generated from rescaling one player's payoff in S° only.

3.2. The Case with Zero Scale Ratio Elasticity

With zero scale ratio elasticity, PO cannot be implied by SIR, IIA, and CSRE. For example, the egalitarian solution is not necessarily PO when the frontier has flat segments.¹⁹ For this reason, we narrow the class of bargaining problems to $\mathcal{B}^\circ$. We show that SIR, IIA, CSRE-0, and PO uniquely characterize the *proportional solution* up to specifications of solutions for S° . To this end, notice that CSRE-0 becomes:

CSRE-0: For all hyperplane problems $S \in \mathcal{B}$ and $\lambda \in \mathfrak{R}_{++}^n$,

$$\frac{F_i(\lambda * S)}{F_j(\lambda * S)} = \frac{F_i(S)}{F_j(S)}, \quad i \neq j.$$

The axiom requires that for any two players, the ratio of their payoffs be invariant to linear transformations. Following Kalai (1977b), a solution F is proportional to vector $z \in \mathfrak{R}_{++}^n$ if

$$F(S) = \left(\max_{t \geq 0: t z \in S} t \right) z.$$

The egalitarian solution, denoted by $E(\cdot)$, is proportional to vector $z = (1, \dots, 1)$.

THEOREM 3 *Let $F : \mathcal{B}^\circ \rightarrow \mathfrak{R}^n$ be a solution satisfying PO, SIR, IIA, and CSRE-0. Then, F is proportional to $F(S^\circ)$ on $\mathcal{B}^\circ$.*

PROOF: Let T be an arbitrary hyperplane problem. Then, there exists a vector $\lambda \in \mathfrak{R}_{++}^n$ such that $T = \lambda * S^\circ$. Thus, by CSRE-0,

$$\frac{F_i(T)}{F_j(T)} = \frac{F_i(S^\circ)}{F_j(S^\circ)}.$$

Consequently, $F(T) = kF(S^\circ)$ for $k = F_i(T)/F_i(S^\circ) > 0$. Next, consider an arbitrary problem $S \in \mathcal{B}^\circ$. Let $x \in S$ be the Pareto optimal point of S proportional to $F(S^\circ)$, and let T be a

¹⁹Note that by the well-known limit properties in (3.1), the CES function $H(x; \delta, \varepsilon)$ converges to $\min\{x_1, x_2, \dots, x_n\}$ as $\varepsilon \rightarrow 0$, whose maximizers over the choice set of a bargaining problem include the egalitarian solution.

hyperplane problem determined by a supporting hyperplane of S at x . Since S does not have any flat segment on the IR frontier for all $S \in \mathcal{B}^\circ$ and $F(S^\circ) \gg 0$ because of SIR, such a hyperplane problem T exists. As shown above, CSRE-0 implies that $F(T)$ is proportional to $F(S^\circ)$. This together with PO implies that $F(T) = x$. Since $S \subseteq T$ and $F(T) \in S$, it follows from IIA that $F(S) = F(T)$; hence, $F(S)$ is proportional to $F(S^\circ)$. *Q.E.D.*

When SYM is added, a direct application of Theorem 3 establishes the following corollary.

COROLLARY 2 *Let $F : \mathcal{B}^\circ \rightarrow \mathbb{R}^n$ be a solution satisfying PO, SYM, SIR, IIA, and CSRE-0. Then, F is the egalitarian solution E on $\mathcal{B}^\circ$.*

The SYM axiom implies that $F_i(S^\circ) = 1/n$ for all i . By Theorem 3, F is proportional to $(1, 1, \dots, 1)$ and hence, it is egalitarian.

3.3. Applications

The PO axiom is redundant when the scale ratio elasticity is positive. However, for uniformity, we include it for both the case with zero and the case with positive scale ratio elasticity.

COROLLARY 3 *The Nash solution is the unique solution on $\mathcal{B}$ satisfying SIR, IIA, PO, and CSRE-1 up to specifications of solutions for S° . The proportional solution is the unique solution on $\mathcal{B}^\circ$ satisfying SIR, IIA, PO, and CSRE-0 up to specifications of solutions for S° .*

By considering the corresponding values of the scale ratio elasticity, Corollary 3 provides a unified characterization of the Nash solution, the egalitarian solution, and a variety of alternative solutions.

4. UNIFICATION WITH THE NORMALIZED CES FUNCTION

For $S \in \mathcal{B}$ and $b \in \mathbb{R}_{++}^n$, consider the truncation of S via b given by $\mathcal{T}_b[S] = S \cap (b - \mathbb{R}_+^n)$. Notice that when $b \leq \beta(S)$, $\beta(\mathcal{T}_b[S]) = b$. We refer to vectors $b \in \mathbb{R}_{++}^n$ such that $b \leq \beta(S)$ as truncated ideal points of S . For $\varepsilon \in (0, 1) \cup (1, \infty)$ and $\delta \in \Delta_{++}^{n-1}$, consider the following normalized CES function:

$$\bar{H}_b(x; \delta, \varepsilon) = \left[\delta_1 \left(\frac{x_1}{b_1} \right)^{\frac{\varepsilon-1}{\varepsilon}} + \delta_2 \left(\frac{x_2}{b_2} \right)^{\frac{\varepsilon-1}{\varepsilon}} + \dots + \delta_n \left(\frac{x_n}{b_n} \right)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

Similar to the CES function, the normalized CES function satisfies limiting properties:

$$\lim_{\varepsilon \rightarrow \bar{\varepsilon}} \bar{H}_b(x; \delta, \varepsilon) = \begin{cases} \min\{\frac{x_1}{b_1}, \dots, \frac{x_n}{b_n}\}, & \text{if } \bar{\varepsilon} = 0, \\ \prod_{i=1}^n \left(\frac{x_i}{b_i} \right)^{\delta_i}, & \text{if } \bar{\varepsilon} = 1, \\ \sum_{i=1}^n \delta_i \frac{x_i}{b_i}, & \text{if } \bar{\varepsilon} = \infty. \end{cases} \quad (4.1)$$

We may let $\bar{H}_b(x; \delta, \bar{\varepsilon}) = \lim_{\varepsilon \rightarrow \bar{\varepsilon}} \bar{H}_b(x; \delta, \varepsilon)$ for $\bar{\varepsilon} = 0, 1, \infty$.

Given $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon \in [0, \infty)$, define

$$\bar{C}(S; \delta, \varepsilon) = \operatorname{argmax}_{x \in S: x \geq 0} \bar{H}_{\beta(S)}(x; \delta, \varepsilon), \quad S \in \mathcal{B}. \quad (4.2)$$

For $\varepsilon \neq 0, \infty$, $\bar{H}_{\beta(S)}(\cdot; \delta, \varepsilon)$ is strictly quasi-concave for $S \in \mathcal{B}$; hence, $\bar{C}(\cdot; \delta, \varepsilon)$ is well-defined and single-valued on $\mathcal{B}$. We refer to $\bar{C}(\cdot; \delta, \varepsilon)$ as the normalized CES solution with distribution and substitution parameters δ and ε on $\mathcal{B}$, for $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon \in [0, \infty)$. When there is no danger of confusion, we will suppress δ and ε from $\bar{C}(\cdot; \delta, \varepsilon)$ and refer to it simply as the normalized CES solution. Note that the normalized CES solution coincides with the Nash solution when $\varepsilon = 1$ and with the Kalai-Smorodinsky solution when $n = 2$ and $\varepsilon = 0$. We exclude the case of $\varepsilon = \infty$ for the reason that our key unifying axiom cannot be formulated in this limit case.

For our unification via the normalized CES function, we need the following weaker version of IIA in Roth (1977).

RIIA (Restricted Independence of Irrelevant Alternatives): For $S, T \in \mathcal{B}$ with $S \subseteq T$ and $\beta(S) = \beta(T)$, $F(T) \in S$ implies $F(S) = F(T)$.

The following is our second key unifying axiom.

CTRE (Constant Truncation Ratio Elasticity): There exists a constant $\varepsilon \geq 0$ such that for all $b \in (0, 1]^n$ with $\sum_{i=1}^n b_i > 1$ and $i \neq j$,

$$\frac{F_i(\mathcal{T}_b[S^\circ])/b_i}{F_j(\mathcal{T}_b[S^\circ])/b_j} \leq \left(\frac{b_j}{b_i}\right)^\varepsilon \frac{F_i(S^\circ)}{F_j(S^\circ)}, \quad (4.3)$$

whenever $F_j(\mathcal{T}_b[S^\circ]) < b_j$ (i.e., whenever truncation is non-binding for player j).

Denote this axiom by **CTRE- ε** when ε is explicitly specified.

The left-hand-side of (4.3) is the ratio of player i 's share of his truncated ideal payoff to that of player j , which they capture in the solution of the truncated S° . We refer to this ratio as the truncated ideal payoff share ratio and b_j/b_i as the truncation ratio between players i and j . The right-hand-side of (4.3) has elasticity ε with respect to the truncation ratio. We refer to this elasticity as the truncation ratio elasticity.

Let $\varepsilon > 0$ be given. To understand CTRE- ε , assume first that truncation is non-binding for both players i and j . Then, (4.3) becomes

$$\frac{F_i(\mathcal{T}_b[S^\circ])/b_i}{F_j(\mathcal{T}_b[S^\circ])/b_j} = \left(\frac{b_j}{b_i}\right)^\varepsilon \frac{F_i(S^\circ)}{F_j(S^\circ)}.$$

The truncated ideal payoff share ratio between players i and j stays closer to the ratio $F_i(S^\circ)/F_j(S^\circ)$ as ε decreases to 0. It follows that, as ε decreases, the solution tends to assign less payoff to player i than to player j if $b_i < b_j$ or equivalently, if S° is more truncated with respect to player i 's payoff than with respect to player j 's. This indicates that the solution

tends to be more individually oriented (promoting more equal truncated ideal payoff shares). In contrast, with $b_i < b_j$, the truncated ideal payoff share ratio increases as ε increases. That is, as ε increases, player i 's share of his truncated ideal payoff becomes larger relative to that of player j . Since the same increment in payoff results in a larger share of truncated ideal payoff to payer i than to player j , the solution tends to be more collectively oriented as ε increases (promoting bigger total truncated ideal payoff share). Thus, as with the non-normalized case, ε could be understood as a degree of the balance between individually (small values of ε) and collectively oriented (large values of ε) solutions.

Notice that with $b_i < b_j$, truncation will eventually be binding for player i as ε further increases, because $(b_j/b_i)^\varepsilon$ increases unboundedly while payoffs of players i and j are bounded by b_i and b_j , respectively. With binding truncation, the axiom requires that the truncated ideal payoff share ratio stay closest to the right-hand-side of (4.3) having truncation ratio elasticity ε .

To be more specific, let $n = 2$ and let $b \in \mathbb{R}_{++}^2$ with $b_1 < b_2$ and $F_i(S^\circ) < b_i$ for $i = 1, 2$ be given. Figure 1 illustrates how the truncated ideal payoff share ratio in the solution of the truncated S° varies as the truncation ratio elasticity ε changes.²⁰ The shaded area in Figure 1 consists of vectors of players' truncated ideal payoff shares. Three dotted rays representing share ratios with truncation ratio elasticities $\varepsilon = 0, 1$, and $\varepsilon \gg 1$, respectively, are depicted. As the player with respect to whose payoff S° is more truncated, the truncated ideal payoff share of player 1 decreases (increases) relative to that of player 2 when ε decreases (increases). Truncation will eventually be binding for player 1 as ε further increases. For those large values of ε , the truncated ideal payoff share ratio remains unchanged with player 1 getting one hundred percent of his truncated ideal payoff.

REMARK 4 Given ε , the CSRE- ε and CTRE- ε axioms are fundamentally different. The former relates payoff ratios in the solutions of transformed hyperplane problems to the ratios of the payoff scales resulting from the transformations. In comparison, the latter specifies how truncated payoff share ratios in the solutions of the truncated divide-the-dollar problem react to ratios of truncated ideal payoffs. For both axioms, the constant elasticity captures a degree of balance of individual versus collective orientations.

A geometric property similar to that of the CES solution with positive scale ratio elasticity summarized in Remark 3 also holds for the normalized CES solution with positive truncation ratio elasticity. However, This property is not valid when the truncation ratio elasticity is zero. For this reason, we analyze the cases with positive and zero truncation elasticities separately.

4.1. The Case with Positive Truncation Ratio Elasticity

Given $\varepsilon > 0$, a parallel result to Lemma 1 can be established. We summarize this result in the following lemma. Its proof is similar to the proof of Lemma 1 and is thus omitted.

²⁰For expositional reasons, we assume that the solution satisfies PO which will be imposed as a separate axiom.

FIGURE 1.— The shaded area is $(\frac{1}{b_1}, \frac{1}{b_2}) * \mathcal{T}_b[S^\circ]$ and the triangle with corners at A, B , and origin is $(\frac{1}{b_1}, \frac{1}{b_2}) * S^\circ$.

LEMMA 3 Given $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon \in (0, \infty)$, the normalized CES solution $\bar{C}(\cdot; \delta; \varepsilon)$ satisfies SIR on $\mathcal{B}$.

The following lemma shows that the normalized CES solution satisfies the rest of the axioms for our second unification of bargaining solutions.

LEMMA 4 Given $\varepsilon \in (0, \infty)$ and $\delta \in \Delta_{++}^{n-1}$, the normalized CES solution $\bar{C}(\cdot; \delta, \varepsilon)$ satisfies PO, SC, RIIA, and CTRE- ε .

PROOF: See Appendix B.

Lemma 3 and Lemma 4 together establish the existence of a solution satisfying the axioms of SIR, PO, SC, RIIA, and CTRE- ε for given $\delta \in \Delta_{++}^{n-1}$ and $\varepsilon > 0$. We show next that given $\varepsilon > 0$, the normalized CES solution is the unique solution up to specifications of δ or equivalently, solutions for S° under these axioms. To this end, we first show that given $\varepsilon > 0$, CTRE- ε together with PO imply the uniqueness of the solution on the class of truncated divide-the-dollar problems $\mathcal{T}_b[S^\circ]$ with $b \in (0, 1]^n$ such that $\sum_{i=1}^n b_i > 1$. We then apply SC and RIIA to complete the rest of the proof. We begin with the following technical lemma.

LEMMA 5 *Let $b \in (0, 1]^n$ such that $\sum_{i=1}^n b_i > 1$ and $K \in \mathfrak{R}_{++}^n$ be given. Then, the following system has at most one solution:*

$$\sum_{i=1}^n x_i = 1, \tag{i}$$

$$x_i \geq 0, \tag{ii} \quad i = 1, \dots, n,$$

$$x_i \leq b_i, \tag{iii} \quad i = 1, \dots, n,$$

$$x_i/K_i \leq x_j/K_j, \tag{iv} \quad \text{for all } i \text{ and } j \text{ such that } x_j < b_j.$$

PROOF: See Appendix B.

As formulated in the following remark, a similar property to the one summarized in Remark 3 holds for the normalized CES solution. In addition, the normalized CES solution of a problem coincides with the solution of the problem, which is constructed by first transforming S° via a linear transformation and then truncating the ideal point of the transformed problem to be the same as that of the original problem. Both properties are useful for the proof of the uniqueness of the solution with explicitly given truncation ratio elasticity.

REMARK 5 Given $\varepsilon \geq 0$, $\delta \in \Delta_{++}^{n-1}$, and $S \in \mathcal{B}$, define

$$\bar{p}_i = \delta_i \left(\frac{1}{b_i} \right)^{\frac{\varepsilon-1}{\varepsilon}} (\bar{C}_i(S; \delta, \varepsilon))^{\frac{-1}{\varepsilon}}, \quad i = 1, 2, \dots, n.$$

By Lemma 3, $\bar{C}(S; \delta, \varepsilon) \in \mathfrak{R}_{++}^n$, implying that $\bar{p}_i$ is well defined for all i . By Lemma 4, $\bar{C}(S; \delta, \varepsilon)$ is PO. Thus, since the normalized CES function $\bar{H}_{\beta(S)}(\cdot; \delta, \varepsilon)$ is strictly quasi-concave for $\delta \in \Delta^{n-1}$ and $\varepsilon > 0$, the hyperplane that separates S and the upper contour of $\bar{H}_{\beta(S)}(\cdot; \delta, \varepsilon)$ at point $\bar{C}(S; \delta, \varepsilon)$ is given by

$$\bar{p} \cdot x = \sum_{i=1}^n \delta_i \left(\frac{1}{b_i} \right)^{\frac{\varepsilon-1}{\varepsilon}} (\bar{C}_i(S; \delta, \varepsilon))^{\frac{\varepsilon-1}{\varepsilon}}.$$

Consider hyperplane problem,

$$T = \left\{ x \in \mathfrak{R}_+^n \mid \bar{p} \cdot x \leq \sum_{i=1}^n \delta_i \left(\frac{1}{b_i} \right)^{\frac{\varepsilon-1}{\varepsilon}} (\bar{C}_i(S; \delta, \varepsilon))^{\frac{\varepsilon-1}{\varepsilon}} \right\}$$

and its truncation $\mathcal{T}_{\beta(S)}[T]$. Since $S \subseteq \mathcal{T}_{\beta(S)}[T]$, $\bar{C}(S; \delta, \varepsilon)$ is also the maximizer of $\bar{H}_{\beta(S)}(\cdot; \delta, \varepsilon)$ over $\mathcal{T}_{\beta(S)}[T]$. This shows that $\bar{C}(S; \delta, \varepsilon) = \bar{C}(\mathcal{T}_{\beta(S)}[T]; \delta, \varepsilon)$.

Furthermore, $T = \lambda * S^\circ$ for some $\lambda \in \mathfrak{R}_{++}^n$ such that $\lambda > \beta(S)$. Consequently, $\bar{C}(S; \delta, \varepsilon) = \bar{C}(\mathcal{T}_{\beta(S)}[T]; \delta, \varepsilon) = \bar{C}(\mathcal{T}_{\beta(S)}[\lambda * S^\circ]; \delta, \varepsilon)$.

We are now ready to show the uniqueness of the solution for explicitly given positive truncation ratio elasticity.

THEOREM 4 *Let $\varepsilon > 0$ be given and let $F : \mathcal{B} \rightarrow \mathbb{R}^n$ be a solution satisfying SIR, PO, SC, RIIA, and CTRE- ε . Then, $F(S) = \bar{C}(S; \delta, \varepsilon)$ for $S \in \mathcal{B}$ where*

$$\delta_i = \frac{(F_i(S^\circ))^{\frac{1}{\varepsilon}}}{\sum_{j=1}^n (F_j(S^\circ))^{\frac{1}{\varepsilon}}}, \quad i = 1, 2, \dots, n. \quad (4.4)$$

PROOF: Choose $\delta \in \Delta_{++}^{n-1}$ such that $\bar{C}(S^\circ; \delta, \varepsilon) = F(S^\circ)$. Then, since $\beta(S^\circ) = (1, \dots, 1)$, we have $\bar{C}(S^\circ; \delta, \varepsilon) = C(S^\circ; \delta, \varepsilon)$ and hence, δ must satisfy (4.4). We complete the rest of the proof in two steps.

Step 1: For $b \in (0, 1]^n$ with $\sum_i b_i > 1$, we show that $F(\mathcal{T}_b[S^\circ]) = \bar{C}(\mathcal{T}_b[S^\circ]; \delta, \varepsilon)$.

For $i = 1, \dots, n$, define K_i by

$$K_i = \frac{1}{b_i^\varepsilon F_i(S^\circ)}.$$

By SIR, $K_i (> 0)$ is well-defined. By PO, the coordinates of $F(\mathcal{T}_b[S^\circ])$ sum to 1. Next, with K_i specified as above, SIR and CTRE- ε imply that $F(\mathcal{T}_b[S^\circ])$ must be a solution for the system in Lemma 5. Consequently, by Lemmas 3–5, $F(\mathcal{T}_b[S^\circ]) = \bar{C}(\mathcal{T}_b[S^\circ]; \delta, \varepsilon)$.

Step 2: For $S \in \mathcal{B}$, we show that $F(S) = \bar{C}(S; \delta, \varepsilon)$.

Let $S \in \mathcal{B}$ be an arbitrary bargaining problem. Construct a truncated hyperplane problem $\mathcal{T}_{\beta(S)}[\lambda * S^\circ]$ for some $\lambda \in \mathbb{R}_{++}^n$ with $\lambda > \beta(S)$ such that $\bar{C}(\mathcal{T}_{\beta(S)}[\lambda * S^\circ]; \delta, \varepsilon) = \bar{C}(S; \delta, \varepsilon) \in S$ as in Remark 5. By SC and Step 1,

$$\begin{aligned} F(\mathcal{T}_{\beta(S)}[\lambda * S^\circ]) &= \lambda * F(\mathcal{T}_{(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}) * \beta(S)}[S^\circ]) \\ &= \lambda * \bar{C}(\mathcal{T}_{(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}) * \beta(S)}[S^\circ]; \delta, \varepsilon) \\ &= \bar{C}(\mathcal{T}_{\beta(S)}[\lambda * S^\circ]; \delta, \varepsilon). \end{aligned}$$

Since $\beta(\mathcal{T}_{\beta(S)}[\lambda * S^\circ]) = \beta(S)$ and $S \subseteq \mathcal{T}_{\beta(S)}[\lambda * S^\circ]$, RIIA then implies $F(S) = \bar{C}(S; \delta, \varepsilon)$. *Q.E.D.*

By positive CTRE we mean the CTRE axiom with the corresponding truncation ratio elasticity being positive. With Theorem 4 in place, we establish our second unification of bargaining solutions via the normalized CES function in the next theorem.

THEOREM 5 *Let $F : \mathcal{B} \rightarrow \mathbb{R}^n$ be a solution satisfying SIR, PO, SC, RIIA, and positive CTRE. Then, $F(S) = \bar{C}(S; \delta, \varepsilon)$ for $S \in \mathcal{B}$ with δ recovered from F as in (4.4) and ε recovered from F as follows*

$$\bar{b}_h^2 \sum_{i \neq h} \left(\frac{1}{\bar{b}_h^2} \frac{F_i(S^\circ)}{F_h(S^\circ)} \right)^{\frac{2}{\varepsilon+1}} = \frac{1 - (F_h(\mathcal{T}_{\bar{b}}[S^1]))^2}{(F_h(\mathcal{T}_{\bar{b}}[S^1]))^2}, \quad (4.5)$$

where $S^1 = \{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i^2 \leq 1\}$, h is a player for whom $F_h(S^\circ) = \min_i F_i(S^\circ)$, and $\bar{b} \in \mathbb{R}^n$ is such that $\bar{b}_h \in (0, 1)$, $\bar{b}_h^2 + (n-1)\bar{b}_h^4 \geq 1$, and $\bar{b}_i = 1$ for $i \neq h$.²¹

PROOF: Let $\varepsilon > 0$ be the constant elasticity such that $F(\cdot)$ satisfies CTRE- ε . Then, with $\delta \in \Delta_{++}^{n-1}$ given in (4.4), $\bar{C}(S^\circ; \delta, \varepsilon) = F(S^\circ)$. Since $F_i(S^\circ)/F_h(S^\circ) \geq 1$, $0 < \bar{b}_h < 1$, and $0 < \varepsilon/(\varepsilon+1) < 1$, we have

$$\bar{b}_h^2 + \sum_{i \neq h} \left(\frac{F_i(S^\circ)}{F_h(S^\circ)} \right)^{\frac{2}{\varepsilon+1}} \bar{b}_h^{\frac{4\varepsilon}{\varepsilon+1}} > \bar{b}_h^2 + (n-1)\bar{b}_h^4.$$

Since $\bar{b}_h^2 + (n-1)\bar{b}_h^4 \geq 1$, the above inequality implies that

$$\frac{1}{\sqrt{1 + \sum_{i \neq h} \left(\frac{F_i(S^\circ)}{F_h(S^\circ)} \right)^{\frac{2}{\varepsilon+1}} \bar{b}_h^{\frac{2(\varepsilon-1)}{\varepsilon+1}}}} < \bar{b}_h. \quad (4.6)$$

By (4.4),

$$\frac{\delta_h^{\frac{\varepsilon}{\varepsilon+1}}}{\sqrt{\delta_h^{\frac{2\varepsilon}{\varepsilon+1}} + \sum_{i \neq h} \delta_i^{\frac{2\varepsilon}{\varepsilon+1}} \bar{b}_h^{\frac{2(\varepsilon-1)}{\varepsilon+1}}}} = \frac{1}{\sqrt{1 + \sum_{i \neq h} \left(\frac{F_i(S^\circ)}{F_h(S^\circ)} \right)^{\frac{2}{\varepsilon+1}} \bar{b}_h^{\frac{2(\varepsilon-1)}{\varepsilon+1}}}}. \quad (4.7)$$

Next, with $\bar{b}_i = 1$ for $i \neq h$, it follows from simple calculation together with (4.6) and (4.7) that

$$\bar{C}_h(\mathcal{T}_{\bar{b}}[S^1]; \delta, \varepsilon) = \frac{1}{\sqrt{1 + \sum_{i \neq h} \left(\frac{F_i(S^\circ)}{F_h(S^\circ)} \right)^{\frac{2}{\varepsilon+1}} \bar{b}_h^{\frac{2(\varepsilon-1)}{\varepsilon+1}}}} < \bar{b}_h. \quad (4.8)$$

Equation (4.5) follows from (4.8) and the equality that $\bar{C}_h(\mathcal{T}_{\bar{b}}[S^1]; \delta, \varepsilon) = F_h(\mathcal{T}_{\bar{b}}[S^1])$ as established in Theorem 4. With δ and ε recovered from F in (4.4) and (4.5), it follows from Theorem 4 that $F(S) = \bar{C}(S; \delta, \varepsilon)$ for all $S \in \mathcal{B}$. Q.E.D.

By Theorem 4 and Theorem 5, the truncation ratio elasticity associated with the CTRE axiom is identical to the elasticity of substitution for the normalized CES function whose maximization gives rise to the solution. Notice that as with the non-normalized case, we cannot pin down δ (i.e., the solution $F(S^\circ)$ for S°) and ε from the axioms. It is in this sense that the uniqueness of the solution is up to specifications of the distribution and substitution parameters. Notice also that while the elasticity of substitution measures a degree of balance of individual versus collective orientation, the distribution parameter specifies asymmetries between the players.

When the symmetry axiom is added, a direct application of Theorem 5 establishes the following corollary.

²¹As a numerical example, let $\bar{b}_h = \sqrt{3}/2$. Then, $\bar{b}_h$ satisfies $\bar{b}_h^2 + (n-1)\bar{b}_h^4 \geq 1$.

COROLLARY 4 *Let $F : \mathcal{B} \rightarrow \mathfrak{R}^n$ be a solution satisfying SIR, PO, SYM, SC, RIIA, and positive CTRE. Then, $F(S) = \bar{C}(S; \delta, \varepsilon)$ for $S \in \mathcal{B}$ with*

$$\delta_i = \frac{1}{n}, \quad i = 1, 2, \dots, n$$

and

$$(n-1)\bar{b}_h^{\frac{2(\varepsilon-1)}{\varepsilon+1}} = \frac{1 - (F_h(\mathcal{T}_{\bar{b}}[S^1]))^2}{(F_h(\mathcal{T}_{\bar{b}}[S^1]))^2}, \quad (4.9)$$

where $S^1 = \{x \in \mathfrak{R}_+^n \mid \sum_{i=1}^n x_i^2 \leq 1\}$, h is a player for whom $F_h(S^\circ) = \min_i F_i(S^\circ)$, and $\bar{b} \in \mathfrak{R}^n$ such that $\bar{b}_h \in (0, 1)$, $\bar{b}_h^2 + (n-1)\bar{b}_h^4 \geq 1$, and $\bar{b}_i = 1$ for $i \neq h$.

The symmetry axiom simplifies the solution by making the distribution parameters symmetric across the players and by having ε determined by the solution for problem S^1 alone. There is nevertheless still a variety of choices for the substitution parameter.

4.2. The Case with Zero Truncation Ratio Elasticity

It is known that the Kalai-Smorodinsky solution cannot be extended to bargaining problems with more than two players (see [Roth, 1979b](#)). For this reason, we restrict attention to 2-person bargaining problems in this subsection. Notice first that with zero truncation ratio elasticity, CTRE becomes:

CTRE-0: For all $b \in (0, 1]^2$ with $b_1 + b_2 > 1$ and for any $i \neq j$,

$$\frac{F_i(\mathcal{T}_b[S^\circ])/b_i}{F_j(\mathcal{T}_b[S^\circ])/b_j} \leq \frac{F_i(S^\circ)}{F_j(S^\circ)} \quad \text{whenever } F_j(\mathcal{T}_b[S^\circ]) < b_j. \quad (4.10)$$

By replacing SYM with RIIA, [Dubra \(2001\)](#) provides an axiomatization for a single parameter family of asymmetric Kalai-Smorodinsky solutions. We show that this family can be axiomatized using the CTRE-0 axiom along with the other axioms. To this end, for $S \in \mathcal{B}$, let $\text{PO}(S)$ and $\text{WPO}(S)$ denote the set of Pareto optimal and the set of weakly Pareto optimal points in S , respectively. Following [Dubra \(2001\)](#), for $\ell > 0$, let $D_\ell(S)$ be an element in $\text{WPO}(S)$ such that

$$\frac{D_{\ell 2}(S)}{D_{\ell 1}(S)} = \ell \frac{\beta_2(S)}{\beta_1(S)} \quad (4.11)$$

and let $KS_\ell(S) \in \text{PO}(S)$ be such that $KS_\ell(S) \geq D_\ell(S)$.

LEMMA 6 *For $\ell > 0$, KS_ℓ satisfies PO, SIR, SC, RIIA, and CTRE-0.*

PROOF: See Appendix [B](#).

We are now ready to establish an axiomatization of KS_ℓ using CTRE-0.

THEOREM 6 *Let $n = 2$. A solution F satisfies, PO, SIR, SC, RIIA, and CTRE-0 if and only if $F = KS_\ell$ where $\ell = F_2(S^\circ)/F_1(S^\circ)$.*

PROOF: The sufficiency follows from Lemma 6. Now suppose F is a solution satisfying the axioms. We need to show that $F = KS_\ell$ with $\ell = F_2(S^\circ)/F_1(S^\circ)$. To this end, consider $b \in (0, 1]^2$ such that $b_1 + b_2 > 1$.

Case 1: $F(\mathcal{T}_b[S^\circ]) \ll b$.

By CTRE-0, (4.11) and the fact that $\beta(\mathcal{T}_b[S^\circ]) = b$, we have

$$\frac{F_2(\mathcal{T}_b[S^\circ])}{F_1(\mathcal{T}_b[S^\circ])} = \frac{b_2 F_2(S^\circ)}{b_1 F_1(S^\circ)} = \ell \frac{\beta_2(\mathcal{T}_b[S^\circ])}{\beta_1(\mathcal{T}_b[S^\circ])} = \frac{D_{\ell 2}(\mathcal{T}_b[S^\circ])}{D_{\ell 1}(\mathcal{T}_b[S^\circ])}.$$

The above equalities together with PO satisfied by solution F implies $F(\mathcal{T}_b[S^\circ]) = D_\ell(\mathcal{T}_b[S^\circ])$. Since $D_\ell(\mathcal{T}_b[S^\circ]) \in \text{PO}(\mathcal{T}_b[S^\circ])$, we have $D_\ell(\mathcal{T}_b[S^\circ]) = KS_\ell(\mathcal{T}_b[S^\circ])$. Hence, $F(\mathcal{T}_b[S^\circ]) = KS_\ell(\mathcal{T}_b[S^\circ])$.

Case 2: $F_i(\mathcal{T}_b[S^\circ]) = b_i$ for some i

Without loss of generality, suppose $i = 2$. Then, $F_1(\mathcal{T}_b[S^\circ]) < b_1$ because $F(\mathcal{T}_b[S^\circ]) \in \mathcal{T}_b[S^\circ]$ implies $F_1(\mathcal{T}_b[S^\circ]) + F_2(\mathcal{T}_b[S^\circ]) = 1 < b_1 + b_2$. By CTRE-0 and (4.11),

$$\frac{F_2(\mathcal{T}_b[S^\circ])}{F_1(\mathcal{T}_b[S^\circ])} \leq \frac{b_2 F_2(S^\circ)}{b_1 F_1(S^\circ)} = \frac{b_2}{b_1} \ell = \frac{D_{\ell 2}(\mathcal{T}_b[S^\circ])}{D_{\ell 1}(\mathcal{T}_b[S^\circ])}, \quad (4.12)$$

with $\ell = F_2(S^\circ)/F_1(S^\circ)$. Because $F_2(\mathcal{T}_b[S^\circ]) = b_2 \geq D_{\ell 2}(\mathcal{T}_b[S^\circ])$, (4.12) implies $F(\mathcal{T}_b[S^\circ]) \geq D_\ell(\mathcal{T}_b[S^\circ])$. Since $KS_\ell(\mathcal{T}_b[S^\circ])$ is the only Pareto optimal point in $\mathcal{T}_b[S^\circ]$ that Pareto dominates $D_\ell(\mathcal{T}_b[S^\circ])$, we must have $F(\mathcal{T}_b[S^\circ]) = KS_\ell(\mathcal{T}_b[S^\circ])$. By definition of ℓ , $KS_\ell(S^\circ)$ is proportional to $F(S^\circ)$. This together with PO satisfied by both F and KS_ℓ implies that $F(S^\circ)$ coincides with $KS_\ell(S^\circ)$.

In summary, we have shown that $F = KS_i$ on the collection of S° and its truncations. Using SC, the coincidence is established on all hyperplane problems and their truncations. The rest of the proof applies RIIA and follows similar arguments as those in the proof of Theorem 3. *Q.E.D.*

If the symmetry axiom is imposed, then $F(S^\circ) = (1/2, 1/2)$. A direct application of Theorem 6 establishes the following corollary.

COROLLARY 5 *Let $n = 2$. A solution F satisfies, PO, SIR, SC, RIIA, SYM, and CTRE-0 if and only if $F = KS_1$.*

The symmetry axiom implies that $\ell = 1$. Hence, by Theorem 6, the solution must be KS_1 if in addition it satisfies SYM.

4.3. Applications

Theorem 5 and Theorem 6 together provide a unified characterization of the Nash, Kalai-Smorodinsky, and a variety of other bargaining solutions. We summarize the result in the following corollary. Like for the Nash solution, by the Kalai-Smorodinsky solution we mean both the symmetric and asymmetric versions of the solution.

COROLLARY 6 *The Nash solution is the unique solution satisfying PO, SIR, SC, RIIA, and CTRE-1 up to specifications of solutions for S° . For $n = 2$, the Kalai-Smorodinsky solution is the unique solution satisfying PO, SIR, SC, RIIA, and CTRE-0 up to specifications of solutions for S° .*

Condition (4.3) for the CTRE axiom is formulated in terms of the elasticity of the truncated ideal payoff share ratio with respect to the truncation ratio between any two players. It is worth pointing out that the condition can be alternatively formulated as

$$\frac{F_i(T_b[S^\circ])}{F_j(T_b[S^\circ])} \leq \left(\frac{b_j}{b_i}\right)^{\varepsilon-1} \frac{F_i(S^\circ)}{F_j(S^\circ)} \text{ whenever } F_j(T_b[S^\circ]) < b_j. \quad (4.13)$$

This alternative formulation requires that the payoff ratio, instead of the truncated ideal payoff share ratio, between any two players have elasticity $\varepsilon - 1$ with respect to the truncation ratio. When truncation is non-binding for both player i and player j so that the inequality in (4.13) can be replaced with equality, it follows from Corollary 6 that the payoff ratio under the Nash solution have zero truncation ratio elasticity. This means that 1% increase in player i 's truncated payoff relative to that of player j does not change player i 's payoff relative to player j 's. In contrast, the payoff ratio under the Kalai-Smorodinsky solution is -1, so that 1% increase in player i 's truncated payoff relative to that of player j always increases player i 's payoff relative to player j 's by exactly 1%, whenever truncation is non-bonding for both player i and player j . We end this section by revisiting Example 1.

EXAMPLE 2 Consider the bargaining problem in Example 1, which is given by $S = \lambda * \mathcal{T}_b[S^\circ]$ with $\lambda = (60, 60)$ and $b = (2/3, 1)$. By SC, $\bar{C}(S; \delta, \varepsilon) = \lambda * \bar{C}(\mathcal{T}_b[S^\circ]; \delta, \varepsilon)$. The CTRE axiom implies that the truncated ideal payoff share ratio in S has elasticity ε w.r.t. the truncation ratio. Recall that the symmetric Nash and Kalai-Smorodinsky solutions are (30, 30) and (24, 36), respectively. Maintaining the symmetry axiom, simple calculation shows that the normalized CES solution is given by

$$\bar{C}(\varepsilon) := \bar{C}(S; (1/n, \dots, 1/n), \varepsilon) = \begin{cases} \left(\frac{60}{1+(60/40)^{1-\varepsilon}}, \frac{60}{1+(40/60)^{1-\varepsilon}} \right), & \varepsilon \lesssim 2.71 \\ (40, 20), & \varepsilon \gtrsim 2.71 \end{cases}$$

Notice that $\bar{C}(1)$ coincides with the (symmetric) Nash solution while $\bar{C}(0)$ equals the (symmetric) Kalai-Smorodinsky solution. The larger ε is, the larger the total ideal payoff (claim) share becomes. For instance, starting from the Kalai-Smorodinsky solution ($\varepsilon = 0$), the total

share is $3/5 + 3/5 = 6/5$, while it is $3/4 + 1/2 = 5/4$ in the Nash solution ($\varepsilon = 1$). For large enough ε ($\varepsilon \geq 2.71$), $\bar{C}(\varepsilon) = (40, 20)$ with total share equal to $1 + 1/3 = 5/3$.²² Notice that the bargaining problem is more truncated with respect to creditor 1's payoff due to the smaller size of his claim. Increasing player 1's payoff leads to a larger total share of ideal payoffs (claims). In this sense, the larger ε is, the more collectively oriented the CES solution becomes. As the solution becomes more collectively oriented, it is more favorable to creditor 1.

Given that creditor 1 stands to lose less than creditor 2 in case they fail to reach any agreement, creditor 1 is a stronger bargainer. It is therefore reasonable for creditor 1 to obtain a bigger share of his maximum claim, but not as extreme as having his full claim. That is, the normalized solutions with small positive truncation ratio elasticities seem to be more reasonable than the normalized CES solutions with truncation ratio elasticities of 0 and 1, which correspond to the Kalai-Smorodinsky and Nash solutions, respectively.

5. CONCLUSION

We identified axioms under which a bargaining solution can be found by the maximization of the CES or normalized CES function and is unique up to specifications of the distribution and substitution parameters. The Nash and the egalitarian solutions are special cases of the CES solution. In contrast, the Nash and Kalai-Smorodinsky solutions are special cases of the normalized CES solution.

For the CES solution, the key unifying axiom specifies how the ratios of players' payoff gains in the solutions of the transformed problems via affine transformations respond to the ratios of players' payoff scales resulting from the transformations. In comparison, the key unifying axiom for the normalized CES solution specifies how the ratios of players' truncated ideal payoff shares in the solutions of the truncated divide-the-dollar problem respond to the ratios of their truncated ideal payoffs. For both solutions, the substitution parameter measures a degree of balance between individual and collective orientations, while the distribution parameter captures asymmetries among the players. It is worth remarking that the Nash solution is the only one that is simultaneously both the CES and the normalized CES solutions. As the CES solution, it is the only solution that satisfies the SC axiom. In contrast, as the normalized CES solution, it is the only solution that satisfies the IIA axiom.

Our paper contributes to bargaining theory by integrating the axioms for various bargaining solutions into unified systems, so that the solutions can be realized as the maximizers of the CES or normalized CES function with specific values of the distribution and substitution parameters. Among other applications, our unification results are useful for characterizing and comparing various solutions.

APPENDIX A: PROOFS OF PROPOSITIONS

We need the following auxiliary result for the proof of Proposition 1.

LEMMA 7 *Let $\varepsilon > 0$ be given and let $F : \mathcal{B} \rightarrow \mathbb{R}^n$ be a solution satisfying SIR, IIA, and CSRE- ε . Then, F satisfies PO on the subclass of hyperplane problems.*

²²Precisely, this holds for $\varepsilon \geq \ln(1/3)/\ln(2/3) \approx 2.7095$.

PROOF: Let $T \in \mathcal{B}$ be a hyperplane bargaining problem. Suppose $F(T)$ is not Pareto optimal in T . Then, $F(T) \in \lambda * T$ for some $\lambda \in \mathfrak{R}_{++}^n$ such that $0 < \lambda_i < 1$ for some player i and $\lambda_j = 1$ for $j \neq i$. Since $\lambda * T \subset T$, IIA implies $F(\lambda * T) = F(T)$. By SIR, $F(T) \gg 0$. Consequently, by CSRE- ε ,

$$\frac{F_i(T)}{F_j(T)} = \frac{F_i(\lambda * T)}{F_j(\lambda * T)} = \lambda_i^\varepsilon \frac{F_i(T)}{F_j(T)}, j \neq i. \quad (\text{A.1})$$

Since $0 < \lambda_i < 1$, $\varepsilon > 0$, and $F(T) \gg 0$, (A.1) cannot hold – a contradiction.

Q.E.D.

Proof of Proposition 1(a): For $\lambda \in \mathfrak{R}_{++}^n$, let $T^\lambda = \{y \in \mathfrak{R}_+^n \mid \lambda \cdot y \leq \lambda \cdot z\}$ be the supporting hyperplane game of S at $z \in z(\lambda) = \operatorname{argmax} \{\lambda \cdot x \mid x \in S\}$ with normal vector λ . Set $m(\lambda) = \max \{\lambda \cdot x \mid x \in S\}$. Since there exists $x \in S$ with $x \gg 0$, $m(\lambda) > 0$ and $T^\lambda = (m(\lambda)/\lambda_1, \dots, m(\lambda)/\lambda_n) * S^\circ$. By CSRE- ε ,

$$\frac{F_i(T^\lambda)}{F_j(T^\lambda)} = \left(\frac{1/\lambda_i}{1/\lambda_j} \right)^\varepsilon \frac{F_i(S^\circ)}{F_j(S^\circ)}. \quad (\text{A.2})$$

Define $\ell : \mathfrak{R}_{++}^n \rightarrow \mathfrak{R}_{++}^n$ by

$$\ell(\lambda) = \left(\left(\frac{1}{\lambda_1} \right)^\varepsilon F_1(S^\circ), \dots, \left(\frac{1}{\lambda_n} \right)^\varepsilon F_n(S^\circ) \right).$$

By (A.2) and SIR, ℓ maps each element $\lambda \in \mathfrak{R}_{++}^n$ into $\ell(\lambda) \in \mathfrak{R}_{++}^n$ that is proportional to $F(T^\lambda)$. Next, set $c(\lambda) = \frac{m(\lambda)}{\lambda \cdot \ell(\lambda)} > 0$ and $f(\lambda) = c(\lambda)\ell(\lambda)$. Then, $\lambda \cdot f(\lambda) = m(\lambda)$. Consequently, $f(\lambda)$ is on the Pareto boundary of T^λ . On the other hand, as shown in Lemma 7, $F(T^\lambda)$ is also Pareto optimal. The Pareto optimality of $f(\lambda)$ and $F(T^\lambda)$ together with the property that $f(\lambda)$ and $F(T^\lambda)$ are proportional establishes that $f(\lambda) = F(T^\lambda)$.

Define a correspondence $Z : \mathfrak{R}_{++}^n \rightarrow \mathfrak{R}^n$ by²³

$$Z(\lambda) = f(\lambda) - z(\lambda). \quad (\text{A.3})$$

In particular, if $0 \in Z(\lambda)$ for some λ , then there exists $z \in z(\lambda)$ such that $z = f(\lambda) = F(T^\lambda)$. In that case, $F(T^\lambda) \in S$ and $F(T^\lambda)$ is a tangency point between S and T^λ , implying in particular $S \subseteq T^\lambda$. Hence, by IIA, $F(T^\lambda) = F(S)$. Thus, it suffices to show that $0 \in Z(\lambda)$ for some $\lambda \in \mathfrak{R}_{++}^n$.

For $\lambda \in \mathfrak{R}_{++}^n$, as intersection of a hyperplane and the convex set S , $z(\lambda)$ is convex. By the Maximum Theorem, z is also upper-semi continuous (usc in short). Notice also that by definition, ℓ is automatically continuous and the continuity of m follows from the Maximum Theorem. Hence, f is continuous which together with the upper-semi continuity of z implies that Z is usc. Next, by constructions, $f(r\lambda) = f(\lambda)$ and $z(r\lambda) = z(\lambda)$ for $r > 0$. It follows that $Z(r\lambda) = Z(\lambda)$ for $r > 0$. For $x \in Z(\lambda)$, there exists $z \in z(\lambda)$ such that $x = f(\lambda) - z$. Consequently,

$$\lambda \cdot x = \lambda \cdot f(\lambda) - \lambda \cdot z = m(\lambda) - m(\lambda) = 0.$$

By SIR, $f_j(\lambda) > 0$ for all j and $z(\lambda)$ is bounded above by the ideal point $\beta(S)$. Hence, $Z(\lambda)$ is bounded below.

Let $(\lambda^t)_{t=1}^\infty$ be any sequence in $\mathfrak{R}_{++}^n$ such that $\lim_{t \rightarrow \infty} \lambda^t = \bar{\lambda}$ for some $\bar{\lambda} \in \mathfrak{R}_+^n \setminus \mathfrak{R}_{++}^n$ and $\bar{\lambda} \neq 0$. We show that $\lim_{t \rightarrow \infty} \|f(\lambda^t)\| = \infty$. To this end, notice that by definition, for $k = 1, \dots, n$,

$$f_k(\lambda^t) = \frac{m(\lambda^t)F_k(S^\circ)}{\sum_{i:\lambda_i^t > 0} \lambda_i^t \left(\frac{\lambda_k^t}{\lambda_i^t} \right)^\varepsilon F_i(S^\circ) + \sum_{j:\lambda_j^t > 0} \lambda_j^t \left(\frac{\lambda_k^t}{\lambda_j^t} \right)^\varepsilon F_j(S^\circ)}. \quad (\text{A.4})$$

For $t = 1, 2, \dots, \infty$, let $k(t)$ be the player for whom $\lambda_{k(t)}^t = \min_i \lambda_i^t$.²⁴ Then, $\lim_{t \rightarrow \infty} \lambda_{k(t)}^t = 0$. By (A.4) and the fact that $F(S^\circ) \gg 0$, $\lim_{t \rightarrow \infty} f_{k(t)}(\lambda^t) = \infty$, which implies that $\|f(\lambda^t)\| \rightarrow \infty$ as $t \rightarrow \infty$.

²³Given $A \subseteq \mathfrak{R}^n$ and $b \in \mathfrak{R}^n$, $b - A$ is understood as the set $\{b - a \mid a \in A\}$.

²⁴If there are multiple players satisfying this condition, let $k(t)$ be any one of them.

In summary, considering λ as a price vector, we have shown that the correspondence $Z(\cdot)$ satisfies all the conditions required by the Debreu-Gale-Nikaido Lemma (see [Debreu, 1982](#), Theorem 8, p. 722). Consequently, there exists λ such that $0 \in Z(\lambda) = f(\lambda) - z(\lambda)$.

Proof of Proposition 1(b): By Lemma 7, the solution satisfies PO on the class of hyperplane problems. Now let $S \in \mathcal{B}$ be an arbitrary bargaining problem. Then, by Proposition 1(a), $F(T) = F(S)$ for a hyperplane problem T . Since $F(T)$ is Pareto optimal and $S \subseteq T$, $F(S)$ is optimal in S . *Q.E.D.*

APPENDIX B: PROOFS OF LEMMAS

Proof of Lemma 1:

PROOF: Let $\delta \in \Delta_{++}^{n-1}$, $\varepsilon \in (0, \infty)$, and $S \in \mathcal{B}$ be given. Set $\bar{x} = C(S; \delta, \varepsilon)$. Next, define $A = \{x \in \mathbb{R}^n : H(x; \delta, \varepsilon) > H(\bar{x}; \delta, \varepsilon)\}$. Then, A is convex and $A \cap S = \emptyset$. Consequently, since S is closed, the Separating Hyperplane Theorem implies that there exists a pair $(p, I) \in \mathbb{R}^n \times \mathbb{R}$ such that $p \neq 0$ and

$$p \cdot x \leq I < p \cdot y, \quad x \in S, \quad y \in A. \quad (\text{B.1})$$

Since $H(\cdot; \delta, \varepsilon)$ is strictly monotonic, A is not bounded above along any direction. It follows that p cannot have any negative component. With $p \in \mathbb{R}_+^n$ such that $p \neq 0$ and $S \cap \mathbb{R}_{++}^n \neq \emptyset$, (B.1) implies that $I > 0$. We claim $H(\bar{x}; \delta, \varepsilon) = \max_{x \in T} H(x; \delta, \varepsilon)$ where $T = \{x \in \mathbb{R}_+^n : p \cdot x \leq I\}$. Suppose this is not true. Then, there exists $\bar{y} \in T$ such that $H(\bar{y}; \delta, \varepsilon) > H(\bar{x}; \delta, \varepsilon)$. By the construction of A , $\bar{y} \in A$ which contradicts $T \cap A = \emptyset$. Notice also that if $p_i = 0$ for some $i \in N$, then $\bar{x} + k\chi^i \in T$ for any $k \geq 0$ where χ^i denotes the i -th unit vector in $\mathbb{R}^n$. By the strict monotonicity of $H(\cdot; \delta, \varepsilon)$, $\max_{x \in T} H(x; \delta, \varepsilon) = \infty$ which contradicts $\max_{x \in T} H(x; \delta, \varepsilon) = H(\bar{x}; \delta, \varepsilon) < \infty$.

In summary, we have shown that $p \gg 0$, $I > 0$, and $\bar{x}$ solves

$$\max H(x; \delta, \varepsilon) \text{ s. t. } p \cdot x \leq I, \quad x \in \mathbb{R}_+^n.$$

Since $H(\cdot; \delta, \varepsilon)$ is strictly quasi-concave, $\bar{x}$ is the unique solution for the above maximization problem. Furthermore, it follows from the functional form of the CES function $H(\cdot; \delta, \varepsilon)$ that $\bar{x} \gg 0$. *Q.E.D.*

Proof of Lemma 2: As the solution for maximization problem (3.2), $C(\cdot; \delta, \varepsilon)$ automatically satisfies IIA. Let $S \in \mathcal{B}$ be such that $S = \{x \in \mathbb{R}_+^n | p \cdot x \leq I\}$ for some $p \in \mathbb{R}_{++}^n$ and $I > 0$. Then,

$$\lambda * S = \{y \in \mathbb{R}_+^n | \frac{p_1}{\lambda_1} y_1 + \dots + \frac{p_n}{\lambda_n} y_n \leq I\}.$$

Observe that

$$C(S; \delta, \varepsilon) = \left(\frac{\delta_1^\varepsilon p_1^{-\varepsilon} I}{\sum_{i=1}^n \delta_i^\varepsilon p_i^{1-\varepsilon}}, \frac{\delta_2^\varepsilon p_2^{-\varepsilon} I}{\sum_{i=1}^n \delta_i^\varepsilon p_i^{1-\varepsilon}}, \dots, \frac{\delta_n^\varepsilon p_n^{-\varepsilon} I}{\sum_{i=1}^n \delta_i^\varepsilon p_i^{1-\varepsilon}} \right),$$

and

$$C(\lambda * S; \delta, \varepsilon) = \left(\frac{\delta_1^\varepsilon (\frac{p_1}{\lambda_1})^{-\varepsilon} I}{\sum_{i=1}^n \delta_i^\varepsilon (\frac{p_i}{\lambda_i})^{1-\varepsilon}}, \frac{\delta_2^\varepsilon (\frac{p_2}{\lambda_2})^{-\varepsilon} I}{\sum_{i=1}^n \delta_i^\varepsilon (\frac{p_i}{\lambda_i})^{1-\varepsilon}}, \dots, \frac{\delta_n^\varepsilon (\frac{p_n}{\lambda_n})^{-\varepsilon} I}{\sum_{i=1}^n \delta_i^\varepsilon (\frac{p_i}{\lambda_i})^{1-\varepsilon}} \right).$$

It follows from the preceding expressions that

$$\frac{C_i(\lambda * S; \delta, \varepsilon)}{C_j(\lambda * S; \delta, \varepsilon)} = \left(\frac{\lambda_i}{\lambda_j} \right)^\varepsilon \frac{C_i(S; \delta, \varepsilon)}{C_j(S; \delta, \varepsilon)}, \quad (\text{B.2})$$

showing that $C(\cdot; \delta, \varepsilon)$ satisfies CSRE- ε . *Q.E.D.*

Proof of Lemma 4: As the solution for (4.2), $\bar{C}(\cdot; \delta, \varepsilon)$ automatically satisfies PO and SC. Notice that $\bar{C}(\cdot; \delta, \varepsilon)$ also satisfies RIIA because for $S, T \in \mathcal{B}$ such that $\beta(S) = \beta(T) = b$, the normalized CES solutions for S and T are determined by the same maximization problem.

Let $b \in (0, 1]^n$ with $\sum_{i=1}^n b_i > 1$ be given. Consider the truncated divide-the-dollar problem $T = \mathcal{T}_b[S^\circ]$. Notice that $\beta(T) = b$. Thus, $\bar{C}(T; \delta, \varepsilon) = \operatorname{argmax}_{x \in T: x \geq 0} \bar{H}_b(x)$. Applying the Kuhn-Tucker first-order conditions,

$$\begin{aligned} \bar{H}_b^{\frac{1}{\varepsilon-1}}(x) \delta_i \left(\frac{1}{b_i} \right)^{\frac{\varepsilon-1}{\varepsilon}} x_i^{-\frac{1}{\varepsilon}} + \rho_0 + \rho_i &= 0 & (i = 1, \dots, n) \\ \rho_i (x_i - b_i) &= 0 & (i = 1, \dots, n) \\ x_1 + \dots + x_n &= 1 \end{aligned}$$

where ρ_0 is the multiplier associated with the constraint that the coordinates sum to 1 on the frontier of S° and for $i = 1, 2, \dots, n$, ρ_i is the associated multiplier for $x_i \leq b_i$ due to truncation.²⁵ The strict quasiconcavity ensures that the necessary conditions are also sufficient and that there is a unique maximizer $\bar{x} = \bar{C}(T; \delta, \varepsilon)$ with multipliers $\bar{\rho}_0, \bar{\rho}_1, \dots, \bar{\rho}_n$. Since $\sum_i b_i > 1$, there is at least one j such that $\bar{x}_j < b_j$; hence,

$$\bar{\rho}_0 = -\bar{H}_b^{\frac{1}{\varepsilon-1}}(x) \delta_j \left(\frac{1}{b_j} \right)^{\frac{\varepsilon-1}{\varepsilon}} x_j^{-\frac{1}{\varepsilon}}$$

which in turn implies that

$$\bar{\rho}_i = \bar{H}_b^{\frac{1}{\varepsilon-1}}(x) \delta_j \left(\frac{1}{b_j} \right)^{\frac{\varepsilon-1}{\varepsilon}} x_j^{-\frac{1}{\varepsilon}} - \bar{H}_b^{\frac{1}{\varepsilon-1}}(x) \delta_i \left(\frac{1}{b_i} \right)^{\frac{\varepsilon-1}{\varepsilon}} x_i^{-\frac{1}{\varepsilon}}.$$

It follows that

$$\begin{aligned} \bar{\rho}_i \geq 0 &\iff \frac{\delta_i \left(\frac{1}{b_i} \right)^{\frac{\varepsilon-1}{\varepsilon}} \bar{x}_i^{-1/\varepsilon}}{\delta_j \left(\frac{1}{b_j} \right)^{\frac{\varepsilon-1}{\varepsilon}} \bar{x}_j^{-1/\varepsilon}} \geq 1 \iff \frac{\bar{x}_i}{\bar{x}_j} \leq \left(\frac{b_i}{b_j} \right)^{1-\varepsilon} \left(\frac{\delta_i}{\delta_j} \right)^\varepsilon \\ &\iff \frac{\bar{C}_i(\mathcal{T}_b[S^\circ]; \delta, \varepsilon)}{\bar{C}_j(\mathcal{T}_b[S^\circ]; \delta, \varepsilon)} \leq \left(\frac{b_i}{b_j} \right)^{1-\varepsilon} \frac{\bar{C}_i(S^\circ; \delta, \varepsilon)}{\bar{C}_j(S^\circ; \delta, \varepsilon)}. \end{aligned}$$

The last inequality is due to the fact that

$$\bar{C}(S^\circ; \delta, \varepsilon) = C(S^\circ; \delta, \varepsilon) = \left(\frac{\delta_1^\varepsilon}{\sum_{i=1}^n \delta_i^\varepsilon}, \frac{\delta_2^\varepsilon}{\sum_{i=1}^n \delta_i^\varepsilon}, \dots, \frac{\delta_n^\varepsilon}{\sum_{i=1}^n \delta_i^\varepsilon} \right).$$

This completes the proof that $\bar{C}(\cdot; \delta, \varepsilon)$ satisfies CTRE- ε .

Q.E.D.

Proof of Lemma 5: Let $x', \bar{x}$ be two different solutions for system (i)-(iv).

Case 1: For all i , $\bar{x}_i = b_i$ if and only if $x'_i = b_i$.

In this case, since the coordinate sum is constant and since $\bar{x} \neq x'$, there are i_0, j_0 with $\bar{x}_{i_0} < x'_{i_0} < b_{i_0}$ and $x'_{j_0} < \bar{x}_{j_0} < b_{j_0}$. By (iv), $\bar{x}_{i_0}/K_{i_0} = \bar{x}_{j_0}/K_{j_0}$ and $x'_{i_0}/K_{i_0} = x'_{j_0}/K_{j_0}$ which contradicts $\bar{x}_{i_0} < x'_{i_0} < b_{i_0}$ and $x'_{j_0} < \bar{x}_{j_0} < b_{j_0}$.

Case 2: For some j_0 , $x'_{j_0} = b_{j_0} > \bar{x}_{j_0}$.

²⁵Thanks to monotonicity and quasiconcavity of the maximand, we may omit non-negativity conditions and assume that the coordinates in $\bar{x}$ sum to 1.

Notice that $\sum_i b_i > \sum_i x'_i$, there exists i_0 such that $x'_{i_0} < b_{i_0}$. But, for each such i_0 ,

$$\frac{x'_{i_0}}{K_{i_0}} \stackrel{\text{(iv)}}{\geq} \frac{x'_{j_0}}{K_{j_0}} = \frac{b_{j_0}}{K_{j_0}} > \frac{\bar{x}_{j_0}}{K_{j_0}} \stackrel{\text{(iv)}}{\geq} \frac{\bar{x}_{i_0}}{K_{i_0}}.$$

That is, $\bar{x}_{i_0} < x'_{i_0}$ whenever $x'_{i_0} < b_{i_0}$, which together with Case 1 implies

$$\sum_i \bar{x}_i = \bar{x}_{j_0} + \sum_{i_0 \neq j_0: x'_{i_0} < b_{i_0}} \bar{x}_{i_0} + \sum_{i_0 \neq j_0: x'_{i_0} = b_{i_0}} \bar{x}_{i_0} < b_{j_0} + \sum_{i_0 \neq j_0: x'_{i_0} < b_{i_0}} x'_{i_0} + \sum_{i_0 \neq j_0: x'_{i_0} = b_{i_0}} b_{i_0} = \sum_i x'_i.$$

This contradicts $\sum_i x_i = \sum_i x'_i$.

Case 1 covers the case in which (iii) is binding for i at solution $\bar{x}$ if and only if it is binding for i at solution x' . In comparison, Case 2 covers the rest. As such, Case 1 and Case 2 are mutually exclusive and jointly exhaustive. The proof is thus completed. *Q.E.D.*

Proof of Lemma 6:

PROOF: By construction, KS_ℓ satisfies PO, SIR, SC, RIIA. To show that KS_ℓ satisfies CTRE-0, first observe that any weakly Pareto optimal point in S° is strictly Pareto optimal. Thus, $KS_\ell(S^\circ) = D_\ell(S^\circ)$ and $KS_{\ell 2}(S^\circ)/KS_{\ell 1}(S^\circ) = \ell$. For $b \in (0, 1]^2$ with $b_1 + b_2 > 1$, we have $(1 - b_1)(1 - b_2) < b_1 b_2$. Set $T = \mathcal{T}_b[S^\circ]$.

$$\text{Case 1: } \frac{1-b_1}{b_1} < \ell \frac{b_2}{b_1} < \frac{b_2}{1-b_2}$$

Since $\beta(T) = b$, it follows from (4.11) that $D_{\ell 2}(T)/D_{\ell 1}(T) = \ell b_2/b_1$. Thus, $(1 - b_1)/b_1 < \ell b_2/b_1 < b_2/(1 - b_2)$ implies that $D_{\ell 2}(T)/D_{\ell 1}(T)$ falls between the ratios on the Pareto frontier of $\mathcal{T}_b[S^\circ]$ that result when truncation is binding for player 1 and player 2, respectively. This in turn implies that D_ℓ is Pareto optimal. Consequently, $KS_\ell(T) = D_\ell(T) \ll b$ and

$$\frac{KS_{\ell 2}(T)}{KS_{\ell 1}(T)} = \frac{D_{\ell 2}(T)}{D_{\ell 1}(T)} = \ell \frac{b_2}{b_1} = \frac{b_2}{b_1} \frac{KS_{\ell 2}(S^\circ)}{KS_{\ell 1}(S^\circ)},$$

showing that KS_ℓ satisfies CTRE-0 in this case.

$$\text{Case 2: } \frac{1-b_1}{b_1} \geq \ell \frac{b_2}{b_1}$$

Note that $x_2/x_1 > (1 - b_1)/b_1$ whenever $x_1 < b_1$ for any x in $\text{WPO}(T)$. Thus, (4.11) can never be satisfied at points in WPO with $x_1 < b_1$. Consequently, $D_{\ell 1}(T) = b_1$ implying $D_{\ell 2}(T) \leq b_2$. Since $KS_\ell \geq D_\ell$ and $x_1 \leq b_1$ for all $x \in T$, we have $KS_{\ell 1}(T) = D_{\ell 1}(T) = b_1$ and $KS_{\ell 2}(T) = 1 - b_1 < b_2$. It follows that

$$\frac{KS_{\ell 2}(T)}{KS_{\ell 1}(T)} \leq \frac{D_{\ell 2}(T)}{D_{\ell 1}(T)} = \ell \frac{b_2}{b_1} = \frac{b_2}{b_1} \frac{KS_{\ell 2}(S^\circ)}{KS_{\ell 1}(S^\circ)}. \quad (\text{B.3})$$

This shows that KS_ℓ satisfies CTRE-0.

$$\text{Case 3: } \ell \frac{b_2}{b_1} \geq \frac{b_2}{1-b_2}$$

The same reasoning as in Case 2 by interchanging player 1 with player 2 shows that KS_ℓ satisfies CTRE-0 in this case.

In summary, we have shown that KS_ℓ satisfies CTRE-0 in Cases 1-3. The proof of the lemma is completed by noticing that cases 1-3 are mutually exclusive and jointly exhaustive. *Q.E.D.*

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