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A Note on Manipulability in School Choice with Reciprocal Preferences

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Abstract

We show that the Boston school choice mechanism (BM), the student proposing deferred acceptance algorithm (DA) and the top trading cycles algorithm (TTC) generate the same outcome when the colleges' priorities are modified according to students' preferences in a "first preferences first" manner. This outcome coincides with the BM outcome under original priorities. As a result, the DA and TTC mechanism that are non-manipulable under original priorities become vulnerable to strategic behavior.

1 Introduction

The model of a *college admission problem*, as it was termed by Gale and Shapley (1962), and a popular variation, the *school choice problem* (Abdulkadiroğlu and Sönmez, 2003), are widely used in practice to organize the allocation of students to colleges or schools. In the practical implementation, there are three predominant matching mechanisms: the Boston school choice mechanism (BM), the student proposing deferred acceptance algorithm (DA) and the top trading cycles algorithm (TTC). A common feature of all three mechanisms is that students have to submit their preferences over colleges so that strategic behavior is an issue that may have to be taken care of.

In the classical school choice models students' preferences and colleges' priorities are assumed to be independent. The colleges' priorities are typically based on objective criteria like already enrolled siblings or walking distances, leaving no room to strategize. With respect to strategic concerns the deferred acceptance algorithm or the top trading cycles algorithm are superior to the Boston school

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choice mechanism, as the two former are strategy-proof for students, while the latter was shown to be manipulable (Ergin and Sönmez, 2006).

However, in reality there are matching practices in school choice where preferences are not completely independent. For example, in France students are matched to secondary schools with a variant of the deferred acceptance algorithm. The schools' priorities are score-based but may vary across districts. In some districts, schools give points to a student if she or he has ranked the school first (Hiller and Tercieux, 2013). Such a dependence raises an interesting question:

> When colleges' priorities are not independent from students' preferences, will a matching mechanism be manipulable by the students, even if it is strategy-proof for students under independent priorities (like the DA algorithm)?

In this work we answer this question by considering a matching market, in which the colleges' priorities reciprocate the students' preferences. We start with a classical school choice problem and introduce a reciprocity condition on priorities. Within this setup, we show that all three algorithms (DA, BM, TTC) produce the same matching (Theorem 1). As a consequence, the application of the DA or TTC in this context leaves room for strategic behavior on the students' side (Corollary 1).

2 School Choice with Reciprocation

Following Abdulkadiroğlu and Sönmez (2003) a school choice problem is defined as follows: There are two disjoint finite sets, $S = \{s_1, \ldots, s_n\}$, the set of students, and $C = \{c_1, \ldots, c_m\}$, the set of schools or colleges. Each college c has a number of available seats or quota q_c, where $\sum_{c \in C} q_c \geq |S|$. Thus, there is at least one available seat for each student s. We denote the vector of quotas by $q = (q_{c_1}, \ldots, q_{c_m})$.

Each student s is looking for exactly one seat according to his strict preferences over colleges C, denoted P_s, i.e., $c_l P_s c_j$ means that student s strictly prefers college c_l over c_j. The preferences are independent of the other students' assignments and of a college's behavior within the mechanism.

Each college c has a strict priority structure over the students that we refer to as college c's (original) preferences P_c^o, as they are usually termed within the college admission framework (see Ergin and Sönmez, 2006).

A matching $\mu : S \to C$ is a mapping from the set of students to the set of colleges so that the number of students mapped to a college does not exceed its quota. $\mu(s)$ denote the assignment of student s under matching μ and $\mu^{-1}(c)$ is the set of students assigned to college c.

The Boston school choice mechanism is widely used in school choice problems where students submit preferences over colleges and the colleges then accept the students according to some fixed priority ordering (see, e.g., Abdulkadiroğlu et al., 2005b) . In the first round only the students' first choices are considered. Each college starts assigning seats according to its preference to students who have ranked it first and until either the college's quota is reached or all such students are admitted. In general, in round k only the students' k-th choices are considered. Each college starts filling its remaining seats with students who have ranked this college as their k-th choice according to its priorities and until either it is filled or all students are admitted in this round. The procedure terminates when each student is assigned a seat at a college.

Non-manipulability guarantees that no student can take advantage of misreporting his true preferences. A matching mechanism is manipulable, if telling the truth is not a Nash equilibrium in the non-cooperative game induced by the matching mechanism with reported preferences as strategies and payoffs determined by the outcome of the mechanism. Ergin and Sönmez (2006) theoretically show that the BM is manipulable, so that submitting true preferences might not be in a student's best interest. Experimental evidence for manipulation can be found in Pais and Pintér (2008). Observe that colleges are excluded from strategic considerations as in classical school choice and are rather treated as objects.

The most prevalent alternative to overcome the shortcomings is the student proposing deferred acceptance algorithm (Gale and Shapley, 1962), a version of which is, e.g., applied in the New York City high school match (Abdulkadiroğlu et al., 2005a). In the first round students propose to the colleges they rank first. If the colleges receive more proposals than they have seats available, they reject the least priorized students without finally accepting the others. In each next round, all rejected students propose to their next best colleges according to their preferences. The colleges reject the students they least prefer, whenever they have more applications than remaining seats. The algorithm terminates when no student is rejected anymore. All proposals being held at this moment are now finally accepted. The outcome is the so-called student-optimal stable matching. From a strategic perspective, there is no incentive for a student not to propose according to his preferences. Otherwise he runs the risk of being stuck at a college that was not next on his lists.

The second strategy-proof alternative to use is a variant of the top trading cycles algorithm (Shapley and Scarf, 1974), adapted to school choice problems in Abdulkadiroğlu and Sönmez (2003). In each round a student points to his most preferred college and a college points to its most preferred student. Then in any cycle, each student is matched to the college he is pointing to and leaves the market. Colleges' quotas are reduced accordingly.

One can easily verify that the three discussed mechanisms may produce different results. But what happens, if the colleges' preferences not only depend on preferences such as the students' grades or whether they live in walking distance to the college but also on the students' preferences over colleges? It appears pretty natural that a college may prefer to allocate its seats to students who really favor this college.

To model this type of reciprocity, we modify college preferences in the following way. Let $I_k(c)$ be the set of students who rank college c at the k-th position, i.e.,

$$I_k(c) = \left\{ s \in S \;\middle|\; |\{c' \in C \mid c' P_s c\}| = k - 1 \right\}. \tag{1}$$

To capture reciprocity of college preferences, we construct college c's reciprocal preferences P_c from its original preferences P_c^o as follows: For all $s, s' \in S$ with $s \in I_k(c)$, $s' \in I_{k'}(c)$

$$s P_c s' \quad \text{if and only if} \quad \text{(a) } k < k' \text{ or (b) } k = k' \text{ and } s P_c^o s'. \tag{2}$$

In other words, college c's original ranking over two students s and s' applies only when both s and s' rank c the same. Otherwise, c prefers the student who ranks it higher, independent of its original preferences P_c^o.

By $M^o := \{C, S, (P_c^o)_{c \in C}, (P_s)_{s \in S}\}$ and $M := \{C, S, (P_c)_{c \in C}, (P_s)_{s \in S}\}$ we denote the matching market under original and under reciprocal preferences, respectively. Further, denote by $\mu^{BM}(M^o)$ and $\mu^{BM}(M)$ the matchings resulting from the Boston mechanism with original and reciprocal preferences, respectively. Similarly, with reciprocal preferences, $\mu^{DA}(M)$ and $\mu^{TTC}(M)$ are the matchings produced by the student proposing deferred acceptance algorithm and the top trading cycle algorithm, respectively.

Theorem 1. *In a matching market M with reciprocal preferences, the matchings resulting from the Boston school choice mechanism, the student proposing deferred acceptance algorithm, and the top trading cycles algorithm are equal and coincide with the matching produced by the Boston mechanism under original preferences. That means*

$$\mu^{BM}(M^o) = \mu^{BM}(M) = \mu^{DA}(M) = \mu^{TTC}(M).$$

Proof. We start by reviewing the Boston mechanism in M^o.

Under original preferences, the Boston school choice mechanism produces the final matching along the following procedure:

In the first round, each college c accepts students who rank c as their first choice, i.e., it selects students from the set $I_1(c)$ according to preferences $(P_c^o)_{c\in C}$ in M^o until either the quota is filled, $|I_1(c)| \geq q_c$, or all students who rank c first are accepted, if $|I_1(c)| \leq q_c$. Selected students are finally matched to that college and leave the market. If $|I_1(c)| \geq q_c$, then college c also leaves the market. Otherwise, c goes to the next round with correspondingly reduced quota. All unmatched students also proceed to the next round.

In round k each college c, which has not yet left the market, chooses among all unmatched students in $I_k(c)$, i.e., among those who rank c as their k-th best choice, according to its preferences P_c^o. All students who are accepted by c are matched to it and leave. A college leaves the market, as soon as all its seats are filled.

The algorithm terminates when all students are accepted at some college, which happens after finitely many rounds, since there are fewer students than total seats.

Note that once a student s is accepted by college c, $\mu^{BM}(M^o)(s) = c$ in the final outcome. In other words, students who are rejected by college c will never be accepted by c in later rounds.

Step 1: $\mu^{BM}(M^o) = \mu^{BM}(M)$

In each round of the Boston mechanism, the students, among which college c may choose, belong to the same set $I_k(c)$ for appropriate k. Therefore, only college preferences restricted to sets $I_k(c)$ take effect, which are the same in M^o and M. Hence, the final matchings of the Boston mechanism coincide.

Step 2: $\mu^{BM}(M^o) = \mu^{DA}(M)$

We now assume that in the deferred acceptance algorithm colleges act according to reciprocal preferences $(P_c)_{c\in C}$ in matching market M. During the first round, all students in $I_1(c)$, who rank the college c first, propose to c. This is the same situation as in the first round of the Boston school choice mechanism. If the number of students proposing to c is not greater than the college's quota, $|I_1(c)| \leq q_c$, c accepts all students. Otherwise, c accepts students according to reciprocal preferences P_c until all seats are filled.

As defined above, c's original preferences P_c^o are identical to the reciprocal preferences P_c restricted to the students in $I_1(c)$. Thus, after the first round, c has accepted the same set of students, a subset of $I_1(c)$, in both algorithms. The only difference is that acceptance in the DA algorithm is tentative, so that in principle an accepted student may be rejected in later rounds.

The key observation, however, that this cannot happen, is due to the reciprocity condition. To see this, assume that k' is the first round in the DA algorithm, in which a student s, who was tentatively accepted at college c in round $k < k'$ is now rejected by c, because student s' has now proposed to c round k'. Since

k' is minimal, both s and s' have never been rejected before in the algorithm, and hence, $s \in I_k(c)$ and $s' \in I_{k'}(c)$. Due to reciprocity, sP_cs' and student s will not be rejected, a contradiction. Therefore, like in the BM, once a student is tentatively accepted, he will also be finally matched to that college in μ^{DA}. As a consequence, in each round, at most colleges with unfilled seats newly accept proposing students. Further, a student, who was rejected in round $k-1$ must have been rejected in all other previous rounds, so that in round k he proposes to his k-th best choice.

To sum, comparing the BM and DA algorithm, in each round k the same set of students consider their k-th best choice. Hence, each college c faces students in $I_k(c)$. If a college's seats have not yet been filled, it starts accepting students according to preferences P_c^o until either all students in $I_k(c)$ are accepted, or the quota is reached. This is common to both algorithms. If all seats of c are filled, due to the key observation above, it rejects all future proposing students, meaning that all accepted students are finally matched to c as in the BM. Hence, the final matchings of BM and DA are equal under reciprocal preferences.

Step 3: $\mu^{BM}(M^o) = \mu^{TTC}(M)$

In the first step of the TTC algorithm under reciprocal preferences, each student points to his or her most preferred college and each college points to its most preferred student. All students included in a cycle are matched to the college they have pointed to and leave the market. The corresponding colleges' quotas are reduced by one. When the updated quota reaches zero, a college leaves the market. Otherwise, the algorithm continues with the next round.

The key observation here is that with reciprocal preferences, any cycle that is formed includes exactly one student and one college. To see this, suppose there is a cycle $s_1 \to c_1 \to s_2 \to c_2 \to \ldots \to c_j \to s_1$ with j different students and colleges. Let $k_1, \ldots k_j$ be defined by $s_i \in I_{k_i}(c_i)$. That means, k_i is college c_i's rank in student s_i's preferences. Analogously, define $k'_1, \ldots, k'_j$ by $s_{i+1} \in I_{k'_i}(c_i)$ and $s_1 \in I_{k'_j}(c_j)$ so that k'_i is student s_{i+1}'s rank of college c_i. Since in each round, agents point to their best available match, student s_i pointing to college c_i implies that s_i prefers c_i to any other college appearing in the cycle, i.e., $k'_{i-1} > k_i$ (indices modulo j). College c_i pointing to s_{i+1} implies $k_i \geq k'_i$. Put together, we get

$$k_1 \geq k'_1 > k_2 \geq k'_2 > k_3 \geq \ldots > k_j \geq k'_j > k_1,$$

a contradiction.

As a consequence, in the TTC algorithm with reciprocal preferences students point to their first best choices and each college c sequentially chooses its best students until either the quota is down to zero, or all students in $I_1(c)$ are admitted. In the latter case, the college next points to its best student s among those in $I_2(c)$. Thereby, c points to s as long as either s is matched to his best college c', or c' was filled and s now has to point to his second best choice c. Thus, c waits for s so that it can now sequentially choose best students from those students in $I_2(c)$ that have not been matched to their first choices. Consequently, the set of students finally matched to a college under reciprocal preferences is exactly the same as in the Boston mechanism. □

As mentioned above, the BM is manipulable and therefore a direct implication of Theorem 1 answers our question on manipulability, when preferences are reciprocal.

Corollary 1. In a matching market M with reciprocal preferences, the BM, the DA, and the TTC mechanisms are manipulable.

3 Conclusion

We provide new insights in the connection between the three most discussed algorithms in school choice. When colleges' preferences reciprocate students' preferences, the three algorithms yield the same result and this is the same as the Boston mechanism produces under original (independent) preferences. In the deferred acceptance algorithm the colleges "see" students according to the students' preferences and choose the best students at any stage. This is exactly how the Boston school choice mechanism works. In the TTC algorithm there are only cycles that consist of one college and one student so that both get the best still available match. This is exactly what happens in the DA algorithm under reciprocal preferences and in the BM under either version of preferences.

The main consequence in our view is on strategic behavior. If the colleges' preferences depend on the students' preferences in a "first preference first" manner, then desirable strategic properties get lost. It is no longer a dominant strategy for students to submit their true preferences to the DA or TTC algorithm. Still, the algorithms produce stable matchings in each Nash equilibrium of the preference revelation game, but that requires to solve a challenging coordination problem regarding the typically large number of students in such mechanisms. However, telling the truth does not necessarily result in a stable matching.

In essence, announcing that preferences depend on the other side's preferences opens a door to strategic manipulation.

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