Universität Paderborn
Fachgebiet Rechnernetze

Summer term 2017

Empiric performance evaluation

Homework 3: Poisson processes

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Note: Order of questions swapped; it makes more sense to start with the equivalency proof of interarrival times and counting process characterization.

Note 2: think of this as solution sketches. There are a couple of details missing (e.g., we don't explicitly prove independence), but that should be enough to get you an idea.

1. Show that the two definitions of Poisson process (via interarrival times and via number of events in time interval) are equivalent.

   Lösung:

   - Assume that X_i are i.i.d. with $X_i \sim \exp(\lambda)$. We have to show that the increments of $N(t)$ follow a Poisson distribution.
     - We note that the process is time-invariant because of memorylessness.
     - We have hence to look at $P(N(t) = n)$.
     - Brute force: follows immediately from the quantiles of the n-fold Gamma distribution.
     - Proof directly: Trivial for $n = 0$.

     $$\begin{aligned} P(N(t) = 0) &= P(X_1 > t) \\ &= e^{-\lambda t} \end{aligned} \tag{1}$$

     For $n = 1$, look at convolution of X_1, X_2, $P(X_1 < t \wedge X_1 + X_2 > t)$.
     We have to show: $P(N(t) = 1) = \frac{(\lambda t)^1}{1!} e^{-\lambda t}$

     $$\begin{aligned} P(X_1 < t \wedge X_1 + X_2 > t) &= \int_0^t P(X_1 + X_2 > t | X_1 = s) f_{X_1}(s) \mathrm{d}s \\ &= \int_0^t P(X_2 > t - s) f_{X_1}(s) \mathrm{d}s \\ &= \int_0^t e^{-\lambda(t-s)} \lambda e^{-\lambda s} \mathrm{d}s \\ &= \int_0^t \lambda e^{-\lambda t} \mathrm{d}s \\ &= \lambda e^{-\lambda t} \int_0^t 1 \mathrm{d}s \\ &= \lambda t e^{-\lambda t} \end{aligned} \tag{2}$$

     ...and so on. Use Gamma density!

   - Assume that $N(t)$ follows a Poisson distribution. I.e., we know that $P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$. We have to show that the interarrival times are i.i.d. exponential .
     - Independence follows from independent increments of the counting process.
     - First look at $P(X_1 \leq t)$; this is exponential immediately from the definition of Poisson distribution.

More in detail: $P(X_1 \leq t) = 1 - P(X_1 > t)$. This event occurs when the interval $(0,t)$ has no events, i.e., $N(t) = 0$. Hence:

$$P(X_1 \leq t) = 1 - P(X_1 > t) = P(N(t) = 0) = 1 - e^{\lambda t}$$

by definition of the Poisson distribution.

- Now look at $P(X_2 > t)$. Condition this on the time of the first arrival $X_1 = s$.
  $P(X_2 > t) = \int_{s=0}^{\infty} P(X_2 > t | X_1 = s) f(s) ds$
  (where f is the density of the exponential distribution, which we have already established for X_1).
  For the first event to happen at s and the second event to happen more than t units after event 1, there must be no event occurring in interval $(s, s+t)$. This interval has length t, and because of independent increments, it doesn't matter whether we look at $(s, s+t)$ or $(0,t)$.
  Hence, the above equation simplifies to
  $P(X_2 > t) = \int_{s=0}^{\infty} P(N(t) = 0) f(s) ds = P(N(t) = 0) \int_{s=0}^{\infty} f(s) ds = P(N(t) = 0) = e^{-\lambda t}$
  Note that the second-to-last equality follows because we integrate over the entire support of the density, which must by definition give 1.
- Continue inductively for $X_3, \ldots$.

2. Show that, for the special case where A and B are Poisson processes, their superposition is again a Poisson process, the rate of which is the sum of the rates of A and B.

   Lösung: Easiest approach: Count process!

   Let N_1 and N_2 be the count processes for the two individual, stochastically independent processes to be superimposed, with rates λ_1 and λ_2, respectively. Obviously, $N(t) = N_1(t) + N_2(t)$; the number of events that have happened up to t in the superimposed process is the sum of the number of events in the two constituting processes.

   We have to show: $N(t)$ follows a Poisson distribution with rate $\lambda_1 + \lambda_2$. We use law of total probability and condition on N_1.

   $P(N(t) = k) = P(N_1(t) + N_2(t) = k) = \sum_{n_1=0}^{k} P(N_1(t) + N_2(t) = k | N_1(t) = n_1) P(N_1(t) = n_1)$

   Simplifying the first term gives: $\ldots = \sum_{n_1=0}^{k} P(N_2(t) = k - n_1) P(N_1(t) = n_1)$.

   Plugging in definition of Poisson PMF:

   $\ldots = \sum_{n_1=0}^{k} \frac{(\lambda_2 t)^{k-n_1}}{(k-n_1)!} e^{-\lambda_2 t} \frac{(\lambda_1 t)^{n_1}}{(n_1)!} e^{-\lambda_1 t}$

   Rearrange terms and multiply by $\frac{k!}{k!}$:

   $\ldots = e^{-\lambda_1 t} e^{-\lambda_2 t} \frac{1}{k!} \sum_{n_1=0}^{k} \frac{k!}{(n_1)!(k-n_1)!} (\lambda_2 t)^{k-n_1} (\lambda_1 t)^{n_1} = e^{-\lambda_1 t} e^{-\lambda_2 t} t^k \frac{1}{k!} \sum_{n_1=0}^{k} \frac{k!}{(n_1)!(k-n_1)!} \lambda_2^{k-n_1} \lambda_1^{n_1}$

   But the sum is just $(\lambda_1 + \lambda_2)^k$!

   Hence we obtain:

   $\ldots = \frac{((\lambda_1 + \lambda_2) t)^k}{k!} e^{-(\lambda_1 + \lambda_2) t}$

   which is just what we had to show!

   Note: In passing we have shown the sum of two independent Poisson random variables is again Poisson with the sum of the rates as parameter.

3. Let A be a Poisson process with rate λ. For each event in A, accept it with probability p and reject with probability $1 - p$ (*splitting* the Poisson process). Show that the sequence of accepted events is again a Poisson process with rate $p\lambda$.

   Lösung:

- Probably easiest approach: Count process again.

  Let $N_s(t)$ be the count process for the splitted process, $N(t)$ the original process, p the acceptance probability for each event.

  Fairly straightforward, condition $P(N_s(t) = k)$ on the number of events that have happened in the original process in this interval. Obviously, for k events to remain after splitting, the original process must have generated at least k events in the same time interval.

  $P(N_s(t) = k) = \sum_{m=k}^{\infty} P(N_s(t) = k | N(t) = m) P(N(t) = m)$

  Now $P(N(t) = m) = \frac{(\lambda t)^m}{m!} e^{-\lambda t}$ by definition of Poisson process.

  $P(N_s(t) = k | N(t) = m)$ occurs when out of these m events generated by the original process, any k were chosen. Each choice is an independent Bernoulli trial with success probability p, hence the number of successes is just Binomially distributed.

  Hence: $P(N_s(t) = k | N(t) = m) = \binom{m}{k} p^k (1-p)^{m-k}$

  Putting this together give us:

  $P(N_s(t) = k) = \sum_{m=k}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} \frac{(\lambda t)^m}{m!} e^{-\lambda t}$

  Rearrange terms and shift the start index from m to 0:

  $P(N_s(t) = k) = p^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} \sum_{m=k}^{\infty} (1-p)^{m-k} \frac{(\lambda t)^{m-k}}{(m-k)!}$

  Look at the sum term: $\sum_{m=k}^{\infty} (1-p)^{m-k} \frac{(\lambda t)^{m-k}}{(m-k)!} = \sum_{m=0}^{\infty} (1-p)^m \frac{(\lambda t)^m}{(m)!}$, which is just the Taylor expansion for $e^{(1-p)\lambda t}$

  Bringing this back together: $P(N_s(t) = k) = p^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} e^{(1-p)\lambda t} = \frac{(p\lambda t)^k}{k!} e^{-p\lambda t}$

  This is just what we had to show! QED

- Note that the solution via the IAT characterization is also instructive. We have to compute sums of exponentially distributed random variables, giving Gamma distributed RVs, which we sum up again weighted by their occurrence, which is geometrically distributed. After some basic algebraic drudgery, we end up with the same result. But this is not necessarily fun ....