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Abstract

The main purpose of this paper is the development of iterative plug-in algorithms for local polynomial estimation of the trend and its derivatives in macroeconomic time series. In particular, a data-driven lag-window estimator for the variance factor is proposed so that the bandwidth is selected without any parametric assumption on the stationary errors. Further analysis of the residuals using an ARMA model is discussed briefly. Moreover, confidence bounds for the trend and its derivatives are conducted using some asymptotically unbiased estimates and applied to test possible linearity of the trend. These graphical tools also provide us further detailed features about the economic development. Practical performance of the proposals is illustrated by quarterly US and UK GDP data.

Keywords: Macroeconomic time series, semiparametric modelling, nonparametric regression with dependent errors, bandwidth selection, misspecification test

1 Introduction

Local polynomial regression (Ruppert and Wand, 1994; Cleverland and Loader, 1996; Fan and Gijbels, 1996; Loader, 2012), including the well-known Henderson-filter (Henderson, 1916) and LOESS or LOWESS (locally weighted regression, Cleverland, 1979) as special cases, is one of the most widely used nonparametric techniques and a very popular approach for extracting trend from macroeconomic time series (Alexandrov et al., 2012). Great advantage of this approach are that it has automatic boundary correction under regularity conditions and that the derivatives of the trend can be easily estimated. Local polynomial regression is also an automatic kernel carpentry (Hastie and Loader, 1993) and most well-known results on kernel regression can hence be adapted to this approach. Moreover, it is also widely applied to models with short-range (Opsomer, 1997; Opsomer et al., 2001; Francisco-Fernandez and Vilar-Fernandez, 2001; Proietti and Luati, 2008) and long-range (see e.g. Beran and Feng, 2002a) dependent errors. Modified local polynomial approaches in the presence of short-range dependent errors are proposed e.g. by Vilar-Fernandez and Francisco-Fernandez (2002) using adjusted weights and by Xiao et al. (2003) through a pre-whitening transformation. Recently, difference-based local polynomial estimation of derivatives is studied by De Brabanter et al. (2013), Wang and Lin (2015) and Dai et al. (2016). This paper focuses on the development of some new data-driven algorithms for local polynomial estimation of the trend and its derivatives under short-memory errors without any parametric assumption on the stationary part. Further parametric analysis can be done based on the residuals.

A suitable bandwidth selection criterion in local polynomial with dependent errors is the plug-in (PI), because now the asymptotic effect of the dependence structure is completely reflected in the asymptotically optimal bandwidth. Two well-known special cases of the PI-rule are the IPI (iterative PI, Gasser et al., 1991) and the DPI (direct PI, Ruppert et al., 1995). The former is adapted in local polynomial regression with long-memory errors by Beran and Feng (2002a) under the SEMIFAR (semiparametric fractional autoregressive, Beran and Feng, 2002b) framework and is further extended to the estimation of the first derivative in Feng (2007). See also Herrmann et al. (1992) and Ray and Tsay (1997) for related research. A DPI-algorithm for local linear regression under an AR(1) assumption on the errors is e.g. proposed by Francisco-Fernandez and Vilar-Fernandez (2001) and Francisco-Fernandez et al. (2004). The simulation study of Francisco-Fernandez and Vilar-Fernandez (2005) shows that, according to the goodnessof-fit, the DPI-algorithm usually outperforms all of the other bandwidth selectors under consideration. On the other hand, the simulation study in Beran et al. (2009) for kernel regression with independent errors shows that the IPI works better than the DPI. In this paper only the IPI-rule will be considered.

In nonparametric regression with short-memory errors, only the variance factor in the asymptotically optimal bandwidth is affected by the correlation. To estimate this factor by means of an AR(1) model is usually a misspecification, because now the estimate is only determined by the first sample autocovariance. On the other hand, the use of a general AR(p) model or an ARMA model may still be a misspecification and the resulting estimate is sometimes unstable, in particular when local cubic regression is considered. To solve this problem Opsomer (1997) proposed a DPI lag-window estimator of the variance factor with a piecewise quadratic pilot estimate of the spectral density. In this paper a data-driven lag-window estimator of this quantity is proposed following the IPI-algorithm of Bühlmann (1996) and is combined with the main IPI-algorithm for local polynomial regression. In each main iteration, the variance factor is estimated from the residuals of a pilot estimate. Results in Altman (1990) and Francisco-Fernandez et al. (2004) for residual-based autocovariances indicate that the asymptotically optimal bandwidth for estimating the autocovariances should be larger than that for the regression function. Further, the relationship between those two bandwidths is the same as that given in Feng and Heiler (2009) for models with independent errors. This finding is employed to improve the quality of the estimated variance factor. Consequent analysis of the residuals in the final iteration by an ARMA model and its properties are discussed briefly. The proposed algorithm is then extended to select bandwidths for estimating the first and second derivatives. Furthermore, the use of the so-called asymptotically unbiased estimates based on bandwidths deduced from the corresponding selected bandwidths for conducting the confidence bounds is proposed. This idea is applied to test possible linearity of the trend in a macroeconomic time series in different ways. These tools also provide us further detailed features about the economic development. The proposed IPI-algorithm does not involve the use of any pilot estimation procedures and is hence self-contained. It is also shown that the effect of the initial bandwidth is usually negligible so that the proposed algorithms run fully automatically. Our proposals can be easily adapted to cases with independent errors or extended to models with long-memory errors.

The remaining part of the paper is organized as follows. The models and the estimators are proposed in Sections 2 and 3. The IPI-algorithms are developed in Section 4 and applied to real data in Section 5. In Section 6 the calculation of the confidence bounds and its application to linearity test are discussed. Final remarks in Section 7 conclude.

2 The proposed non- and semiparametric models

We consider the modeling of a macroeconomic time series Y_t, $t = 1, ..., n$. Given observations y_t, $t = 1, ..., n$, our goal is to analyze this time series as well as possible. The data under consideration can e.g. be the logarithmic transformation of the GDP of some country. Such time series are usually non-stationary with a deterministic trend function caused by economic growth, which can usually not be well fitted by any known parametric function. In this paper we will propose the use of the following nonparametric time series model for analyzing those data:

$$Y_t = m(x_t) + \xi_t, \tag{1}$$

where $x_t = t/n$ denotes the rescaled time, m is a smooth (deterministic) trend function and ξ_t is a zero mean stationary process with ACF (autocovariances) $\gamma_\xi(l) = \mathrm{cov}\,(\xi_1, \xi_{l+1}) = E(\xi_1 \xi_{l+1})$. This model defines a nonparametric approach for macroeconomic time series. In the sequel local polynomial estimation of m or its ν-th derivative $m^{(\nu)}$ based on model (1) will be investigated and our focus is on the development of suitable data-driven algorithms for estimating $m^{(\nu)}$ under short-range dependent errors without any parametric assumptions on ξ_t so that the estimation procedure at this stage is fully nonparametric. For this purpose, it is assumed that $\gamma_\xi(l)$ converge to zero quickly such that $\sum_{l=0}^{\infty} (l+1)^4 |\gamma_\xi(l)| < \infty$. And a data-driven IPI-algorithm for the variance factor in the asymptotically optimal bandwidth can be developed following Bühlmann (1996).

After estimating and removing the nonparametric trend, any suitable parametric model can be fitted to the residuals to carry out further econometric analysis. If an ARMA (autoregressive moving average) model is assumed, the following semiparametric ARMA model, called Semi-ARMA, is then defined:

$$\xi_t = \varphi_1 \xi_{t-1} + ... + \varphi_r \xi_{t-r} + \psi_1 \varepsilon_{t-1} + ... + \psi_s \varepsilon_{t-s} + \varepsilon_t, \tag{2}$$

where ε_t are i.i.d. (independent identically distributed) innovations.

Models (1) and (2), and closely related models are very well studied in the literature (see e.g. Altman, 1990; Hart, 1991; Francisco-Fernandez and Vilar-Fernandezs, 2001, and

Vilar-Fernandezs and Francisco-Fernandez, 2002). Application of this model for modeling growth curves based on longitudinal data is e.g. investigated by Ferreira et al. (1997). The Semi-AR model (2) can be thought of as a special case of the well known SEMIFAR (semiparametric fractional AR, Beran and Feng, 2002a) with short memory only. If model (2) is used, the above mentioned variance factor can be easily calculated from the fitted ARMA-model. Now, the fully data-driven procedure is clearly simplified. However, the use of the ARMA-assumption in the nonparametric stage has some disadvantages. And this idea will not be further studied in the current paper.

3 The semiparametric estimation procedure

3.1 Local polynomial estimation of the trend

Local polynomial regression is an automatic and flexible kernel method. The boundary problem is now well solved. This approach also clearly simplifies the use of higher order kernel functions and the estimation of derivatives. This approach is also very well studied. In this paper a local polynomial estimator of $m^{(\nu)}$, the ν-th derivative of m will hence be considered. For more details on this topic see e.g. Ruppert and Wand (1994) and Fan and Gijbels (1996). Assume that m is at least $(p+1)$-times differentiable at a point x_0. Let $\mathbf{y} = (y_1, ..., y_n)^{\mathrm{T}}$, the local polynomial estimator of $m^{(\nu)}(x)$ $(\nu \le p)$ at a point x is obtained by solving the locally weighted least squares problem

$$Q = \sum_{t=1}^{n} \left\{ y_t - \sum_{j=0}^{p} \beta_j (x_t - x)^j \right\}^2 W\left(\frac{x_t - x}{h}\right) \Rightarrow \min, \tag{3}$$

where h is the bandwidth and W is the weight function, i.e. a second order kernel function on $[-1, 1]$. Let $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, ..., \hat{\beta}_p)^{\mathrm{T}}$. Following (3) we can see that $\hat{m}^{(\nu)}(x) = \nu!\hat{\beta}_\nu$, denoted by $\hat{m}_h^{(\nu)}(x)$, is an estimator of $m^{(\nu)}(x)$, $\nu = 0, 1, ..., p$. Note that $\hat{m}^{(\nu)}(x)$ is a linear smoother with the weighting system $\mathbf{w}^\nu(x) = (w_1^\nu, ..., w_n^\nu)^{\mathrm{T}}$, where $w_i^\nu \neq 0$ only if $|x_t - x| \le h$. The weighting system and the definition of $\hat{m}^{(\nu)}$ do not depend on the error dependence structure. Assume that $p - \nu$ is odd, $\hat{m}^{(\nu)}(x)$ has automatic boundary correction and its bias is uniformly of the same order. This nice feature ensures that local polynomial regression achieves the point-wise as well as the global optimal rates of convergence in nonparametric regression (Stone, 1980, 1982; Feng and Beran, 2013).

For simplicity the weight function $W(u)$ is assumed to be a second order kernel with compact support $[-1, 1]$ of the particular form

$$W(u) = C_\mu (1 - u^2)^\mu \mathbb{I}_{[-1,1]}(u), \mu = 0, 1, ..., \tag{4}$$

where μ is the order of smoothness of a kernel function and $C_\mu = [\int_1^1 (1 - u^2)^\mu du]^{-1}$. This class of (second order) kernel functions includes the uniform kernel, the Epanechnikov or the so-called optimal kernel, the bisquare kernel and the triweight kernel for $\mu = 0, 1, 2$ or 3, as special examples. It is well-known that the Gaussian kernel is a limit case of the above mentioned class as $\mu \to \infty$. The use of other weight functions is straightforward.

3.2 Summary of known asymptotic properties

The remaining part of this paper focuses on the data-driven estimation of $m^{(\nu)}$ for $\nu = 0$, 1 or 2, with $p = 1$, 2 or 3, respectively, as well as local cubic estimation of the regression function under correlated errors. Some works on related topics are Altman (1990), Hart (1991), Robinson (1997), Opsomer et al. (2001) and Li and Li (2009), among others. In the following, we will give a brief summary of necessary asymptotic properties on $\hat{m}^{(\nu)}$ according to the comprehensive study on nonparametric regression with long-memory, short-memory and antipersistent errors of Beran and Feng (2002a). However, only the related results for models with short-memory errors will be represented.

The quality of $\hat{m}^{(\nu)}$ is usually assessed by the MISE (mean integrated squared error):

$$\text{MISE}(h) = \int_{c_b}^{d_b} [\hat{m}^{(\nu)}(x) - m^{(\nu)}(x)]^2 dx, \tag{5}$$

where $0 < c_b < d_b < 1$ are introduced to reduce the boundary effects on this measure, when the selection of the bandwidth is considered. The bandwidth that minimizes MISE(h), h_{opt} say, is hence called the optimal bandwidth. The asymptotic results of local polynomial regression can be represented by means of the so-called (asymptotically) equivalent kernel of the corresponding estimator (see e.g. Ruppert and Wand, 1994, Beran and Feng, 2002a). Let $p - \nu$ be odd, $k = p + 1$ and $K_{(\nu,k)}(u)$ denote the k-th order equivalent kernel obtained for estimating $m^{(\nu)}$ in the interior. For $\nu = 0$ and $p = 1$, $K_{(0,2)}(u)$ is e.g. the weight function itself. It is a corresponding fourth order kernel for $\nu = 0$ and $p = 3$. See e.g. Table 5.7 in Müller (1988) for explicit forms of those kernels with $\mu = 0$, 1, 2 or 3. Note in particular that in nonparametric regression for equidistant time series the asymptotic results of local polynomial estimators and kernel estimators are the same, provided that corresponding boundary kernels in the latter are employed.

Define $I[m^{(k)}] = \int_{c_b}^{d_b} [m^{(k)}(x)]^2 dx$, $\beta_{(\nu,k)} = \int_{-1}^{1} u^k K(u) du$ and $R(K) = \int K^2(u) du$. Under regularity conditions (see Theorem 2 in Beran and Feng, 2002a) the AMISE (asymptotic MISE) of $\hat{m}^{(\nu)}$ is given by:

$$\mathrm{AMISE}(h) \;=\; h^{2(k-\nu)} \frac{I[m^{(k)}]\beta^2}{[k!]^2} + \frac{2\pi c_f (d_b - c_b) R(K)}{nh^{2\nu+1}}, \tag{6}$$

where $c_f = f(0)$ and

$$f(\lambda) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \gamma_\xi(l) e^{-il\lambda}, \quad -\pi \le \lambda \le \pi, \tag{7}$$

is the spectral density of ξ_t. The asymptotically optimal bandwidth for estimating $m^{(\nu)}$ that minimizes the AMISE is given by

$$h_{\mathrm{A}} = C_{\mathrm{A}} n^{-1/(2k+1)}\,, \tag{8}$$

where

$$C_{\mathrm{A}} = \left[\frac{2\nu+1}{2(k-\nu)} \, \frac{2\pi c_f [k!]^2 (d_b - c_b) R(K)}{I[m^{(k)}]\beta^2_{(\nu,k)}} \right]^{1/(2k+1)}, \tag{9}$$

provided that $I[m^{(k)}] > 0$. The use of a bandwidth $h = O(h_{\mathrm{A}})$ leads to MISE=$O(n^{-2(k-\nu)/(2k+1)})$.

3.3 Fitting a parametric model to the residuals

After obtaining a consistent estimate of m, one can calculate the residuals $r_t = \hat{\xi}_t = y_t - \hat{m}(x_t)$. Based on these results a parametric time series model, e.g. an AR or ARMA, can be estimated and applied for further econometric analysis. Denote by θ the parameter vector of the unknown model for the stationary part. An important question in a semi-parametric time series model is to discuss the effect of the errors in $\hat{m}$ on $\hat{\theta}$. This question is studied in Beran and Feng (2002b). The authors showed that, the additional variance in $\hat{\theta}$ caused by the errors in $\hat{m}$ is always negligible, if only $\hat{m}$ is consistent. However, both, the variance and the bias in $\hat{m}$ will introduce additional bias into $\hat{\theta}$. In the current context, assume that a bandwidth of the form $h = O(n^{-\alpha})$ with $1/(4k) < \alpha < 1/2$ is used. It is shown that $\hat{\theta}$ is still $\sqrt{n}$-consistent as in a parametric model. Here, the requirement $\alpha > 1/(4k)$ ensures that the additional bias due to the bias of $\hat{m}$ is negligible and the condition $\alpha < 1/2$ ensures the negligibility of the bias of $\hat{\theta}$ due to the variance of $\hat{m}$. Note in particular that these conditions are fulfilled by the optimal bandwidth.

4 The proposed iterative plug-in algorithms

The unknown quantities in the formula of h_{A} are c_f and $I[m^{(k)}]$. If they can be estimated suitably, we can insert their estimates into (9) to obtain a selected bandwidth following the plug-in rule. For a given bandwidth, $\hat{I}[m^{(k)}]$ is not affected by the error correlation, which can be estimated using known approaches proposed for models with independent errors. Hence, only the estimation of c_f will be discussed in detail.

4.1 Data-driven estimation of c_f

Let $r_{t,\mathrm{V}} = \hat{m}_{\mathrm{V}}$ denote the residuals obtained based using a bandwidth h_{V} and $\hat{\gamma}(l)$ die sample autocovariances obtained from the residuals. In this paper we propose the use of the following lag-window estimator of c_f:

$$\hat{c}_{f,\mathrm{M}} = \frac{1}{2\pi} \sum_{l=-M}^{M} w_l \hat{\gamma}(l), \tag{10}$$

where M is the window-width (of the lag-window) and $w_l = l/(M + 0.5)$ are weights calculated according to the Bartlett-window. The IPI-algorithm proposed by Bühlmann (1996) with some minor adjustments will be used to choose the optimal window-width. The proposed data-driven IPI-algorithm for estimating c_f reads as follows.

i) Let $M_0 = [n/2]$ be the initial window-width, where $[\cdot]$ denotes the integer part.

ii) Global steps: In the j-th iteration estimate $\int (f(\lambda)^2) d\lambda$ following Bühlmann (1996). Denote the integral of the first generalized derivative of $f(\lambda)$ by $f^{(1)}(\lambda)$. Put $M_j' = [M_{j-1}/n^{2/21}]$ and estimate $\int f^{(1)}(\lambda) d\lambda$ using this window-width, the proposal in Bühlmann (1996) and the Bartlett-window. Insert the estimates into Equation (5) in Bühlmann (1996) and obtain M_j. Iterate the procedure until the selected M converges or until maximal 20 iterations. Denote the selected M by $\hat{M}_G$.

iii) Local adaptation at $\lambda = 0$: Let $M' = [\hat{M}_G/n^{2/21}]$ and calculate $\int f^{(1)}(\lambda) d\lambda$ using this window-width. Put the estimates into the formula of the local optimal bandwidth at $\lambda = 0$ in (5) of Bühlmann (1996) to obtain the finally selected window-width $\hat{M}$.

This algorithm will result in an estimate with the rate of convergence $O(n^{(-1/3)})$. An important adjustment is that we propose to use the inflation factor $n^{-2/21}$ instead of

$n^{-4/21}$ used in the original proposal. Our experiment with simulated and data examples shows that the current inflation factor works much better then the original theoretical one. But the use of this adjusted inflation factor will increase the required number of iterations. To ensure the quality of $\hat{M}$, we propose to run the iterative procedure until convergence is achieved. Compared to the original procedure, we propose the use of a large starting window-width so that the resulting estimate is more stable. If the selected window-width converges, it will usually not be affected so much by M_0. Hence, the proposed procedure can be started using any suitable M_0. Furthermore, we propose to use the Bartlett-window only throughout the whole procedure, so that it is much simpler.

In nonparametric regression with independent errors it is well-known that the asymptotically optimal bandwidth $h_{\mathrm{V,opt}}$ for calculating the residual-based variance should be larger than that for estimating the regression function. See Feng and Heiler (2009) and references therein. Comparing the asymptotic results on residual-based sample autocovariances in Francisco-Fernandez et al. (2004) with those in Feng and Heiler (2009), it can be shown that the bandwidth correction factors (denoted by CF) as given in Table 1 in Feng and Heiler (2009) are not affected by the correlation. Hence, the (asymptotically) optimal bandwidth for calculating the residuals $r_{t,\mathrm{V}}$ used in $\hat{c}_f$ should be $h_{\mathrm{V,opt}} = \mathrm{CF}h_{\mathrm{A}}$. This finding will be employed to improve the quality of $\hat{c}_f$.

4.2 The IPI-algorithm for estimating m

In this sub-section a data-driven IPI-algorithm for selecting the bandwidth for local linear and local cubic estimators $\hat{m}$ under correlated errors will be developed. Let $k = p+1$ and $p_d = p+2$. The proposed algorithm is a further development of the idea in Beran and Feng (2002a, b) in the current context, which reads as follows.

i) Start with an initial bandwidth h_0 given beforehand.

ii) For $j = 1, ...,$ estimate m using $\mathrm{CF}h_{j-1}$, calculate the corresponding residuals and obtain $\hat{c}_f$ from those residuals using the data-driven lag-window estimator.

iii) Set $h_{d,j} = h_{j-1}^{\alpha}$ with $\alpha = 5/7$ or $5/9$ for $p = 1$, and $\alpha = 9/11$ or $9/13$ for $p = 3$. Estimate $I[m^{(k)}]$ with $h_{d,j}$ and a local polynomial of order p_d, respectively. Let

$$h_j = \left(\frac{[k!]^2}{2k\beta^2} \frac{2\pi\hat{c}_f(d_b - c_b)R(K)}{I[\hat{m}^{(k)}]} \right)^{1/(2k+1)} \cdot n^{(-1)/(2k+1)}, \quad (11)$$

vi) Increase j by 1. Repeat Steps ii) and iii) until convergence is reached at some j^0 or a given number of iterations, J say, is achieved. Set $\hat{h} = h_{j^0}$ or $\hat{h} = h_J$ respectively.

In this paper we propose to use the initial bandwidths $h_0 = 0.1$ for $p = 1$ and $h_0 = 0.2$ for $p = 3$. Although local polynomial has automatic boundary correction, the estimation quality at the two ends of the time series is poorer than that in the middle part. Therefore, the values $c_b = 1 - d_b = 0.05$ are chosen so that the bandwidth is selected using 90% of the observations. The effect of h_0 will be discussed in detail later and we will see that $\hat{h}$ is usually not affected by h_0, because the IPI is a fixpoint-search procedure. The bandwidth $h_{d,j} = h_{j-1}^{\alpha}$ for estimating the k-th derivative is (roughly) calculated using a so-called EIM (exponential inflation method) instead of the original MIM (multiplicative inflation method, Gasser et al., 1991), because the MIM may fail to work in models with correlated errors. It is important to mention, that the bandwidth for estimating the derivative needs to be lager than that for estimating $\hat{m}$.

Some known results in an unpublished work on the use of different inflation factors are summarized here. The inflation rates $\alpha_{1,\mathrm{O}} = 5/7$ for $p = 1$ and $\alpha_{3,\mathrm{O}} = 9/11$ for $p = 3$ are the optimal choice so that the MSE of $I[\hat{m}^{(k)}]$ is minimized. The Algorithm using these inflation factors is called **AlgA**. The rate of convergence of a bandwidth selected by AlgA achieves the highest (relative) rate of convergence of a PI-bandwidth selector in local polynomial regression with independent or short-range dependent errors such that

$$(\hat{h} - h_{\mathrm{A}})/h_{\mathrm{A}} = O_p[n^{-2/(2k+3)}].$$

For $k = 2$ this result reduces to the well-known rate of convergence $O_p(n^{-2/7})$ as given by Ruppert et al. (1995) and Francisco-Fernandez et al. (2004). The two inflation rates $\alpha_{1,\mathrm{N}} = 5/9$ for $p = 1$ and $\alpha_{3,\mathrm{N}} = 9/13$ for $p = 3$ are chosen to minimize the MISE of $\hat{m}^{(k)}$, which are stronger than the optimal ones. Those two inflation factors are especially interesting for practical purpose with a small sample size and a complex trend function. The algorithm using these inflation factors will be called **AlgB**. The rate of convergence of a bandwidth selected by AlgB in local polynomial regression with independent or short-range dependent errors is given by

$$(\hat{h} - h_{\mathrm{A}})/h_{\mathrm{A}} = O_p[n^{-2/(2k+5)}].$$

There is another inflation factor $\alpha_{\mathrm{V}} = 1/2$, which can be used so that the variance of $I[\hat{m}^{(k)}]$ will be $\sqrt{n}$-consistent and the resulting bandwidth is most stable with a lower rate of convergence. We will see that this inflation factor might sometimes be a suitable choice. In all of the cases, the rate of convergence of $\hat{h}$ is not affected by the errors in $\hat{c}_f$. It should particularly indicate that, the rate of convergence of the resulting regression estimates is not affected by the rate of convergence of the selected bandwidth, which stays the same, as far as the selected bandwidth is consistent. The choice of different bandwidth selectors will affect the finite sample behavior of the nonparametric regression approach.

4.3 Data-driven estimation of m' and m''

The proposed IPI-algorithm can be easily adapted to select bandwidths for estimating $\hat{m}^{(\nu)}$. This is interesting in an economic sense in particular for log-linear growth trends. Two particular cases with $\nu = 1$ or 2 will be considered. In the following an data-driven algorithm for this purpose will be developed, where c_f is estimated by means of a data-driven pilot estimate of m of the order p_p, say. Assume that we would like to estimate $m^{(\nu)}$ with local polynomial of order $p = \nu + 1$ and hence $k = \nu + 2$. As before, assume that $m^{(k)}$ will be estimated with $p_d = p + 2$. The following procedure is proposed.

i) Carry out the algorithm for estimating m with $p_p = 1$ or $p_p = 3$ to obtain $\hat{c}_f$;

ii) Carry out a similar IPI-procedure as proposed in the last section but with fixed $\hat{c}_f$ obtained in i) to select the bandwidth for estimating $m^{(\nu)}$ according to (8).

Again, only the four weight functions with $\mu = 0$, 1, 2 and 3 are considered. Explicit forms of the equivalent kernels for estimating $m^{(\nu)}$ may be found in Table 5.7 of Müller (1988). The corresponding inflation factor can be calculated from p or k.

5 Practical performance

An R-package for local polynomial smoothing of a time series was developed, which applies to $\hat{m}^{(\nu)}$ of any order ν with $p - \nu$ odd, a suitable bandwidth h_ν chosen beforehand and a kernel weighting function defined in (5) with any $\mu \geq 0$. The proposed data-driven algorithms for local linear and local cubic estimators of $\hat{m}(x)$, local quadratic estimates of m' and local cubic estimates of m'' are built-in. For selecting the bandwidth in local linear regression any kernel weighting function defined in (5) can be used. Due to the required calculation of the corresponding kernel constants, data-driven estimators in the other cases are only defined for $\mu = 0$, 1, 2 and 3.

5.1 Application to real data examples

The quarterly US-GDP (million dollars in 2009 dollars) for the first quarter of 1947 to the first quarter of 2016 and the quarterly UK-GDP (million pounds) for the first quarter of 1955 to the first quarter of 2016 are chosen to show the practical performance of the proposed data-driven algorithms. The selected bandwidths for estimating the regression function with $p = 1$ and 3 and $\mu = 1$, 2 and 3, are displayed in Table 1. Results using the uniform kernel as the weight function are omitted. For local linear regression both

Table 1: Selected bandwidths for estimating m in all cases

		$p=1$			$p=3$		
μ		1	2	3	1	2	3
AlgA	US	0.112	0.134	0.155	0.150	0.165	0.180
	UK	0.122	0.142	0.167	0.142	0.163	0.178
AlgB	US	0.110	0.134	0.155	0.168	0.197	0.220
	UK	0.122	0.148	0.170	0.191	0.214	0.243

of AlgA and AlgB work well. In this case the optimal choice is $\alpha_{1,\mathrm{O}} = 5/7$, because it has the highest rate of convergence. However, for local cubic regression, the selected bandwidths for the examples under consideration is sometimes unstable. And the use of the optimal choice $\alpha_{3,\mathrm{O}} = 9/11$ may result in a clear underestimation. The reason might be that the sample size is not large enough for the higher order of the polynomial. Thus, we propose the use of the stronger inflation factor $\alpha_{3,\mathrm{N}} = 9/13$. It is important to note that the Epanechnikov kernel is theoretically the optimal weight function. Moreover, it is although possible to use the bisquare or the triweight kernel which are smoother. In our example, the difference using different kernel functions is minor. Estimation results with $\mu = 2$, AlgA for $p = 1$ and AlgB for $p = 3$ together with the observations are displayed in the upper and lower panels in Figure 1 for the US and UK data sets, respectively. From Figures 1(a) and 1(b) we can see that the automatically smoothing results with $p = 1$ and $p = 3$ almost coincide each other. The differences between the two estimates are minor. What one can discover by eye is e.g. the difference between them at some left endpoints, in particular for the US data. Also here the difference is unclear. These features confirm that the proposed data-driven algorithm works very well in practice. Comparing the curves for the US and the UK we can see that the development trend in the log-data of the GDP in both countries share many similar common features. A detailed discussion on further economic meanings of those empirical results is beyond the aim of this paper.

For analyzing the parametric part, we propose the use of the residuals of the more stable local linear estimate. By means of the BIC (Bayesian information criterion) the following AR(3) and ARMA(3, 1) models

$$\xi_{t,\mathrm{US}} = 1.219\xi_{t-1,\mathrm{US}} - 0.182\xi_{t-2,\mathrm{US}} - 0.159\xi_{t-3,\mathrm{US}} + \varepsilon_{t,\mathrm{US}} \text{ and}$$

$$\xi_{t,\mathrm{UK}} = 1.603\xi_{t-1,\mathrm{UK}} - 0.494\xi_{t-2,\mathrm{UK}} - 0.165\xi_{t-3,\mathrm{UK}} - 0.702\varepsilon_{t-1,\mathrm{UK}} + \varepsilon_{t,\mathrm{UK}}$$

are obtained for the two examples, respectively. These parametric models can be applied for further econometric analysis, in particular for carrying out short-term forecasting.

Again, $\hat{c}_f$ obtained by $p = 1$ will be used for selecting the bandwidth for the derivatives. Moreover, the estimation of the derivatives usually requires a relatively larger bandwidth. It is found that now the theoretically optimal inflation rate does not work well in practice. Therefore, we switch to the inflation factors discussed in section 4.2. According to our experience, we will propose to estimate m' with the naive inflation factor $\alpha_{2,\mathrm{N}} = 7/11$ and to estimate m'' with the corresponding most stable inflation factor $\alpha_{3,\mathrm{V}} = 1/2$, respectively. Using these specifications we have $\hat{h}_1^d = 0.211$ and $\hat{h}_2^d = 0.277$ for the US data, and $\hat{h}_1^d = 0.242$ and $\hat{h}_2^d = 0.275$ for the UK data, respectively. One can see that those selected bandwidths are quite reasonable. The resulting estimates are also satisfactory, which are omitted to save space. Related results may be found in Figures 3 and 4 below.

5.2 Effect of the initial bandwidth

The IPI-algorithm is a fixpoint-search procedure. Theoretically, the finally selected bandwidth hence does not depend on the initial bandwidth h_0, if this is chosen from a suitable interval. This fact was studied in detail e.g. by Herrmann and Gasser (1994) and Feng (2013). Due to this observation the use of $h_0 = 0.1$ for $p = 1$, $h_0 = 0.15$ for estimating m' with $p = 2$ and $h_0 = 0.2$ for $p = 3$ is simply proposed. It is clear that the original idea of Gasser et al. (1991) using a very small initial $h_0 = n^{-1}$ does not work in our context, because we already need to estimate c_f at the beginning from the residuals. Usually, the finally selected bandwidth $\hat{h}$ will not be affected by h_0. But sometimes, different h_0 may lead to a few different selected bandwidths, caused by the fact that for some data sets there may be some local minimal points of the MISE.

Possible effects of h_0 are shown in Figure 2, where $\hat{h}$ is selected using all initial bandwidths between 0.02 and 0.40 with a grid 0.01, i.e. with $b_0 = 0.02$, 0.03, ..., 0.39, 0.40, the four selected cases are displayed against h_0 in Figure 1. From this figure we can see that in three figures the selected bandwidths using all of those initial bandwidths are always the same, which show that in a typical case h_0 chosen from a suitable range will not affect the final result. An exceptional case is that for the US data with $p = 3$, we see now, if $h_0 < 0.20$, the selected bandwidths are almost the same as given in Table 1. For an initial bandwidth $h_0 \geq 0.21$ however, the selected bandwidth $\hat{h}^*$ becomes immediately 0.339 and stays at this level thereafter, which will lead to a clear overestimation. But this phenomenon will not cause any confusion in our decision, because it is clear that this very large bandwidth should not be used. Moreover, such a phenomenon will occur rarely only when using $p = 3$ and the user will detect this problem easily. In the provided R package, the reader is allowed to run the algorithm using different h_0. If it is found that the selected bandwidths depend on the initial bandwidth, further discussion is necessary to determine, which of them should be used.

6 Some graphical tools for testing the linearity

6.1 Asymptotically unbiased estimators and confidence bounds

An asymptotically (relatively) unbiased estimate of $m^{(\nu)}$, which is sometimes very helpful in practice, e.g. for conducting reasonable confidence bounds, can be obtained by applying a bandwidth $h_{\text{ub}} = o[n^{-1/(2k+1)}]$, which is of a smaller order of magnitude than the optimal bandwidth. Compared to the variance, the bias is now asymptotically negligible and the error in $\hat{m}^{(\nu)}$ is dominated by the variance part. Hence, we have

$$\sqrt{nh}[m^{(\nu)}(x) - \hat{m}^{(\nu)}(x)] \xrightarrow{\mathfrak{D}} N[0, 2\pi c_f R(x)]. \tag{12}$$

Note that there is no unique definition of h_{ub} and hence there is no rule for selecting it. If the estimation of the regression function m is considered, a reasonable choice of a bandwidth for this purpose is with $h = (h_{\text{A}})^{(2k+1)/(2k)} = O[n^{-1/(2k)}]$ such that the bias in $\hat{m}^{(\nu)}$ achieves the parametric convergence rate $O(n^{-1/2})$. This leads to the choices of $h = O(n^{-1/4})$ for a local linear estimator and $h = O(n^{-1/8})$ for a local cubic estimator of m. The use of a bandwidth of an even smaller order is no longer necessary. Based on the data-driven results, a bandwidth $\hat{h}_{\text{ub}} = (\hat{h})^{(2k+1)/(2k)}$ of the above mentioned order can be used to obtain asymptotically unbiased estimates and the corresponding confidence bounds. This idea will also be employed to deduce bandwidths for calculating asymptotically unbiased estimates of m', m'' as well as their confidence bounds from the corresponding selected bandwidths. This leads to $\hat{h}^d_{1,\text{ub}} = (\hat{h}^d_1)^{7/6}$ for estimating m' and $\hat{h}^d_{2,\text{ub}} = (\hat{h}^d_2)^{9/8}$ for estimating m'', respectively.

6.2 The methodology

The most simple econometric approach for analyzing a growth curve is the use of a linear trend function. In growth theory the linear trend is in accordance with Jones (2002) a good starting point. Thus we use the overall linear trend as a benchmark and show that it could be a misspecification for different periods. Our purpose is to use the confidence bounds proposed in the last sub-section as some new graphical tools to test possible linearity of the trend nonparametrically. Related research on nonparametric test of linearity was e.g. done by Hjellvik et al. (1998). Here, the null-hypothesis to be tested is

$$H_0 : y_t = \beta_0 + \beta_1 x_t + \varepsilon_t, \text{ against } H_1 : \text{“}\textit{A linear trend is not suitable.}\text{”}$$

Three variants to test H_0 based on confidence bounds for m, m' and m'', respectively, will be proposed. Firstly, an unbiased local linear estimator $\hat{m}_{\text{ub}}(x)$ and the corresponding confidence bounds at given confidence level $1-\alpha$ can be calculated using the bias-corrected bandwidth $\hat{h}_{\text{ub}} = (\hat{h})^{5/4}$, where $\hat{h}$ is the selected bandwidth for estimating m with $p = 1$. To this end the selected bandwidth with $p = 3$ can also be used, which will however not be considered to save space. Furthermore, a linear trend $\hat{m}_{\text{L}}$ can be fitted to the data as a comparison. Under H_0 the estimated trend $\hat{m}_{\text{L}}$ is unbiased and $\sqrt{n}$-consistent. Those results can be displayed all together in a figure. If the estimated confidence bounds and $\hat{m}_{\text{L}}$ do not intersect each other at clearly more than $100\alpha\%$ of the observation points, H_0 should be rejected at the significance level α, because those confidence bounds are asymptotically unbiased. Note that the fitted linear regression provides an estimation for the ideal development of the macroeconomy under consideration. And the above mentioned figure hence indicates where the economic development was above and where it was below the average level.

Secondly, under H_0 the first derivative of m should always be β_1. Now, we can conduct an asymptotically unbiased estimate of m', $\hat{m}'_{\text{ub}}(x)$ and the corresponding confidence bounds using the bandwidth $\hat{h}^d_{1,\text{ub}} = (\hat{h}^d_1)^{7/6}$. Then $\hat{m}'_{\text{ub}}$ and the corresponding confidence bounds can be displayed and compared to the average level of $\hat{m}'$, which is an estimate of β_1. If the the upper confidence bounds are below this level or the lower confidence bounds are beyond this level at clearly more than $100\alpha\%$ of the observation points, H_0 should be rejected at the significance level α.

Thirdly, an asymptotically unbiased estimate of m'', $\hat{m}''_{\text{ub}}(x)$ and the corresponding confidence bounds can be calculated using the bandwidth $\hat{h}^d_{2,\text{ub}} = (\hat{h}^d_2)^{9/8}$. Then $\hat{m}''_{\text{ub}}$ and the corresponding confidence bounds can be displayed and compared to the level zero, since m'' should be uniformly zero, if H_0 holds. In particular, this method does not involve the use of an auxiliary estimate. Now, if the the upper confidence bounds are below zero or the lower confidence bounds are beyond zero at clearly more than $100\alpha\%$ of the observation points, H_0 should be rejected at the significance level α. Results in the last two cases are not only interesting for carrying out the above mentioned test, but can also indicate the details of the growth rate and the curvature of the trend in a macroeconomic time series under consideration.

6.3 Empirical results for the data examples

The proposed methods are applied to the two data examples. For the regression function the two selected examples in figure 1 with $p = 1$ are used. The estimates of m' and m'' are those mentioned at the end of Section 5.2 with the pilot estimate $\hat{c}_f$ obtained by $p_p = 1$. The bandwidth for calculating the asymptotically unbiased estimates and their confidence bounds are calculated according to the formulas given in Section 6.2. Results for the US example are displayed in Figure 3. From this figure we see that all of the three methods indicate the similar conclusion that a linear trend is not suitable and should not be used. The fact is more clear at both ends of this time series, and in particular at the current end, where a decreasing growth rate of the US economy is observable. We can also see that the economic development between the middle of 1970s and 2000 was likely with a linear trend (in the log-data). Furthermore, also local polynomial regression has automatic boundary correction (of the order of magnitude of the bias), the estimation quality at a boundary point is poorer than that in the interior. As expected, the higher the order of the derivative the more clear this effect so that the length of the confidence bounds for m' and in particular for m'' at points near to the end-points becomes several times of that at an interior point. This leads to slightly different conclusions according to different approaches. Both of the first two panels in Figure 3 indicate that the growth rate in the US is decreasing for more than a decade. However, the confidence bounds for m'' show that a linear trend at the current end cannot be rejected, but this is due to an increase in the variance at the boundary. The results for the UK data are shown in Figure 4. Similar conclusions as for the US example can be drawn here. A clear difference between the two economies might be that the non-linearity in the UK economic development is more obvious than the non-linearity in the US economic development.

7 Conclusions

Some IPI-algorithms for selecting the bandwidth in local regression for the regression function and its derivatives are proposed. Our algorithm for estimating the regression function can be thought of as a data-driven alternative to other well-known detrending methods and provides hence a new approach for the analysis of (macro-)economic time series. The IPI-algorithms for estimating the derivatives contributes to the current research on this interesting topic. Closely related recent studies are for instance De Brabanter et al. (2013), Wang and Lin (2015) and Dat et al. (2016), who proposed data-driven local polynomial estimation of derivatives based on difference sequence under dependent or independent errors. Theoretically, our proposal can achieve the optimal rate of convergence (Stone, 1980, 1982; Feng and Beran, 2013). It is however still unclear, whether derivative estimators based on difference sequence also share this theoretical advantage. Furthermore, it is worth to carry out a simulation study on the theoretical and practical behaviors of the proposed algorithms and to compare the current proposal to other filtering methods. A suitable forecasting procedure based on this model should also be developed to extend the application spectrum of the proposed approach. Finally, a study on the properties of the proposed linearity test methods is also of great interest.

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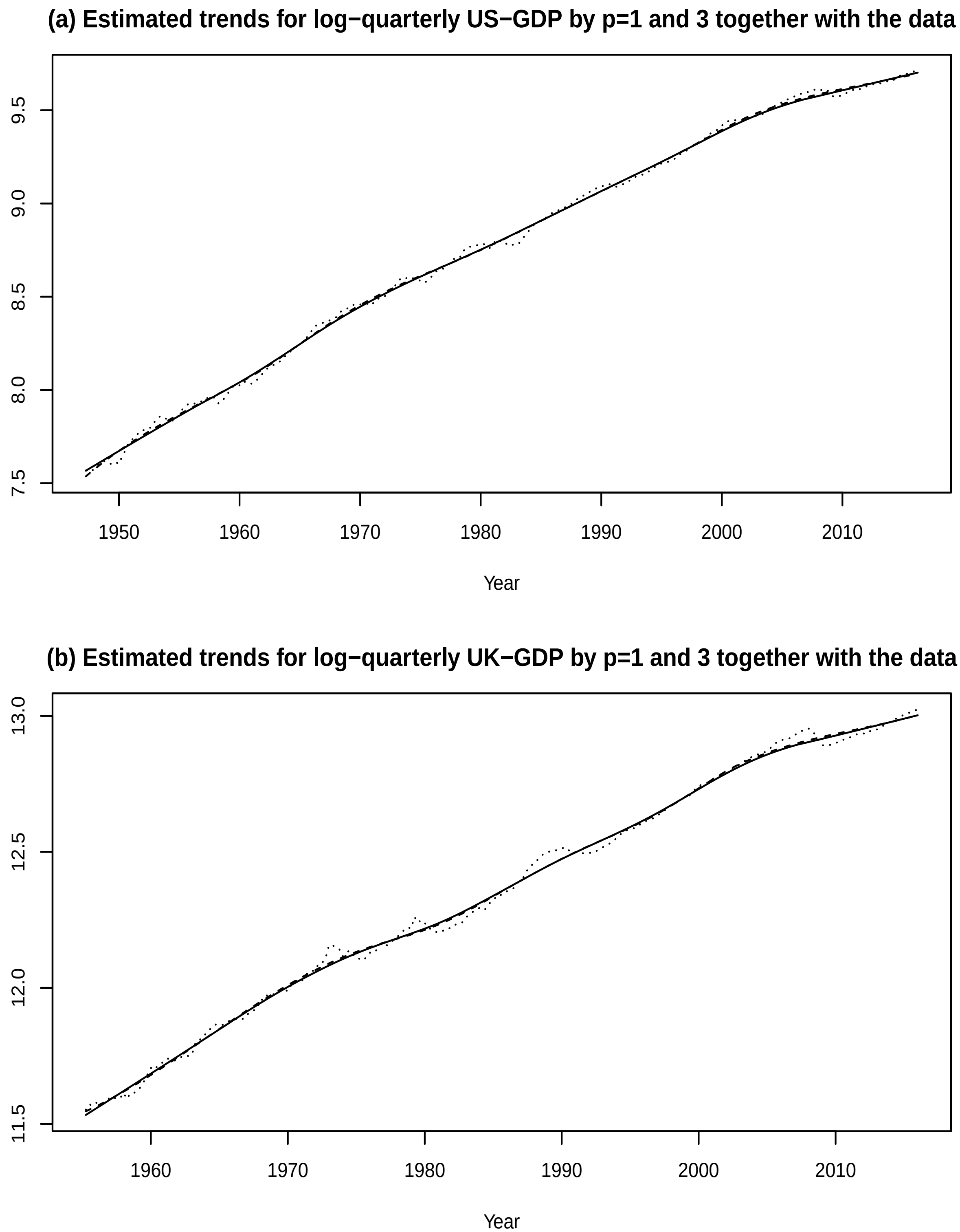

Figure 1: Local linear and local cubic estimates of the trend in the Log-GDP of the US.

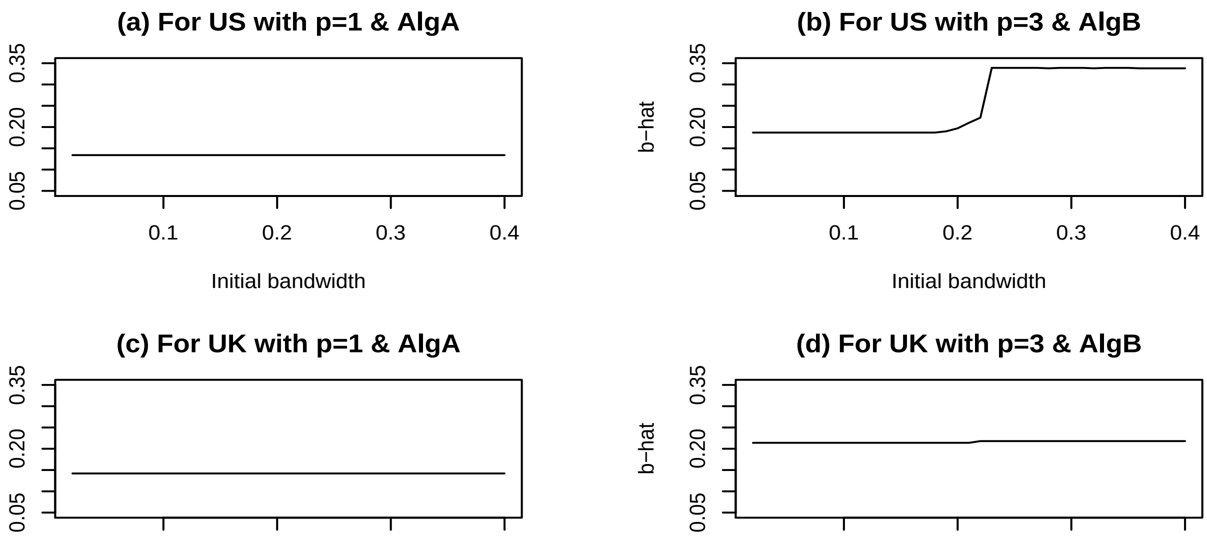

Figure 2: The effect of the initial bandwidth on $\hat{h}$ in the four selected cases.

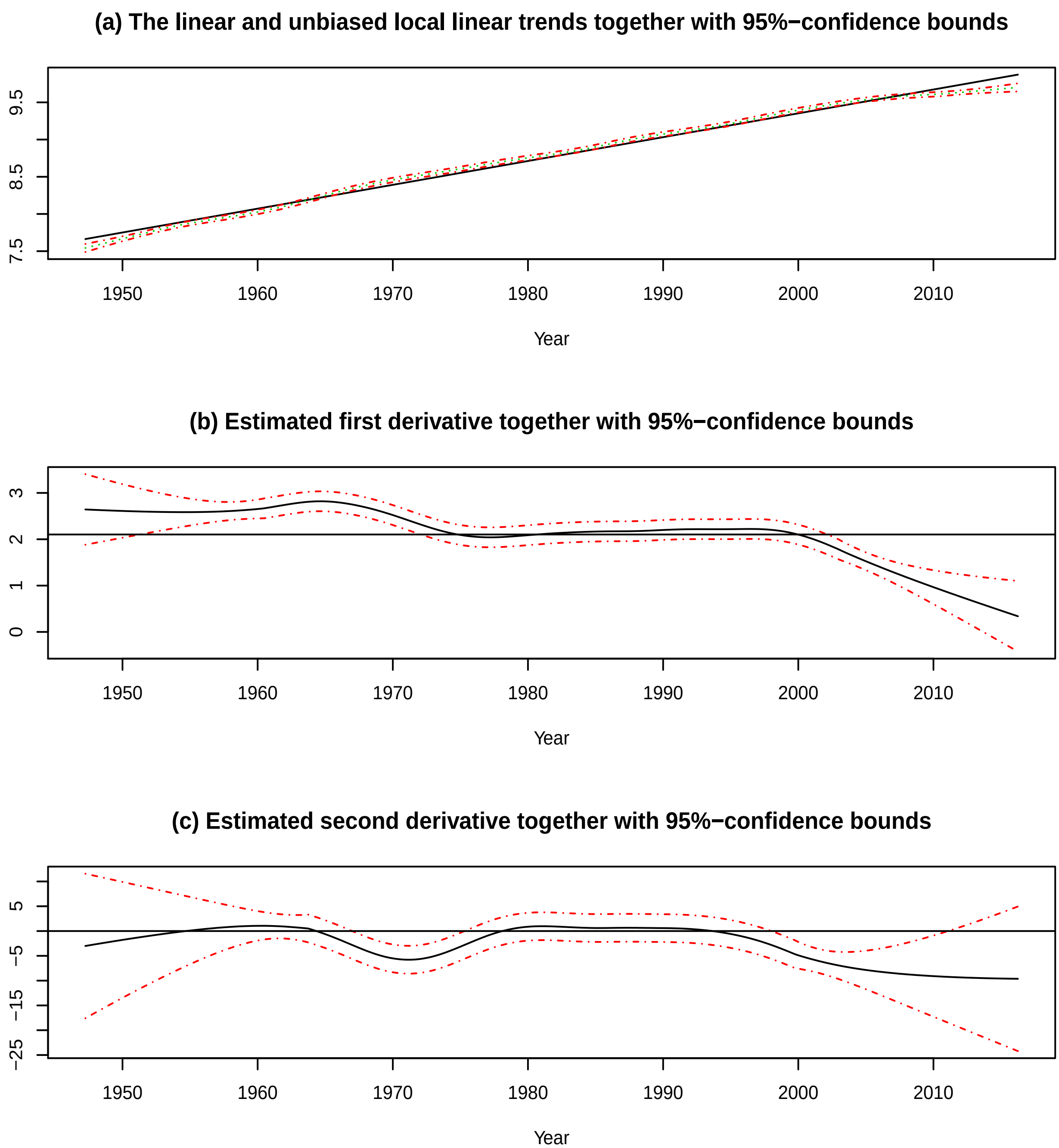

Figure 3: Results of different linearity tests for the US-data.

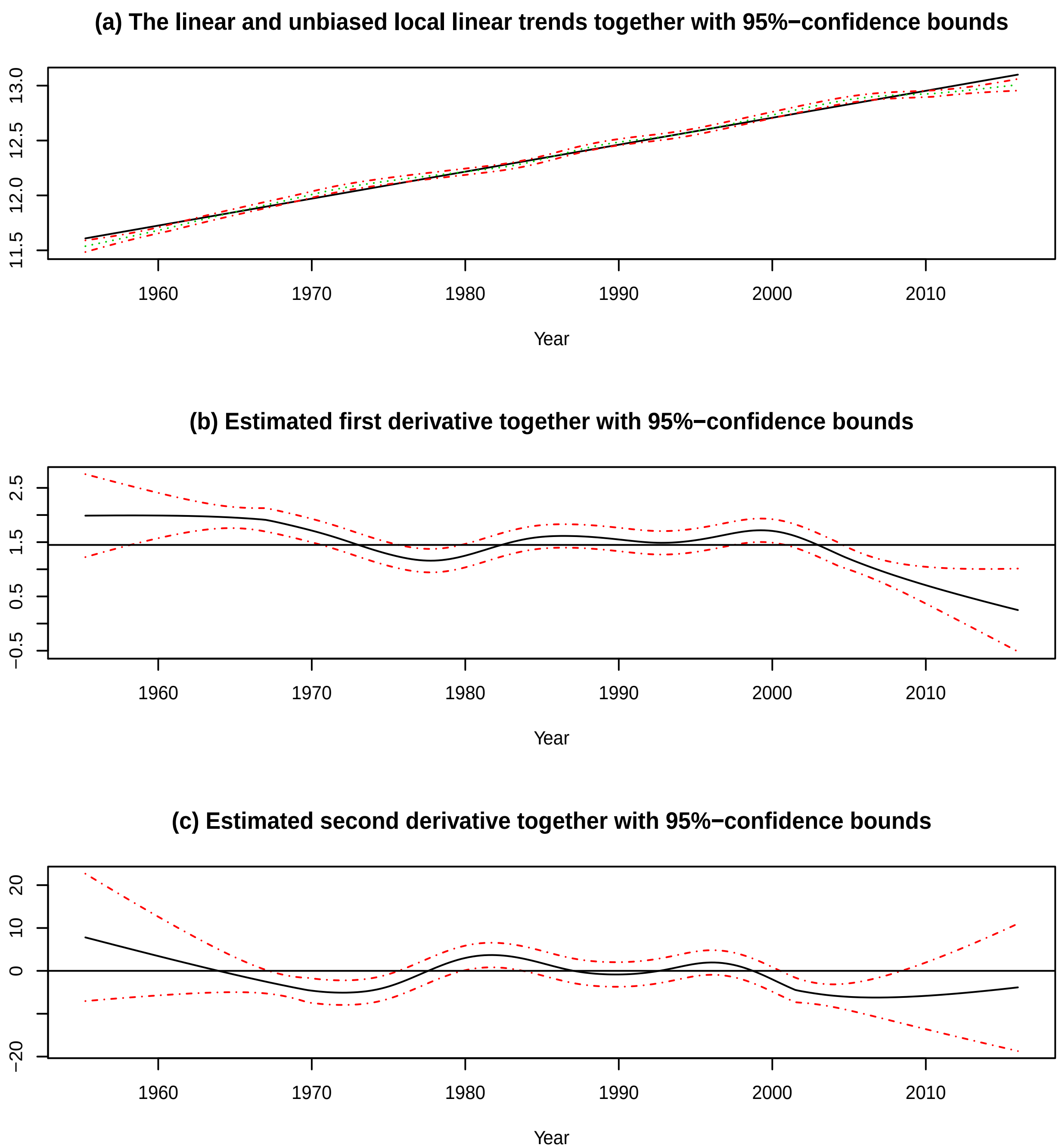

Figure 4: Results of different linearity tests for the UK-data.

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