Robust Equilibria in Location Games

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Abstract

In the framework of spatial competition, two or more players strategically choose a location in order to attract consumers. It is assumed standardly that consumers with the same favorite location fully agree on the ranking of all possible locations. To investigate the necessity of this questionable and restrictive assumption, we model heterogeneity in consumers' distance perceptions by individual edge lengths of a given graph. A profile of location choices is called a "robust equilibrium" if it is a Nash equilibrium in several games which differ only by the consumers' perceptions of distances. For a finite number of players and any distribution of consumers, we provide a full characterization of all robust equilibria and derive structural conditions for their existence. Furthermore, we discuss whether the classical observations of minimal differentiation and inefficiency are robust phenomena. Thereby, we find strong support for an old conjecture that in equilibrium firms form local clusters.

Keywords: spatial competition, Hotelling-Downs, networks, graphs, Nash equilibrium, median, minimal differentiation

JEL Classification: C72, D49, P16, D43

1 Introduction

In his classic example, Harold Hotelling illustrates competition in a heterogeneous market by two firms that consider where to place their shop on a main street (Hotelling, 1929). Ever since, this model of spatial competition has inspired a tremendous amount of research in various disciplines. Starting with Downs (1957), it is used to analyze the positioning of political candidates competing for voters (e.g., Mueller, 2003; Roemer, 2001) and to analyze the positioning of products in order to

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attract consumers (e.g., Carpenter, 1989; Salop, 1979). In the year 2012 alone, Hotelling has been cited more than 430 and Downs even more than 1100 times.[1] Moreover, the model implication of *minimal differentiation* is known far beyond scholarly circles. In this paper we want to challenge a fundamental aspect of the Hotelling-Downs approach.

Throughout the literature (of spatial competition) it has been virtually always assumed that consumers or voters who prefer the same position fully agree upon the ranking of the other alternatives, i.e., they have identical preferences or utility functions. This is hard to justify when we think of voters of the same political party who disagree about the second-best party, or of consumers with the same favorite brand but disagreement about the ordering of two other brands. In fact, this very strong homogeneity requirement of the standard Hotelling-Downs set-up can be considered as driven by the assumption that all consumers/voters use the same distance measure, i.e., there is full agreement on the distance between two positions.

Consider, for example, a poll on a group of voters about their favorite tax rate. The answers can be displayed as locations on a line. Location games that capture this application consider classically two political candidates who strategically choose a tax rate which they propose to the voters. Thereby it is standardly assumed that (a) each voter casts his vote for the candidate that is closest to him and (b) all voters asses the distances between the candidates homogeneously. In combination these two assumptions are not at all innocent. They hide the homogeneity requirement that all voters who consider a tax rate of 10 percent, for example, as their favorite alternative, are supposed to rank any two tax rates, like 2 percent and 20 percent, for example, in exactly the same order. Since this requirement is unnaturally strong, the classical result that two vote maximizing candidates choose the median location (Hotelling, 1929) stands apparently on highly questionable grounds. A way to avoid this issue would be to ask the participants in the poll not only about their favorite tax rate, but about a full ranking of the alternative tax rates. Apart from practical problems, the downside of such an approach is the informational requirement that political candidates know the full assessment of every voter. That is, we have replaced a questionable requirement by another one. A solution to this issue relates back to the seminal contribution of Black (1948). He examined single-peaked preferences on a line, which has the same effect as voters who are allowed to asses the "distances" between different tax rates individually. Black's result that under single-peaked preferences the median voter wins in majority voting against any other alternative has the following implication for the situation of spatial competition outlined above: In *any* location game that is consistent with the poll, both candidates choose the median tax rate in equilibrium. In that sense the classical result is *robust*.

The example on tax rates illustrates that in two-player location games on a line the questionable requirement of homogeneous distance perceptions is not driving the final outcome. However, for all other cases – in particular, for more than two players and for multidimensional spaces – robustness of the results is an open problem. If one can show that the model assumption is not driving the results, then the model is put on a solid foundation. This issue, although fundamental, seems to have been overlooked in the the – rich and exciting – history of location games.

In this paper we want to scrutinize for given outcomes of spatial competition whether they rely on homogeneous distance perceptions or not. To this end we formalize subjective distance perceptions

[1] Google Scholar, February 7, 2013.

as individual edge lengths of a graph.[2] A formal description of consumers/voters of this type leads to a non-cooperative game between p players, which are interpreted as firms or political candidates. In this game players simultaneously choose a location in order to maximize the number of consumers/voters they can attract. An equilibrium is then called *robust* if it is an equilibrium for all possible distance perceptions that are based on the same underlying structure (a line, for example). In other words, our modeling approach boils down to defining a stronger notion of equilibrium which we call *robust equilibrium.* It is defined directly on the situation of spatial competition, i.e., the underlying space and the distribution of agents (such as the poll on tax rates). Formally, several of location games correspond to the same situation of spatial competition, one for each setting of individual distance perceptions; and a robust equilibrium is a Nash equilibrium in any of these games. In particular, it is also a Nash equilibrium in the standard case of homogeneous distances.

A key result in our paper is the characterization of robust equilibria by four conditions which are jointly necessary and sufficient. It is based on partitioning the underlying space into "hinterlands" and "competitive zones". Applying this result allows us first of all to judge which of the standard results are robust. In fact, we find that several outcomes do not depend on the assumption of homogeneous distances, but others do. In the second part of the paper we examine general properties of robust equilibria. Among them is the central issue of minimal differentiation (e.g., d'Aspremont *et al.*, 1979; de Palma *et al.*, 1985, 1990; Eaton and Lipsey, 1975; Economides, 1986; Król, 2012; Meagher and Zauner, 2004). It turns out that robust equilibria satisfy a local variant of minimal differentiation, i.e., they induce reduced games in which the corresponding players are minimally differentiated. This result provides strong support for the "principle of minimal clustering" which has been proposed in the seminal contribution of Eaton and Lipsey (1975). Indeed, for any number of players, any underlying structure, and any distribution of agents, robust equilibria are characterized by clusters of players. That is, the players are jointly located on what we show to be the appropriately defined median of the local area. Based on this result, we discuss the welfare implications for consumers and observe that almost all robust equilibria are not Pareto efficient. Consumers would unambiguously improve if some firm would be relocated appropriately. We finally, elaborate on the conditions for the existence of robust equilibria. We analyze how the spatial structure and the distribution of consumers/voters guarantee, admit, or preclude the existence of robust equilibria. Interestingly, two very common assumptions in the literature – (a) uniform distribution of consumers/voters and (b) one-dimensional space such as cycle or line structures – are mutually exclusive in the sense that for higher numbers of players robust equilibria require that one of them is not satisfied.

Related Literature

There is an immense body of literature on spatial competition. While the original Hotelling-Downs framework is restricted to a one-dimensional space, a uniform distribution of agents, and only two players, many authors have attempted to relax these restrictions. To do so, one branch

[2]This can be shown to be equivalent to the assumption of single-peaked preferences on certain domains. For example, if the underlying structure is a line graph, then this assumption is equivalent to the standard notion of single-peakedness.

of the literature has followed a continuous modeling approach within the Euclidean space $\mathbb{R}^k$ (e.g., d'Aspremont *et al.*, 1979; Economides, 1986), while a second branch replaces the Euclidean space by a graph (e.g., Labbé and Hakimi, 1991). Because the history of both branches is rich and long, providing a summary which covers all of it would exceed the scope of our paper. We restrict ourselves here to list several surveys on the topic and to discuss the most closely related works.

A broad overview and taxonomy of literature on spatial competition can be found in Eiselt *et al.* (1993). Based on five components (the underlying space, the number of players, the pricing policy, the rules of the game, and the behavior of the agents) the authors provide a bibliography for competitive location models. While this summary is not limited to certain subbranches, more specific surveys have been written on spatial models of consumer product spaces (Lancaster, 1990), on spatial competition in continuous space (Gabszewicz and Thisse, 1992), on spatial models of political competition (Mueller, 2003; Osborne, 1995), on competition in discrete location models (Kress and Pesch, 2011; Plastria, 2001), and on one-stage competition in location models (Eiselt and Marianov, 2011).

Although there are many variations and relaxations of spatial competition, virtually all of the models rely on the assumption of homogeneous distance perceptions. In order to examine to which extent this simplification is driving the results we will focus on the first stage of Hotelling's game, i.e., we will investigate the location choices of the players but we will not include additional variables such as prices. Similar approaches have been used, for example, by Eaton and Lipsey (1975), Denzau *et al.* (1985), and Braid (2005) who also concentrate on spatial competition by assuming fixed (and equal) prices. Nevertheless, extending our approach to a two-stage game would be a potential next step for further research. Integrating heterogeneous consumer behavior into a model of spatial competition has been attempted by a few studies only. Among them are de Palma *et al.* (1985, 1990) and Rhee (1996) who find that ambiguity about consumers' (or voters') behavior may lead to minimal differentiation. More specifically, they show that if the consumers' preferences do not only depend on prices and distances but also on inherent product characteristics and, furthermore, the firms have incomplete information about consumers' tastes, then Hotelling's main result can be restored under certain conditions. However, this conclusion is not confirmed in closely related models where the authors assume that the exact position of demand is unknown (see, for example, Meagher and Zauner, 2004 or Król, 2012). Thus, the validity of minimal differentiation under heterogeneous agents is still an open problem and the same holds true for the main implications, like that spatial competition generically does not lead to socially efficient outcomes, for example. The previously cited publications differ from our work in at least two important aspects. First, in these works, players are assumed to have a probability distribution for the behavior of agents. In our work, uncertainty is not explicitly modeled but only enters implicitly as robust equilibria do not depend on specification details about the agents' behavior. Second, the way we model and interpret heterogeneity differs from the approaches of the other authors. In our setting, the agents apply individual distances to compare specific product variations but the preferences do not depend on inherent product characteristics. To model this in a convenient way we use a graph-based approach. We believe that our definitions are more intuitive in discrete spaces than in the plane and that this approach helps highlight the difference between homogeneous and heterogeneous agents. However, the main questions of our work are not restricted to graphs and thus our contribution should also be interesting in a more general context. To the best of our knowledge, this is the first paper that assesses robustness of location games with respect to different distance perceptions.

From a technical point of view, the model from Eiselt and Laporte (1991, 1993) is heavily related to ours. In these publications the authors show for homogeneous agents that the two-player and three-player cases on trees always result in some kind of minimal differentiation. We will check whether this is also true in our more general context of more than two players and arbitrary graphs. More recently, Gur and Stier-Moses (2011) extended the two-player case to arbitrary graphs. Further recent contributions also stem from computer science such as the works of Mavronicolas *et al.* (2008), Godinho and Dias (2010), and Jiang *et al.* (2011). Still, the issue of heterogeneous distances is not addressed in these publications.

2 The Model

Our modeling approach proceeds in two steps. First we consider, as usual, a non-cooperative game between players (the firms/candidates) who are able to occupy a position or object, respectively. The agents (consumers/voters) are still attracted by the player(s) located closest to them but now their distance perceptions may be completely subjective. More specifically, the agents agree on the underlying space which is modeled by means of a graph (Subsection 2.1), but in our setting they may individually asses the similarity between the objects (Subsection 2.2). Then, in the second step, we study whether equilibria of the game are robust with respect to perturbations of the distance perceptions. To this end, roughly speaking, we fully abandon the distances. This means formally that an outcome is called robust if it is an equilibrium for all possible edge lengths of the same underlying graph (Subsection 2.3). If this is satisfied, the outcome is completely independent of subjective distance perceptions and then the standard case of homogeneous distances is a well-justified simplification.

2.1 Definitions of Graphs

An undirected graph (X, E) consists of a set of *vertices* or *nodes* X and a set of *edges* E where each edge is a subset of the vertices of size two. Let X be a finite set of size $\xi \geq 2$. For brevity we write xy or yx for an edge $\{x, y\} \in E$. Given a graph (X, E), we denote by $N_x := \{y \in X \mid xy \in E\}$ the set of *neighbors* of a node x. The number of edges/neighbors is its degree $\deg_x := |N_x(g)|$. Furthermore, $Y \subseteq X \setminus \{x\}$ is neighboring to $x \in X$ if there exists some $y \in Y$ with $xy \in E$.

A *path* from $x \in X$ to $x' \in X$ in (X, E) is a sequence of distinct nodes $(x_1, ..., x_T)$ such that $x_1 = x$, $x_T = x'$, and $x_t x_{t+1} \in E$ for all $t \in \{1, ..., T-1\}$. A set of nodes $Y \subseteq X$ is said to be connected if for any pair $y, y' \in Y$ there exists a path between the two nodes. A set of connected nodes is called a *component* if there is no path to nodes outside of this set, i.e., $C \subseteq X$ is a *component* of (X, E) if it is connected and for all x, x' such that $x \in C$ and $x' \in X \setminus C$ there does not exist any path. A graph that consists of only one component is called connected because then there is a path between any two nodes. Throughout the paper, we will restrict attention to connected graphs. An important class of such graphs is the class of trees. Trees are connected with $\xi - 1$ edges or, equivalently, in a tree each pair of vertices is connected by a unique path.

A node-weighted graph is a triple (X, E, w), where $w := (w_x)_{x \in X} \in \mathbb{R}_+^{\xi}$ is a vector of weights.

We write w_x for the weight of node $x \in X$ and $w(Y) = \sum_{y \in Y} w_y$ for the weight of a set of nodes $Y \subseteq X$. The weight w will be determined later on by the distribution of agents.

Now let (X, E, w) be given. An important operation in graphs is to delete a set of nodes $Y \subseteq X$ and all involved edges: $(X, E) - Y := \left(X \setminus Y, E_{|X \setminus Y}\right)$ with $E_{|X \setminus Y} = \{xy \in E \mid x, y \in X \setminus Y\}$. This is illustrated in Figure 1.

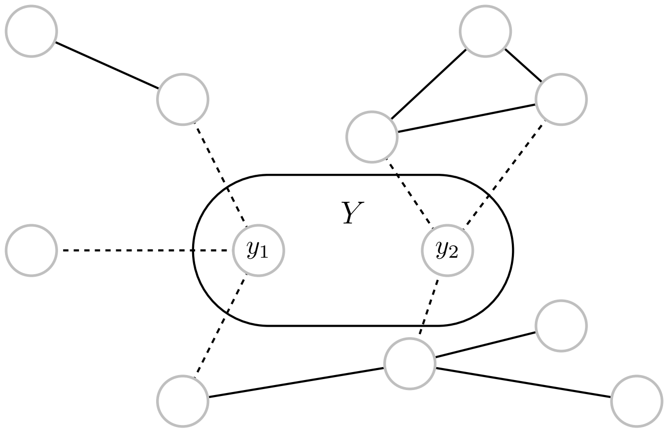

Figure 1: Deletion of nodes.

The operation $(X, E) - Y$ leads to a graph with potentially several components and we denote them by $C_1^Y, C_2^Y, \ldots, C_{l_Y}^Y$ such that $w(C_1^Y) \geq w(C_2^Y) \geq \ldots \geq w\left(C_{l_Y}^Y\right)$. If $|Y| = 1$, say $Y = \{x\}$, such a node is called a *cut vertex* and we will write C_k^x instead of $C_k^{\{x\}}$. In this case, for the number of components it holds that it is smaller than or equal to the degree of x. A set of nodes $B \subseteq X$ is called *block* if there is no cut vertex in $(X, E) - X \setminus B = (B, E_{|B})$ and B is maximal with respect to inclusion, i.e., $B \subsetneq B' \subseteq X$ implies that there exists a cut vertex in $(B', E_{|B'})$. In other words, a set of nodes is a block if the induced subgraph cannot be decomposed into multiple components by deleting single nodes and it is not possible to find a larger subgraph with this feature. Note that $x \in X$ is contained in several blocks if and only if it is a cut vertex. In the following, the set of blocks of a given graph is denoted by $\mathcal{B}$ and $b := |\mathcal{B}|$ is the number of blocks.

2.2 Perceived Distances and Players' Payoffs

In the following, the elements of X are called *objects* and interpreted, according to the three applications, as geographical locations, political platforms or product specifications. Let $N = \{i_1, \ldots, i_n\}$ be a finite set of *agents* who have a favorite object $\hat{x}^i \in X$. As usual, the graph (X, E) is used to represent the relation between the objects as they are perceived by the agents. In order to be as general as possible we impose no further requirements on the structure of the graph, but typical examples from the literature are lines, cycles or lattices, to name but a few. In contrast to previous works, we assume that perceptions are subjective to some extent. Formally, each $i \in N$ is endowed with an tupel of edge lengths $\left(\delta_e^i\right)_{e \in E} > 0$ that represent her individual estimation of distances between the nodes, such that, for example, δ_e^i need not coincide with δ_e^j.[3] Given

[3]The interpretation for geographic locations is as follows: The agents agree on the underlying graph (a road map, for example) but they are heterogeneous in terms of assessing or evaluating the edge lengths (the travel time, for

$\delta := \left(\delta_e^i\right)_{e\in E}^{i\in N}$, agent i's *perceived distance* $d^i(x)$ to an object $x \in X$ is the length of the shortest path(s) from the favorite object $\hat{x}^i$ to x, where the length of a path is the sum of its edge lengths:

$$d^i(x) := \min \left\{ \sum_{t=1}^{T-1} \delta^i_{x_t x_{t+1}} \;\middle|\; (x_1, ..., x_T) \text{ is a path from } \hat{x}^i \text{ to } x \right\}.$$

We set $d^i(\hat{x}^i) = 0$ for all $i \in N$. Note that two agents with the same favorite object, i.e., $\hat{x}^i = \hat{x}^j$, might have different perceptions about the distances to the other objects. As usual, we will assume a "distance-based behavior"of the agents, i.e., agent $i \in N$ weakly prefers an object $x \in X$ over $y \in X$ if and only if $d^i(x) \leq d^i(y)$. In other words: his utility is decreasing in distances. Thus, the preferences of agent $i \in N$ are completely determined by his favorite object $\hat{x}^i$ and his individual edge lengths $\left(\delta_e^i\right)_{e\in E}$.[4] With the assumption that $\delta_e^i = \delta_e^j$ for all $i, j \in N$ and any $e \in E$, we obtain the standard model, where distance perceptions are homogeneous.

In addition to the objects and agents, we consider a set of *players* $P := \{c_1, ..., c_p\}$ of finite size $p \geq 2$. To ease the distinction between agents and players we will use the male form for agents, while players are assumed to be female. Each $c \in P$ is supposed to occupy an object $x \in X$. Formally, the *strategy set* for each player $c \in P$ is $S^c = X$, such that a strategy is an object $s^c \in X$. Let $S = S^{c_1} \times ... \times S^{c_p}$. Given a strategy profile $s \in S$, let $p_x \in \mathbb{N}$ be the number of players whose strategy is $x \in X$. Furthermore, let $\Phi^i(s)$ be the set of players who are perceived as closest by agent $i \in N$, i.e., $\Phi^i(s) = \left\{c \in P \;\middle|\; d^i\left(s^c\right) \leq d^i\left(s^{\bar{c}}\right)\ \forall\, \bar{c} \in P\right\}$. Note that we loosely speak about the perceived distance to a player c instead of the distance to the player's chosen object s^c. We assume that each agent is allocated to the player which is perceived as closest. If multiple players are perceived as closest by some agent, then he is assumed to be uniformly distributed among these players. Thus, given a strategy profile $s \in S$, player c's payoff $\Phi^c(s)$ is the mass of agents who perceive object s^c as closest to their favorite object, i.e., the payoff of $c \in P$ is given by $\pi^c(s) = \sum_{i:c\in\Phi^i(s)} \frac{1}{|\Phi^i(s)|}$. A profile of payoffs is denoted by $\pi_\delta := \left(\pi_\delta^c\right)^{c\in P} := \left(\pi^c\right)^{c\in P}$, where the subscript δ indicates that the payoffs depend on the individual edge lengths $\delta = \left(\delta_e^i\right)_{e\in E}^{i\in N}$.

2.3 Equilibrium Notions

Fix a graph (X, E) and a set of agents N such that for each agent $i \in N$ we have a favorite object $\hat{x}^i \in X$ and individually measured edge lengths $(\delta_e^i)_{e\in E}$. Then a normal form game is given by $\Gamma^\delta = (P, S, \pi_\delta)$. The game is indexed by δ to emphasize that the payoffs, and therefore the game depends on the individual edge lengths. The main goal of our work is to examine to which extent this restriction determines the outcome of the standard setting, that is the special case of homogeneous distances. A Nash equilibrium of the game Γ^δ is also called a locational (Nash)

example). If the graph does not represent geographic distances, but policy spaces or the perception of brands, it seems to be an even more unrealistic assumption that all agents use the same distance measure.

[4]There is a justification for this type of preference which neither deals with differing edge lengths nor with distance-based behavior. Agents can be assumed to have single-peaked preferences on the graph as they were defined for lines (Black, 1948) or trees (Demange, 1982). Such preferences find broad acceptance and play a crucial role in the literature on social choice (see, e.g., Moulin, 1980). The alternative formulation with single-peaked preferences is, in fact, equivalent to the (quite different) formulation here. The proof for this claim can be requested from the authors.

equilibrium (cf. Eiselt and Laporte, 1991, 1993). Thus, $s \in S$ is a *locational equilibrium* if for all $c \in P$ and for all $x \in X$ we have $\pi^c(s^c, s^{-c}) \geq \pi^c(x, s^{-c})$.

Example 1. *Consider a cycle graph on six nodes, i.e., (X, E) with $X = \{x_1, x_2, ..., x_6\}$ and $E = \{x_1x_2, x_2x_3, ..., x_6x_1\}$. Let $N = \{i_1, i_2, ..., i_{12}\}$ be a set of twelve agents with favorite objects $(\hat{x}^1, \hat{x}^2, ..., \hat{x}^{12}) = (x_1, x_1, x_2, x_2, ..., x_6)$. We first assume homogeneous edge lengths, i.e., for all $i \in N$ we have $\delta^i_e = 1$ for any $e \in E$. Together with a set of three players $P = \{c_1, c_2, c_3\}$ this constitutes a game Γ^δ.*
The graph (X, E) is illustrated in Figure 2. The number within a node indicates the number of agents who have this node as the favorite object. The edge lengths are not represented. Finally, the three squares represent the strategy profile $(s^1, s^2, s^3) = (x_1, x_3, x_5)$. We will keep these conventions in the following figures. For this game, results of Mavronicolas et al. (2008) imply that the depicted

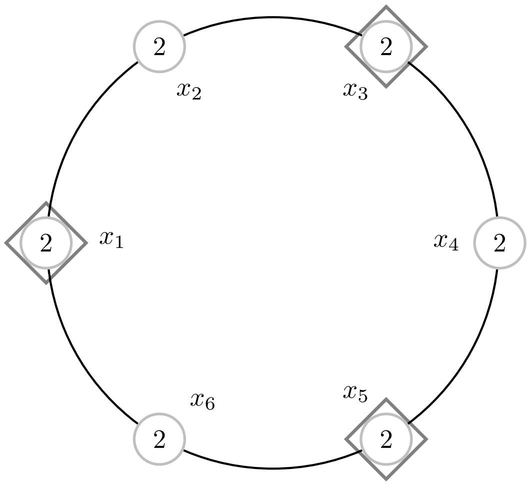

Figure 2: Three players on a cycle graph.

strategy profile s is a locational equilibrium. A player cannot improve by relocating, because her payoff either stays 4 *(when deviating to a neighbor) or decreases.*
This result, however, depends on the specific edge lengths. Consider the situation where one of the twelve agents with favorite object on x_2 assigns a different length to an edge next to him, such as $\tilde{\delta}^3_{x_1x_2} = 1 - \epsilon$ for some $\epsilon > 0$ and $\tilde{\delta}^3_e = 1$ for all other edges. The perceived distances of the other agents are assumed to stay the same. Then the depicted strategy profile s is not a locational equilibrium. The player $c_3 \in P$ with strategy x_3 now has an incentive to deviate to x_2 or x_4 because in both cases she would attract four agents instead of only 3.5. Thus, the strategy profile $s \in S$ is a locational equilibrium in the game Γ^δ but not in the perturbed game $\Gamma^{\tilde{\delta}}$. In some sense the profile is not "robust".

The previous example motivates the following definition.

Definition 1 (Robust equilibrium)**.** $s^* \in S$ is a *robust equilibrium* if it is a locational equilibrium for any collection of individual edge lengths. In other words: $s^* \in S$ is a locational equilibrium in Γ^δ for any $\delta = \left(\delta^i_e\right)^{i \in N}_{e \in E}$.

Certainly, robustness is a strong requirement. But it is a desirable property for at least two reasons. First, a robust equilibrium is independent of the assumption of homogeneous edge lengths but includes this as a special case. Indeed, a robust equilibrium is also a locational equilibrium in the homogeneous case Γ^δ, where $\left(\delta_e^i\right)_{e\in E}$ is the same for all agents $i \in N$. Second, to determine the locational equilibrium one has to specify for each agent her favorite object $\hat{x}^i \in X$ as well as her list of edge lengths $\left(\delta_e^i\right)_{e\in E}$ together with a graph (X, E). On the other hand, to determine robust equilibria it is sufficient to know the graph (X, E) and the distribution of favorite objects $(\hat{x}^i)^{i\in N}$. In fact, it is sufficient to have only information about the node-weighted graph that is induced by $(\hat{x}^i)^{i\in N}$, i.e., it is enough to know (X, E, w) where $w_x := \left|\{i \in N \mid \hat{x}^i = x\}\right|$ is the number of agents having x as their favorite object. We will interpret an exogenously given node-weighted graph (X, E, w) as a situation of spatial competition.

3 Robustness

We will first provide a characterization of robustness which applies to test whether locational equilibria are robust. Then, we will turn to properties of robust equilibria, in particular minimal differentiation and efficiency. Finally, we will reconsider the existence of robust equilibria.

3.1 Characterization

In this subsection we provide the necessary and sufficient conditions for a strategy profile to be a robust equilibrium. For this purpose we need additional definitions.

Definition 2. Let (X, E) be a graph and fix a strategy profile $s \in S$. Furthermore, let $\bar{X} = \bigcup_{c=1}^{p}\{s^c\} \subseteq X$ be the set of occupied nodes in s.

- The *hinterland* $H_x \subseteq X$ of node $x \in \bar{X}$ is the set of nodes that have x on every path to any $x' \in \bar{X}$. In the special case where all players choose the same strategy (i.e., $|\bar{X}| = 1$), say $\bar{X} = \{x\}$, we define $H_x := X$.
- An *unoccupied zone* $Z \subseteq X$ is a component of $(X, E) - \bar{X}$. The set of all unoccupied zones is denoted by $\mathcal{Z}$.
- An unoccupied zone $Y \subseteq X$ is called a *competitive zone* if it is not contained in any hinterland, i.e., $Y \nsubseteq H_x$ for all $x \in \bar{X}$. The set of all competitive zones is $\mathcal{Y}$.
- Two distinct objects $x, x' \in \bar{X}$ are *indirectly neighboring* if there exists a competitive zone to which both nodes are neighboring.
- The *neighboring area* $A_x \subseteq X$ of $x \in \bar{X}$ is the unoccupied zone which would be obtained when removing all players located on x. Formally, that is $A_x = \left(\bigcup_{Z\in\mathcal{Z}_x} Z \cup \{x\}\right)$, where $\mathcal{Z}_x := \{Z \in \mathcal{Z} \mid Z \text{ neighboring to } x\}$.

The notions of hinterland and competitive zone go back to Eiselt (1992) who has defined them for the given positions of two players. The hinterland $H_x \subseteq X$ consists of the node itself and possibly several unoccupied zones that are adjacent to $x \in \bar{X}$ but not to any other occupied node

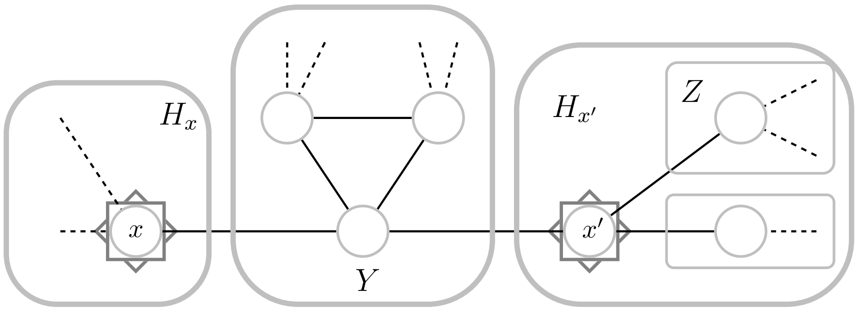

Figure 3: Example for definitions: decomposition into competitive zones and hinterlands.

in $\bar{X}$. Agents who have their favorite object in H_x must be closer to player(s) on node x than to all other players, since any path, and therefore also the shortest one(s), contain this object. This is different for competitive zones. Players surrounding a competitive zone $Y \in \mathcal{Y}$ compete with indirectly neighboring competitors over the agents who have their most favorite object in Y. The definitions are illustrated in Figure 3, where there are two occupied nodes $x, x' \in \bar{X}$, several unoccupied zones, where one of them (Y) is a competitive zone, and another one (Z) belongs to a hinterland. Furthermore, the neighboring area $A_x \subseteq X$ consists of the hinterland H_x and the competitive zone Y, while the neighboring area $A_{x'}$ consists of the other hinterland $H_{x'}$ and the competitive zone Y. Generally, each node either belongs to one hinterland or to one competitive zone. This can be considered as a partition of X into l hinterlands (i.e., $|\bar{X}| = l$) and k competitive zones

$$\Pi(s) = \{H_{x_1}, ..., H_{x_l}, Y_1, ..., Y_k\}. \tag{1}$$

In fact, because every agent with favorite object in $H_x \subseteq X$ is always closer to a player on the corresponding node x than to any other occupied node, $\frac{w(H_x)}{p_x}$ is the "worst-case payoff" that a player who chooses x receives. Conversely, the maximal payoff of a player who chooses x is restricted by the neighboring area $A_x \subseteq X$, i.e., by $\frac{w(A_x)}{p_x}$. These simple considerations lead to the following key proposition.

Proposition 1. Let $s^* \in S$ be a strategy profile on a node-weighted graph (X, E, w) and let $\Pi(s)$ be the corresponding partition as in (1). Furthermore, let $\hat{Z} \in \operatorname{argmax}_{Z \in \mathcal{Z}} w(Z)$ be a heaviest unoccupied zone. Then s^* is a robust equilibrium if and only if the following four conditions are satisfied for all $x \in \bar{X}$:

(1.) $$\frac{w(H_x)}{p_x} \geq w\left(\hat{Z}\right)$$

(2.) $$\frac{w(H_x)}{p_x} \geq \frac{w(A_{x'})}{p_{x'} + 1} \qquad \forall\, x' \in \bar{X} \setminus \{x\}$$

Furthermore, if $p_x = 1$:

(3.) $$w(Y) = 0 \qquad \forall\, Y \in \mathcal{Y}, Y \subseteq A_x$$

(4.) $$w(H_x) \geq \frac{w(A_{x'})}{p_{x'}} \qquad \forall\, x' \text{ ind. neighb. to } x.$$

The proof is relegated to the appendix. Proposition 1 formalizes the requirements for a strategy profile to be a robust equilibrium. It consists of four straightforward conditions. The first one formalizes that deviations into unoccupied zones are never beneficial. Even if the players only receive their worst case payoff, i.e., the weight of their hinterland, they never gain from relocating into any $Z \in \mathcal{Z}$.[5] Similarly, Condition (2.) captures that deviations to already occupied nodes $x' \in \bar{X}$ are not beneficial. The highest possible payoff a deviating player could get is $\frac{w(A_{x'})}{p_{x'}+1}$.[6] These two previous considerations must be strengthened when considering certain deviations of an isolated player because her node becomes unoccupied then. Again, we distinguish between deviations into a neighboring zone and deviations on occupied nodes, which is reflected by Conditions (3.) and (4.). The main intuition is that for some distance perceptions an isolated player would attract only her hinterland, but by deviating she could receive her former hinterland and, in addition, the weight of some competitive zone (Condition (3.)). By deviating on a neighboring occupied node she can not only share the payoffs of the players on this node, but would also regain some share of her former hinterland (Condition (4.)). For competitive zones neighboring a singly occupied node this means that their weight must be zero. We have already seen an example where this condition is violated. In Example 1 there are several singly occupied nodes which are neighboring a non-trivial competitive zone (cf. Figure 2).[7] Thus, we can immediately conclude that the given strategy profile is not a robust equilibrium.
The main importance of Proposition 1 is that it provides a convenient and powerful tool for checking if a strategy profile s constitutes a robust equilibrium. In the remainder of this subsection we will exemplify this for some prominent results from the literature. Hotelling's main result for two players on a continuous line is that both cluster on the so-called median. This finding is driven by the fact that both players tend to the center of the line to steal agents from the other player. This is illustrated for a discrete line in Figure 4 where we can observe the incentive to increase the hinterland by moving to the discrete analogue of the median.

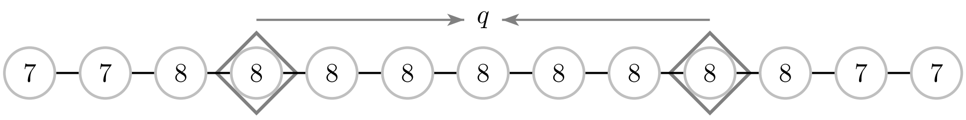

Figure 4: A node-weighted line graph with two players.

Definition 3 (Median)**.** A *median* of a node-weighted graph (X, E, w) is a node $q \in X$ that balances the node weights, i.e., $w(C_1^q) \leq \frac{w(X)}{2} = \frac{n}{2}$, where $C_1^q \in \mathcal{Z}$ is the heaviest component of $(X, E) - \{q\}$.

In general, a median need not exist. For example, if we consider the complete graph where all weights are equal to one, we have $w(C_1^q) = n - 1 > \frac{n}{2}$. Nevertheless, one can show that if (X, E) is a tree, a median always exists.[8]

[5] This requirement also implies that the weight of unoccupied zones can never be higher than the average payoff of the players, i.e., $w(Z) \leq \frac{n}{p}$ for all $Z \in \mathcal{Z}$.

[6] A simple implication of this requirement is that in robust equilibria the number of players on occupied nodes is roughly proportional to the weights of the hinterlands: $\frac{p_x}{p_{x'}+1} \leq \frac{w(H_x)}{w(H_{x'})} \leq \frac{p_x+1}{p_x}$ for all $x, x' \in \bar{X}$.

[7] We say that a competitive zone Y is trivial if no consumer has his favorite object there, i.e., $w(Y) = 0$.

[8] Moreover, for trees a node q is a median if and only if $q \in \operatorname{argmin}\left\{ \sum_{y \in X} d(x, y) w_y \;\middle|\; x \in X \right\}$ for all δ (see

The most direct way to extend Hotelling's model to graphs is to consider trees. Although this is only a special case of our set-up, much attention has been devoted to this class of graphs in the literature. Among others, Eiselt and Laporte (1991) examined this setting and they have shown that in the two-player case for homogeneous distances both players will locate on the median of the tree. Thus, they came to the same conclusion as Hotelling did. In fact, this result had already been established by Wendell and McKelvey (1981) in slightly different terms. In their publication the authors show that for homogeneous distances on a tree the median is always a Condorcet winner.[9] Since a Condorcet winner cannot be beaten in majority voting (by definition), choosing the Condorcet winner constitutes a locational equilibrium in the two-player game.
Now, let us apply Proposition 1 to test whether the two-player results mentioned in the previous paragraph are robust. If both players locate on the same object, say $q \in X$, there is only one hinterland consisting of all the nodes, i.e., $\Pi(s) = \{X\}$. Therefore, only Condition (1.) of Proposition 1 applies and it simplifies to $\frac{n}{2} \geq w(\hat{Z}) = w(C_1^q)$, which is exactly the definition of the median.[10] Now consider the setting where the players choose different positions, say x and $x' \in X$. Eiselt and Laporte (1991) show that this is a locational equilibrium only if the positions are either neighboring or the competitive zone between them has weight 0 and, furthermore, $\frac{n}{2} = w(C_1^x) = w(C_1^{x'})$ holds. Applying conditions (3.) and (4.) of Proposition 1 yields that this is robust, too.

In Eiselt and Laporte (1993) the authors examine the case of three players on a tree. In their main result they distinguish four different cases: (i) type A equilibria (all players cluster on the median $q \in X$), (ii) type B equilibria (two players locate on the median q and one in the heaviest component $C_1^q \in \mathcal{Z}$ on the node that is neighboring to q), (iii) type C equilibria (all three players on different nodes), and (iv) non-existence of equilibria. With the conditions given in Eiselt and Laporte (1993) it is easy to check that type A and type B equilibria are indeed robust. However, type C equilibria are generically not. They are robust only if the hinterland of all players has the same weight because otherwise Condition (4.) of Proposition 1 would be violated.

Note that in the previous examples the equilibria are robust only if some kind of minimal differentiation is satisfied and at least some players choose the median q. Therefore these results raise some questions regarding the general form of robust equilibria.

3.2 Minimal Differentiation

Minimal differentiation is one of the most controversial results and much attention has been devoted to its implications.[11] In the framework of graphs, we define minimal differentiation as follows.

Goldman, 1971), i.e., a median q is a minimizer of the weighted sum of graph distances for all δ. On general graphs there are multiple conventions for the notion 'median': sometimes it is defined (rather than characterized) as the minimizer of the weighted sum of graph distances.

[9] Later Hansen *et al.* (1986) extended this work.

[10] In fact, this has already been shown for the continuous line, although in very different terms, by the seminal contribution of Black (1948). He proved that for single-peaked preferences on a line the median is always a Condorcet winner. As already mentioned in Section 2, single-peaked preferences on a line are equivalent to our assumption of heterogeneous edge lengths on the line graph.

[11] Some works show that it is not satisfied generically (see, e.g., d'Aspremont *et al.*, 1979; Eaton and Lipsey, 1975; Economides, 1986) but others support it for special cases (see, e.g., de Palma *et al.*, 1985, 1990; Hehenkamp and Wambach, 2010). Similar considerations also hold for minimal differentiation on graphs.

Definition 4. A strategy profile $s \in S$ satisfies minimal differentiation if all players locate on the same node, i.e., $s = (x, x, \ldots, x)$ for some $x \in X$.

In the previous section there were already examples for robust equilibria satisfying minimal differentiation for two or three players.[12] These cases can be extended to arbitrary numbers of players in a straightforward way. Consider the strategy profile $s := (x, x, ..., x)$ where all players locate on a node $x \in X$. We then have only one hinterland consisting of all the nodes, i.e., $\Pi(s) = \{X\}$. By using the same arguments as in the two-player case one can see that Conditions (2.), (3.) and (4.) of Proposition 1 do not apply and, furthermore, Condition (1.) simplifies to $\frac{n}{p} \geq w(\hat{Z})$, where $\hat{Z}$ is the heaviest unoccupied zone. Thus, we get the following corollary.

Corollary 1. Let (X, E, w) be a node-weighted graph and $q \in X$. Furthermore, let $C_1^q \in \mathcal{Z}$ be a heaviest component of $(X, E) - \{q\}$. The strategy profile $s = (q, ..., q)$ is a robust equilibrium if and only if the weight of any component of $(X, E) - \{q\}$ is not higher than the average payoff, i.e.,

$$w\left(C_1^q\right) \leq \frac{n}{p}.$$

This result is also easy to prove without Proposition 1 since for s every player earns the average payoff $\frac{n}{p}$, while the most beneficial deviation leads to the heaviest unoccupied zone $w(C_1^q)$. Put differently, if the heaviest component of the graph without $q \in X$ is relatively light, then there exists a robust equilibrium where all players locate on the same node. In particular, this also implies that q has to be a median of the graph. Obviously, Corollary 1 extends previous findings from the two- and three-player case and it shows that for any number of players, it is easy to construct a robust equilibrium. The robust equilibria we have discussed so far all have virtually all players on the median. Therefore one might suspect that in any robust equilibrium the median must be occupied (if it exists) and that the players cluster on or around it. The following example is a counter-example to this conjecture.

Example 2. *Let (X, E, w) be the weighted line graph depicted in Figure 5.*

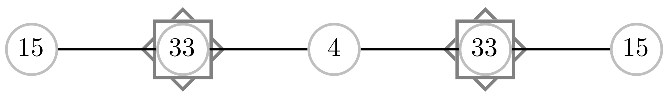

Figure 5: A robust equilibrium with no player on the median and without minimal differentiation.

Furthermore assume that two players locate on each of the nodes with weight 33. As it is easy to check, this strategy profile is a robust equilibrium. The median, however, is the node with a weight four and it belongs to a competitive zone. Thus, neither minimal differentiation is satisfied, nor are players located on the median.
However, consider a reduced game where we remove the two nodes to the right and we remove the two players in this area. In this reduced game the unique robust equilibrium is that the remaining two players both locate on the node with 33 agents such as in the current strategy profile. Moreover,

[12] Definition 4 captures minimal differentiation in a strong sense. A weaker version of minimal differentiation would be the requirement that there is no unoccupied node between any pair of occupied nodes or, equivalently, that there is no competitive zone.

this node is the median of the reduced graph. A similar observation can be made when reducing the game by removing "the left part".

Example 2 shows that in a robust equilibrium it need not be the case that players minimally differentiate on the median. However, it seems that locally, in a kind of reduced game, this is still true. To investigate this issue, let us formally define a reduced game. Given a strategy profile $s \in S$, we define a reduced game for every occupied node $x \in \bar{X}$ by considering the objects and players in the neighboring area $A_x \subseteq X$. Thus, the number of players in the reduced game is p_x and the graph is restricted to $(A_x, E_{|A_x})$. For the payoffs only those agents are considered whose favorite object belongs to the neighboring area A_x such that the node weights of the graph in the reduced game coincide to the node weights of the original game.

Corollary 2 (Reduced Games)**.** Suppose $s^* \in S$ is a robust equilibrium for some (X, E, w) and let $x \in \bar{X}$ be an occupied position such that $p_x \geq 2$. Then, x is the median of the subgraph $(A_x, E_{|A_x})$ and $(x, x, \ldots, x)$ is a robust equilibrium satisfying minimal differentiation in the corresponding reduced game.

Proof. Let $x \in \bar{X}$ be an occupied position in $s^* \in S$ with $p_x \geq 2$. Applying Proposition 1, Condition (1.) implies $\frac{w(A_x(s^*))}{p_x} \geq \frac{w(H_x)}{p_x} \geq w(Z)$ for every unoccupied zone surrounding x. But this is equivalent to the condition of Corollary 1, $w(C_1^x) \geq \frac{w(A)}{p_x}$, which shows that the strategy profile $(x, \ldots, x)$ is a robust equilibrium in the reduced game. Moreover, this condition implies that the weight of the heaviest component of $\left(A_x, E_{|A_x}\right) - \{x\}$ is smaller than $\frac{w(A_x)}{2}$ which shows that $x \in \bar{X}$ is the median of the subgraph $(A_x, E_{|A_x})$. $\square$

Corollary 2 shows that in any robust equilibrium a local variant of minimal differentiation is satisfied. This finding is fully in line with the "principle of local clustering" conjectured in the seminal work of Eaton and Lipsey (1975). Their principle, however, also contains the aspect that players pair, i.e., do not locate away from other firms. This aspect is also true in robust equilibria since it follows from Condition (3.) of Proposition 1 that isolated players do not neighbor a non-trivial competitive zone. This implies that singly occupied nodes must neighbor another occupied node if node weights are strictly positive. Thus, any robust equilibrium can be characterized as a few multiply occupied nodes which are possibly neighbored by some singly occupied nodes. The final question on the extent of differentiation is whether these local clusters can be at a large distance from each other.

In Example 2 only a small share of agents favor the object between the occupied positions. In fact, it holds generally that the weight of competitive zones in robust equilibria must be relatively light.

Proposition 2 (Competitive zones)**.** Let (X, E, w) be a node-weighted graph. Suppose s^* is a robust equilibrium and let $\mathcal{Y}$ be the set of competitive zones. Then $\sum_{Y \in \mathcal{Y}} w(Y) \leq \frac{n}{5}$.

The proof can be found in the appendix. By definition, a strategy profile satisfies minimal differentiation only if there is no competitive zone. In this context Proposition 2 can be interpreted as a weaker form of a global minimal differentiation result: competitive zones might exist in equilibrium, but their weight in sum is bounded by $\frac{n}{5}$, i.e., at most 20% of the agents can have their favorite object in some competitive zone.

3.3 (In-)Efficiency

Traditionally efficiency is measured by aggregating the players' and the agents' surplus. However, from the players' perspective in our setting (i.e., without considering price competition) any strategy profile yields the same aggregated surplus because we study a constant-sum game. Therefore, efficiency will be discussed from the viewpoint of the agents which are interpreted as consumers in this section.[13] The standard result of two firms choosing the median of a line is known to be inefficient because both players choosing the same object leads to unnecessarily high distances for the consumers. In his paper Hotelling complains about this inefficiency:

> "Buyers are confronted everywhere with an excessive sameness [...]" and "[...] competing sellers tend to become too much alike."
>
> (Hotelling, 1929, p. 54)

In fact, to minimize the sum of distances in the original model, where consumers are uniformly distributed along a continuous line, it would be optimal to locate at the two positions which are $\frac{1}{4}$th from the end points of the line.[14] For two players on the line graph we have already observed that the robust equilibrium is as postulated by Hotelling and thus inefficient. We now analyze whether this is a general issue in a robust equilibrium.

Proposition 1 restricts the occurrence of singly occupied nodes. First, Condition (3.) shows that they may not neighbor a non-trivial competitive zone. Moreover, if two singly occupied nodes $x, x' \in \bar{X}$ are neighboring, then Condition (4.) on the RHS only sums up parts of the hinterland of x, such that we get $w(H_x) \geq w(H_{x'})$. Applying the same result to x' versus x we observe that the players on both nodes get the same payoff, which is $w(H_x) = w(H_{x'})$. In a profile without multiply occupied nodes we have a sequence of indirectly neighboring relations between singly occupied nodes. Therefore, we must have $w(H_x) = \frac{1}{p}$ for any occupied node $x \in \bar{X}$, i.e., each player receives just her hinterland as a payoff and all payoffs are equal. Such a profile can be constructed only if a graph has non-generic weights. In particular, it is necessary that the number of agents n is a multiple of the number of players p. We exclude this non-generic case by assumption.

Corollary 3. Let (X, E, w) be a node-weighted graph and suppose that the number of agents n is not divisible by the number of players p. Then in any robust equilibrium $s^* \in S$ at least one node is multiply occupied.

In terms of the sum of distances (of each consumer to a closest player), it cannot be efficient to have two players on one node when there are other nodes with consumers. However, this cardinal approach is not fully justified in our context because we have individual distance perceptions which need not be comparable across consumers. A well-known ordinal criterion is Pareto efficiency. A strategy profile is Pareto efficient if there does not exist another strategy profile such that any

[13] These might be inhabitants that visit a facility or consumers who buy a product. Because we have not specified a second stage like government formation in our model, the discussion of efficiency does not apply to the context of voting.

[14] Similar to minimal differentiation, the discussion of efficiency has been very controversial. For example, in the case of uncertainty Meagher and Zauner (2004) show that the outcome of spatial competition might lead to socially inefficient outcomes, while this is not necessarily true in Król (2012).

consumer is at least as well off and at least one consumer is strictly better off, where better off here means that the perceived distance to the closest player becomes shorter. Note that this is a weak requirement and plenty of strategy profiles satisfy this criterion. However, robust equilibria do not.

Proposition 3 (Pareto efficiency)**.** Let (X, E, w) be a node-weighted graph. Suppose that the number of agents n is not divisible by the number of players p and there are at least p nodes with positive weight $w_x > 0$. Then any robust equilibrium is Pareto dominated (for the consumers).

Proof. By Corollary 3 there must be a multiply occupied object, say $x \in \bar{X}$. Since at least p nodes have a positive weight, there must be an unoccupied node, say $\tilde{x} \in X$ with $w_{\tilde{x}} > 0$. Changing the strategy of one player with $s^c = x$ to $\tilde{s}^c = \tilde{x}$ is a Pareto improvement since consumers with $\hat{x}^i = \tilde{x}$ are better off. □

The conditions for Proposition 3 are very mild. They just exclude non-generic cases. The interpretation of Proposition 3 is simple. Generically, in every robust equilibrium there are two firms who choose the same location respectively, offer the same product, while the consumers would benefit if one of them offered a different product or chose a different location. In fact, because we have a constant-sum game between players, a social-planner could relocate the players and compensate them with transfer payments to keep their payoffs constant. Thus, a socially optimal outcome from the consumers' point of view is possible without changing the payoffs of the players.[15] This shows that, in a much more general form, in our model Hotelling's inefficiency persists.

3.4 (Non-)Existence of Robust Equilibria

So far we examined properties of robust equilibria without explicitly discussing under which conditions they exist. Corollary 1 provides a condition which is sufficient for existence. Intuitively, it is satisfied either if the weight is concentrated on the median or if we have a star-like structure under a more equal weight distribution.[16] Although this condition is necessary and sufficient only for robust equilibria with minimal differentiation, similar considerations also apply in general. Corollary 1 is based on Proposition 1 which characterizes the underlying strategy profiles of robust equilibria.[17] In particular, Condition (1.) states that the hinterland $H_x \subseteq X$ of every occupied node $x \in \bar{X}$ must be heavy enough to carry p_x players. If this weight is not directly on the node x, then it must be on other nodes in its hinterland. Considering the "arms" in the hinterland, i.e., the components in the graph $(H_x, E_{|H_x}) - \{x\}$, each of them is an unoccupied zone. However, for unoccupied zones the weight is bounded, again by Proposition 1 Condition (1.). Thus, in order to be heavy enough, an occupied node $x \in \bar{X}$ must either have sufficiently many arms in its hinterland

[15] However, this result also depends on the exclusion of price competition. If firms do not cluster, i.e., if they have a local monopoly, they might have an incentive to raise prices.

[16] Moreover, as discussed in Subsection 3.1, for two players on tree graphs this condition is always satisfied such that a robust equilibrium must exist.

[17] Proposition 1 provides the necessary and sufficient conditions for existence in the sense that a robust equilibrium exists if and only if there is a strategy profile that satisfies these conditions. Thus, this result transforms the problem of finding a strategy profile that is a robust equilibrium into finding a strategy profile that satisfies the conditions of Proposition 1, but it is not a result on the exogenously given situation of spatial competition, i.e., the node-weighted graph (X, E, w).

(which are heavy in sum) or it must have a relatively high weight itself. This intuition is formalized in Corollary 4.

Corollary 4. For some node-weighted graph (X, E, w), let $s^* \in S$ be a robust equilibrium with heaviest unoccupied zone $\hat{Z} \in \mathcal{Z}$ (and $w(\hat{Z}) > 0$). Let $x \in \bar{X}$ be occupied by $0 < p_x < p$ players. Denote by $a_x \in \mathbb{N}$ the number of arms (i.e., the number of components in the hinterland for $(H_x, E_{|H_x}) - \{x\})$ of x. Then

$$\frac{w_x}{w(\hat{Z})} + a_x \geq p_x.$$

Proof. Let $\hat{Z}_x \in \mathcal{Z}$ be the heaviest unoccupied zone in the hinterland of $x \in \bar{X}$. The result then follows from Proposition 1 Condition (1.):

$$\begin{aligned} w(H_x) \geq p_x w(\hat{Z}) \quad &\Rightarrow \quad w_x + a_x w(\hat{Z}_x) \geq p_x w(\hat{Z}) \\ &\Rightarrow \quad \frac{w_x}{w(\hat{Z})} + a_x \cdot \frac{w(\hat{Z}_x)}{w(\hat{Z})} \geq p_x \quad \Rightarrow \quad \frac{w_x}{w(\hat{Z})} + a_x \geq p_x \end{aligned}$$

□

Corollary 4 shows that in a robust equilibrium the relative weight of an occupied node plus its number of arms must exceed the number of players on it. This result is illustrated in Figure 6 with two occupied nodes x and $x' \in \bar{X}$.

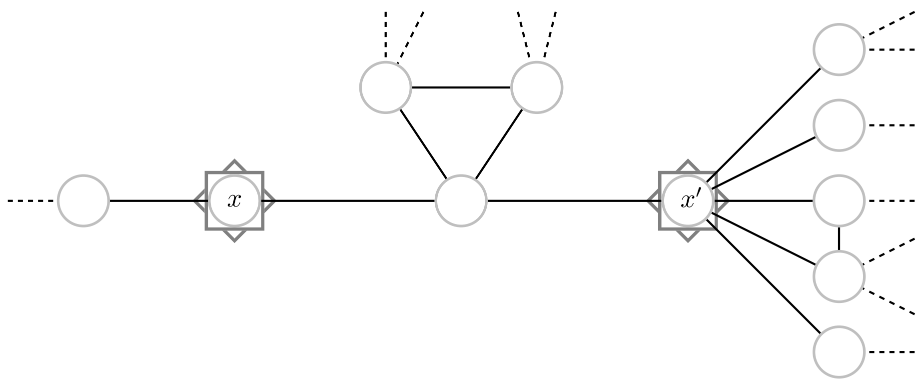

Figure 6: Four players on two nodes. If this is a robust equilibrium, then node x must have high weight. This is not necessarily true for x' because it has a high degree (which leads to several arms in its hinterland).

While x has only one arm in its hinterland, x' has four of them. Therefore, for node x we have $\frac{w_x}{w(\hat{Z})} + 1 \geq 2$, which is equivalent to $w_x \geq w(\hat{Z})$, i.e., the weight of the node must exceed the weight of the heaviest unoccupied zone. Note that this implies an inequality of weights if there are unoccupied zones with many nodes. In contrast to this, x' needs not be as heavy as x, but in order to have four arms it must be a cut vertex and have a degree larger than five. Thus, one interpretation for Corollary 4 is that the weight of occupied nodes and their degree can be interpreted as some kind of substitutes: at least one of them has to be high enough in order to carry p_x players in equilibrium.

This gives a requirement for robust equilibria on the level of single nodes. On the graph level this requirement will translate into (a) structural features of the graph and in (b) conditions on the distribution of weights. To assess the weight distribution, we consider the inequality of weights measured by the variance. In our case it is given by $\mathrm{Var}(w) = \sum_{x\in X} \left(w_x - \frac{n}{\xi}\right)^2 = \frac{1}{\xi}\sum_{x\in X} w_x^2 - \frac{n^2}{\xi^2}$. The variance is the quadratic distance from the uniform distribution. In particular, $\mathrm{Var}(w) = 0$ if and only if $w_x = \frac{n}{\xi}$ for all $x \in X$: that is, w is uniformly distributed (a special case that is predominantly discussed in the literature). To assess structural requirements of a graph we consider its connectedness which is measured by the number of blocks b (cf. Diestel, 2005). If this number is smaller than the number of players p, then it is still impossible to have Corollary 4 trivially satisfied (such as for node x' in Figure 6). For these graphs Corollary 4 has implications on the weight distribution because there must be an occupied node that is similar to node x in Figure 6. As a consequence we have that graphs with a high connectivity (i.e., a relatively small number of blocks) only admit robust equilibria if the weight distribution is far from uniform.

Proposition 4. Let (X, E, w) be a node-weighted graph with $\xi > 3p$. Suppose that the number of blocks is smaller than the number of players, i.e., $b < p$. Then there exists some $\nu > 0$ such that $\mathrm{Var}(w) < \nu$ implies that a robust equilibrium does not exist.

The interpretation of this result is as follows: Suppose the graph is not too small ($\xi > 3p$) and the distribution of agents is sufficiently close to the uniform distribution. Then the existence of robust equilibria requires a low connectivity of the underlying graph in terms of that there must be more blocks than players. Proposition 4 obviously applies to all graphs with just one block ($b = 1$). Those graphs are known as *two-connected* and they are characterized by not containing any cut vertex (see Diestel, 2005). Indeed, for such graphs we have $a_x = 0$ for any occupied node $x \in \bar{X}$ (and for any $s \in S$). A particular example of a two-connected graph is a cycle graph as illustrated in Figure 2. Other two-connected graphs are grids. If those graphs are sufficiently large, they always satisfy the requirements of Proposition 4 and therefore they do not admit robust equilibria if the weight distribution is too close to uniformity.[18]

For tree graphs Proposition 4 does not apply since trees consist of many blocks. However, for this special class the number of arms is also restricted by some structural property. Since there are no cycles in a tree, each arm in any hinterland leads to a node of degree 1, a so-called *loose end*. Therefore, completely analogous to Proposition 4 one can show the following.

Proposition 5. Let (X, E) be a node-weighted tree with $\xi > 3p$. Suppose that $e < p$, where e is the number of loose ends. Then there exists some $\nu > 0$ such that $\mathrm{Var}(w) < \nu$ implies that no robust equilibrium exists.

The number of loose ends is a structural feature that is related with the equality of the degree distribution of the graph. The lowest number of loose ends in a tree is attained in the line graph (which has a highly equal degree distribution), while the highest number is attained in the star graph (which has a highly unequal degree distribution). In that sense, Proposition 5 shows that the existence of a robust equilibrium on a tree requires either an unequal distribution of weight or an unequal distribution of degree.

[18] The result that two-connected graphs require a sufficient inequality of node weights can also be derived from Proposition 2. Since in two-connected graphs any unoccupied node belongs to a competitive zone, Proposition 2 implies that we have $w(\bar{X}) \geq \frac{4}{5}n$ in robust equilibria. Thus, there must be at least one node x with $w_x \geq \frac{4}{5}\frac{n}{p}$. That is, to reach an average payoff $\frac{n}{p}$ it is almost enough to attract all agents with favorite object x.

To sum it up, robust equilibria certainly exist for structures that are similar to a star graph (Corollary 1) or have a highly concentrated distribution of weights. However, for graphs with few cut vertices (i.e., graphs with a low number of blocks) and for tree graphs, robust equilibria can exist only if the weight distribution is not close to uniform. To consider a numerical example for the required inequality: for trees that satisfy the condition $e < p$ of Proposition 5 and for cycle graphs (which always satisfy the condition $b < p$ of Proposition 4) we can show that there only exists a robust equilibrium of three or more players if there is a node $x \in X$ that is at least $\frac{\xi}{p} - 1$ times heavier than some other node $x' \in X$. Thus, if the number of nodes strongly exceeds the number of players in the game (i.e., $\xi \gg p$) those one-dimensional structures do not admit robust equilibria if the weights are uniformly distributed.

4 Discussion

Any model dealing with spatial competition assumes that agents are heterogeneous with respect to their ideal point (i.e., location/policy/product), but homogeneous with respect to the perception of distances. In particular, it must hold that two agents with the same ideal point agree on the ranking of all the other alternatives. In this paper we have introduced a way to relax this strong homogeneity assumption by considering individual distance perceptions. We assess whether model predictions are robust in the sense that they are independent of the perceived distances. In particular, we confirm robustness of the equilibria found for two and three players on a tree graph by Eiselt and Laporte (1991, 1993). And we find strong support for a conjecture of the "principle of local clustering" articulated by Eaton and Lipsey (1975, p. 46) who further explain that "[t]he principle of minimum differentiation is a special case of the principle of local clustering when the number of firms in the market is restricted to two." In fact, we find for graphs that all robust equilibria satisfy local clustering and some of them also minimal differentiation. On the other hand, not all results from models of spatial competition are robust with respect to heterogeneous distance perceptions. Especially in graphs without cut vertices the existence of robust equilibria is highly restricted. We illustrate this in an example of uniform distribution of agents along a cycle graph (discussed by Mavronicolas *et al.*, 2008) and provide general structural conditions for the existence of robust equilibria.

Models of spatial competition predominantly deal with three important applications: (i) firms that strategically locate facilities (e.g., Eiselt and Laporte, 1993), (ii) political candidates who strategically choose a political platform (e.g., de Palma *et al.*, 1990), and (iii) firms that strategically choose a product specification (e.g., Eaton and Lipsey, 1975). For the application of political competition our results predict the minimal differentiation on the median voter if there are two players. For more players, parties still cluster in robust equilibria such that for every party there is at least one other that is very similar. For the application of geographical locations our results suggest that local clusters form, while areas in between are empty. Our result that robust locational choices are not Pareto efficient (Proposition 3) is in line with Hotelling's conjecture and the findings of Eaton and Lipsey (1975). It implies that a relocation of a firm from a cluster to some unoccupied area would increase welfare by decreasing transportation costs of some consumers. For the application of consumer products, this idea of inefficiency carries over. Thus, the prediction for the supply of differentiated products on a market is that in robust equilibria some niches are not served, while other product specifications can be bought from a unnecessarily high number of firms. Of course, this welfare implication rests on the assumption that price competition, which we have not incorporated in our model, does not dominate this effect.

This discussion raises the question whether there exist robust equilibria in the aforementioned applications. As has been shown in the paper, existence of robust equilibria generically requires a highly unequal distribution of agents. For example, Proposition 2 implies that at most 20% of the agents may have their favorite object "between" the players. Interestingly, some empirical data on the geographical distribution of inhabitants suggests that the necessary inequality requirements might just be satisfied. According to the United Nations report from 2012 the rate of urbanization in more developed regions was about 78% in 2011 and it is still increasing.[19] In the US it was even higher than 82%, for example. Thus, the popoulation in more developed regions is quite unequally distributed and this suggests that if firms serve only the major cities this might well be a robust equilibrium (despite the inefficiency for consumers who live outside these cities). In the case of product or policy spaces, the exact distribution of consumers is still an open question. But if it should not meet the requirements of robust equilibria this would lead again to our main motivation that the assumption of homogeneous distances can have a strong impact on the results. In this case, the use of models of spatial competition in these applications has to be reconsidered carefully.

The practical implications of our analysis have to be treated with caution because we have focused on robustness with respect to only one – yet crucial – aspect, while we use several other model specifications that might also play an important role. In particular, we study a simultaneous move game, while models of sequential moves lead to quite different predictions about minimal differentiation (e.g., Loertscher and Muehlheusser, 2011; Prescott and Visscher, 1977), when more than two players are involved.[20] Another major modeling decision is whether continuous or discrete space is considered. We have contributed to bridging the two corresponding literatures, but it is left for future research to clarify the role of this modeling assumption; for example, by approximating a continuous space by a discrete space of shrinking steps.

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[19] United Nations, DESA (2012). World urbanization prospects: The 2011 revision.

[20] Also in the literature on sequential location choices the questionable homogeneity assumption is standard. When relaxing this assumption one can find simple three-player examples where the equilibria are not robust.

A Appendix: Proofs

Proof of Proposition 1

Necessity: Assume $s^* \in S$ is a robust equilibrium with occupied nodes $\bar{X} = \{x_1, \ldots, x_l\}$ and assume $p_x = 1$ where $x \in \bar{X}$. Let $c \in P$ be the player with $s^c = x$.

We first establish that x is not neighboring a non-trivial competitive zone, i.e., $w(Y) = 0$ for all $Y \in \mathcal{Y}$ neighboring to x. To see this suppose the opposite is true. Fix some arbitrary object $y \in Y$. Because $A_x \subseteq X$ is connected, it is possible to find edge lengths $\left(\bar{\delta}^i_e\right)_{e \in E}$ for all $i \in N$ with $\hat{x}^i \in A_x \backslash Y$ such that $d^i(y) < d^i(x')$ for any occupied position $x' \in \bar{X}$ neighboring to A_x. This implies $d^i(x) < d^i(y)$ because every path in A_x from $\hat{x}^i$ to y passes through x. Furthermore, for all $j \in N$ with favorite object in Y one can choose edge lengths $\left(\bar{\delta}^i\right)_{e \in E}$ such that $d^j(y) < d^j(x'') < d^j(x)$, where $x \neq x'' \in \bar{X}$ is some occupied position also neighboring to Y. Then the payoff of player c is $\pi^c_{\bar{\delta}}(s^*) = w(A_x) - w(Y) < w(A_x) = \pi^c_{\bar{\delta}}\left(y, s^{*-c}\right)$. Since she can now beneficially deviate, s^* is not a robust equilibrium.

Furthermore, if $s^* \in S$ is robust, an isolated player $c \in P$ may never have an incentive to deviate to an indirectly neighboring position $x' \in \bar{X}$. Because the weight of all competitive zones surrounding x equals 0, $\pi^c(s^*) = w\left(H_x(s^*)\right) = w(A_x)$ for all perceptions of distances. Suppose c relocates to x'. Similar as before, it is possible to construct individual distances $\left(\bar{\delta}^i_e\right)_{e \in E}$ for all $i \in N$ such that every agent with favorite object in $A_{x'}$ or A_x strictly prefers x' to any other occupied position, i.e., $\pi^c_{\bar{\delta}}\left(x', s^{*-c}\right) = \frac{w(A_{x'})+w(A_x)}{p_{x'}+1}$. But this implies

$$\pi^c(s^*) = w\left(H_x\right) \geq \underbrace{\frac{w(A_{x'}) + w(H_x)}{p_{x'} + 1} \qquad \forall\, x' \text{ indirectly neighboring to } x.}_{\text{highest possible payoff at occupied and indirectly neighboring nodes}}$$

$$\Leftrightarrow \qquad w(H_x) \geq \frac{w(A_{x'})}{p_{x'}} \qquad \forall\, x' \text{ indirectly neighboring to } x. \tag{2}$$

Now let $p_x \geq 1$. Because $s^* \in S$ is supposed to be a robust equilibrium, it is not possible to perturb distances in such a way that a player can increase her payoff. This implies that the payoff she can attain at least has to be greater than the highest possible gain she can reach if she deviates. With similar arguments as in the case $p_x = 1$ this yields

$$\underbrace{\frac{w\left(H_x\right)}{p_x}}_{\text{worst case payoff at } x} \geq \underbrace{w(\hat{Z})}_{\text{best case payoff at unoccupied nodes}},$$

where $\hat{Z} \in Z$ is the heaviest unoccupied zone, and

$$\frac{w\left(H_x\right)}{p_x} \geq \underbrace{\frac{w(A_{x'})}{p_{x'} + 1} \quad \forall\, x' \in \bar{X} \setminus \{x\}.}_{\text{best case payoff at already occupied nodes}} \tag{3}$$

If $p_x = 1$, (2) already implies (3) for indirectly neighboring objects.

Sufficiency: Now assume the requirements from the proposition are satisfied. We have to show that the strategies where p_x players locate at $x \in \bar{X}$ constitute robust equilibria. First consider the case $p_x = 1$, i.e., a singly occupied node. Conditions (3.) and (4.) make sure that the player cannot improve by deviating to a neighboring competitive zone or by deviating to a directly or indirectly neighboring occupied node. Condition (1.) assures that she cannot improve by deviating to any other unoccupied zone and by Condition (2.) she cannot improve by deviating to any other occupied node. Now, let $p_x > 1$. For a player located on $x \in \bar{X}$, Condition (1.) assures that he cannot improve by deviating to any other unoccupied zone and Condition (2.) assures that he cannot improve by deviating to any other occupied node. $\square$

Proof of Proposition 2

Let $s^* \in S$ be a robust equilibrium and $\underline{x} \in \bar{X}$ be the position with lowest worst case payoff, i.e., $\frac{w(H_{\underline{x}})}{p_{\underline{x}}} \leq \frac{w(H_x)}{p_x}$ for all $x \in \bar{X}$. Then Proposition 1 Condition (2.) implies

$$\begin{aligned} w(H_{\underline{x}}) &\geq \frac{p_{\underline{x}}}{p_x+1} w(A_x) = \frac{p_{\underline{x}}}{p_x+1} \left(w(H_x) + \sum_{Y \in \mathcal{Y},\, Y \subseteq A_x} w(Y) \right) \\ &\geq \frac{p_x}{p_x+1} w(H_{\underline{x}}) + \frac{p_{\underline{x}}}{p_x+1} \underbrace{\sum_{Y \in \mathcal{Y},\, Y \subseteq A_x} w(Y)}_{=: w(\mathcal{Y}_x)}. \end{aligned}$$

and, consequently, $w(H_{\underline{x}}) \geq p_{\underline{x}} w(\mathcal{Y}_x)$ for all $x \in \bar{X} \setminus \{\underline{x}\}$, where $w(\mathcal{Y}_x)$ is the aggregated weight of competitive zones surrounding $x \in \bar{X}$.

Case 1: $p_{\underline{x}} = 1$

Here Proposition 1 Condition (3.) implies $w(Y_x) = 0$ for all $Y \subseteq A_{\underline{x}}$ and, thus, $w(H_{\underline{x}}) \geq p_{\underline{x}} w(\mathcal{Y}_{\underline{x}}) = 0$. Then:

$$\begin{aligned} n = \sum_{x \in \bar{X}} w(H_x) + \sum_{Y \in \mathcal{Y}} w(Y) &\geq \sum_{x \in \bar{X}} p_x \cdot \frac{w(H_{\underline{x}})}{p_{\underline{x}}} + \sum_{Y \in \mathcal{Y}} w(Y) \\ &\geq \sum_{x \in \bar{X}} p_x \underbrace{w(\mathcal{Y}_x)}_{=0, \text{ if } p_x = 1} + \sum_{Y \in \mathcal{Y}} w(Y) \\ &\geq 2 \sum_{x \in \bar{X}} w(\mathcal{Y}_x) + \sum_{Y \in \mathcal{Y}} w(Y) \\ &\geq 2 \left(2 \sum_{Y \in \mathcal{Y}} w(Y) \right) + \sum_{Y \in \mathcal{Y}} w(Y) = 5 \sum_{Y \in \mathcal{Y}} w(Y), \end{aligned}$$

where the last inequality is due to the fact that by definition of competitive zones each $Y \in \mathcal{Y}$ is neighboring to at least two occupied positions.

Case 2: $p_{\underline{x}} \geq 2$

If $p_x = 1$ for all $x \in \bar{X} \setminus \{\underline{x}\}$, again Condition (3.) from Proposition 1 implies $w(\mathcal{Y}_x) = 0$ for all $x \in \bar{X} \setminus \{\underline{x}\}$ and there remains nothing to show. Therefore assume that there exists at least one $x' \in \bar{X} \setminus \{\underline{x}\}$ with $p_{x'} \geq 2$. Again one can exploit Proposition 1 Condition (2.):

$$\begin{aligned} w(H_{\underline{x}}) \geq \frac{p_{\underline{x}}}{p_{x'}+1} w(A_{x'}) \quad &\Leftrightarrow \quad w(H_{\underline{x}}) \geq p_{\underline{x}} w(A_{x'}) - p_{x'} w(H_{\underline{x}}) \\ &\Leftrightarrow \quad w(H_{\underline{x}}) \geq p_{\underline{x}} w(H_{x'}) - p_{x'} w(H_{\underline{x}}) + p_{\underline{x}} w(\mathcal{Y}_{x'}) \end{aligned}$$

and, analogously,

$$w(H_{x'}) \geq \frac{p_{x'}}{p_{\underline{x}}+1} w(A_{\underline{x}}) \quad \Leftrightarrow \quad w(H_{x'}) \geq p_{x'} w(H_{\underline{x}}) - p_{\underline{x}} w(H_{x'}) + p_{x'} w(\mathcal{Y}_{\underline{x}}).$$

Now the rest of the proof proceeds similarly to Case 1. According to (1) we can again decompose the graph in hinterlands and competitive zones and by using $w(H_{\underline{x}}) \geq p_{\underline{x}} w(\mathcal{Y}_x)$ for all $x \in \bar{X} \setminus \{\underline{x}\}$ one gets

$$\begin{aligned}
n &= w(H_{\underline{x}}) + w(H_{x'}) + \sum_{x \in \bar{X} \setminus \{\underline{x}, x'\}} w(H_x) + \sum_{Y \in \mathcal{Y}} w(Y) \\
&\geq p_{\underline{x}} w(H_{x'}) - p_{x'} w(H_{\underline{x}}) + p_{\underline{x}} w(\mathcal{Y}_{x'}) + p_{x'} w(H_{\underline{x}}) - p_{\underline{x}} w(H_{x'}) + p_{x'} w(\mathcal{Y}_{\underline{x}}) \\
&\quad + \sum_{x \in \bar{X} \setminus \{\underline{x}, x'\}} p_x \cdot \frac{w(H_{\underline{x}})}{p_{\underline{x}}} + \sum_{Y \in \mathcal{Y}} w(Y) \\
&\geq p_{\underline{x}} w(\mathcal{Y}_{x'}) + p_{x'} w(\mathcal{Y}_{\underline{x}}) + \sum_{x \in \bar{X} \setminus \{\underline{x}, x'\}} p_x \underbrace{w(\mathcal{Y}_x)}_{=0, \text{ if } p_x = 1} + \sum_{Y \in \mathcal{Y}} w(Y) \\
&\geq 2 \sum_{x \in \bar{X}} w(\mathcal{Y}_x) + \sum_{Y \in \mathcal{Y}} w(Y) \\
&\geq 2 \left(2 \sum_{Y \in \mathcal{Y}} w(Y) \right) + \sum_{Y \in \mathcal{Y}} w(Y) = 5 \sum_{Y \in \mathcal{Y}} w(Y)
\end{aligned}$$

Again, the last inequality holds because each $Y \in \mathcal{Y}$ is neighboring to at least two occupied positions. $\square$

Proof of Proposition 4

To show the proposition, assume the opposite is true: that is, assume there exists a robust equilibrium $s^* \in S$. Let $\hat{Z} \in \mathcal{Z}$ be the heaviest unoccupied zone with respect to s^*. Given the requirements of the proposition, we will show that in each robust equilibrium there exists an occupied node which is heavier than $\hat{Z}$. But if the variance becomes small this leads to a contradiction. The proof proceeds in five steps:

Step 1: The ϵ-ν_ϵ-criterion.

Consider the mapping $\|\cdot\|_1 : \mathbb{R}^\xi \longrightarrow \mathbb{R}$ with $\|w\|_1 = \sum_{x \in X} |w_x|$, also known as the Manhattan norm. It is well-know that $\|\cdot\|_1$ is continuous. Thus, for all $\epsilon > 0$ there exists some $\nu_\epsilon > 0$ such that $\|w - w'\|_2 < \nu_\epsilon$ implies $\|w - w'\|_1 < \epsilon$ for all $w, w' \in \mathbb{R}^\xi$, where $\|w - w'\|_2 = \sqrt{\sum_{x \in X} (w_x - w'_x)^2}$ is, as usual, the Euclidean norm. Let $\epsilon := \frac{2p}{5(p+1)} \cdot \frac{n}{\xi}$. Furthermore, in the following let w' be the uniform distribution $w'_x := \frac{n}{\xi}$ for all $x \in X$.[21] Having specified these variables the ϵ-ν_ϵ-criterion from above implies that there exists some $\nu := \nu_\epsilon^2 > 0$ such that from $\sqrt{\mathrm{Var}(w)} = \|w - w'\|_2 < \sqrt{\nu}$ always $\sum_{x \in X} |w_x - \frac{n}{\xi}| < \epsilon = \frac{2p}{5(p+1)} \cdot \frac{n}{\xi}$ follows. Correspondingly, for the rest of the proof it is assumed that there is given a tupel of node weights $(w_x)_{x \in X}$ (i.e., $w \geq 0$ and $\sum_{x \in X} w_x = n$) with $\mathrm{Var}(w) < \nu$.

[21] Because the fraction $\frac{n}{\xi}$ need not be an integer, the uniform distribution cannot always be induced by allocating n agents to nodes. Still, it is possible to study the node-weighted graph (X, E, w').

Step 2: We establish that $\left|w(\hat{X}) - |\hat{X}|\frac{n}{\xi}\right| < \epsilon$ for all $\hat{X} \subseteq X$.

If $\mathrm{Var}(w) < \nu$, Step 1 implies for all subsets $\hat{X} \subseteq X$,

$$\left|w(\hat{X}) - |\hat{X}|\frac{n}{\xi}\right| = \left|\sum_{x\in\hat{X}} \left(w_x - \frac{n}{\xi}\right)\right| \leq \sum_{x\in\hat{X}} \left|w_x - \frac{n}{\xi}\right| \leq \sum_{x\in X} \left|w_x - \frac{n}{\xi}\right| < \epsilon.$$

Step 3: We establish that $\sum_{x\in\bar{X}} a_x \leq b$.

The main intuition of this step is that all unoccupied zones can be covered by blocks of the graph and we will show that minimal covers of different zones have to be disjoint. Let $Z_x \neq Z'_{x'}$ be two unoccupied zones in the hinterland of x and x', respectively, where $x, x' \in \bar{X}$. Note that $x = x'$ is allowed but, nevertheless, the two zones may not be equal. If it is not possible to find such two zones, $\sum_{x\in\bar{X}} a_x \leq 1$ and there remains nothing to show. According to Section 2 let $\mathcal{B}$ be the set of blocks. Obviously $X = \bigcup_{B\in\mathcal{B}} B$ holds. Therefore there exist $\mathcal{B}^{Z_x}, \mathcal{B}^{Z_{x'}} \subseteq \mathcal{B}$ with $Z_x \subseteq \bigcup_{B\in\mathcal{B}^{Z_x}} B$ and $Z_{x'} \subseteq \bigcup_{B\in\mathcal{B}^{Z_{x'}}} B$ such that both sets are minimal with respect to inclusion, i.e., $\hat{\mathcal{B}} \subsetneq \mathcal{B}^{Z_x}$ implies $Z_x \nsubseteq \bigcup_{B\in\hat{\mathcal{B}}} B$ (analogously for $\hat{\mathcal{B}} \subsetneq \mathcal{B}^{Z_{x'}}$). Given the construction of blocks, the two sets $\mathcal{B}^{Z_x}$ and $\mathcal{B}^{Z_{x'}}$ must be disjoint because otherwise there would be a path from Z_x to $Z'_{x'}$ not passing through x and x', which is not possible due to the definition of hinterlands. Thus:

$$\sum_{x\in\bar{X}} a_x = \sum_{x\in\bar{X}} \sum_{Z_x\in\mathcal{Z},\, Z_x\subseteq H_x} 1 \leq \sum_{x\in\bar{X}} \sum_{Z_x\in\mathcal{Z},\, Z_x\subseteq H_x} \left|\mathcal{B}^{Z_x}\right| \leq |\mathcal{B}| = b$$

Step 4: We establish that $w_{x'} \geq w(\hat{Z})$ for some $x' \in \bar{X}$.

As has already been shown in Step 3, the number of hinterlands is bounded by b and, thus, $\sum_{x\in\bar{X}} a_x \leq b < p = \sum_{x\in\bar{X}} p_x$. Therefore there exists some $x' \in \bar{X}$ with $a_{x'} \leq p_{x'} - 1$ and by applying Corollary 4 this yields $w_x \geq w(\hat{Z})$. In words: there necessarily exists an occupied node which is heavier than the heaviest unoccupied zone.

Step 5: The final contradiction.

Because the number of hinterlands is smaller than the number of players and because of Proposition 2 the average weight of unoccupied zones in hinterlands needs to be relatively high:

$$w(\hat{Z}) \geq \underbrace{\frac{\sum_{x\in\bar{X}} w(H_x) - w(\bar{X})}{\sum_{x\in\bar{X}} a_x}}_{\text{average weight of unoccupied zones in hinterlands}} > \frac{\frac{4}{5}n - w(\bar{X})}{p}$$

Moreover, because of Step 4 this implies that x' must be relatively heavy as well, $w_{x'} > \frac{\frac{4}{5}n - w(\bar{X})}{p}$. But then from Step 2 follows

$$\frac{n}{\xi} + \epsilon > \frac{\frac{4}{5}\xi\frac{n}{\xi} - \left(|\bar{X}|\frac{n}{\xi} + \epsilon\right)}{p} \geq \frac{\frac{12}{5}p\frac{n}{\xi} - p\frac{n}{\xi} - \epsilon}{p} = \frac{7n}{5\xi} - \frac{\epsilon}{p}$$

which contradicts $\epsilon = \frac{2p}{5(p+1)} \cdot \frac{n}{\xi}$. Therefore, s^* cannot be a robust equilibrium. $\square$

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