Universität Paderborn
Computer networks group

Winter term 2017/2018

Mobile communications

Homework assignment 4: Spectral efficiency, bit errors in fading channels, DSSS

Due date: 2017-11-15

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1. **Energy per bit vs. SNR: Spectral efficiency**

   (a) Modulation symbols are transmitted with a duration of $T_s = 4\,\mu$s. Each symbol carries 2 bits. Supposing we receive 2 mW, how much energy *per bit* E_b is at most available at the mobile terminal? For simplification, assume that each bit within the symbol carries the same energy. Which unit does your result have?

   Solution: With $P_{\text{rec}} = 2\,\text{mW}$. and note that $1\,\text{W} = 1\,\text{J/s}$. Thus we receive 2 Millijoules if we transmit for one second. In 4 micro seconds, we receive

   $$r = 4 \cdot 2 \cdot 10^{-9}\,\text{J}$$

   As in 4 micro seconds 2 bits are transmitted, a bit has at most $4 \cdot 10^{-9}\,\text{J}$.

   (b) Now relate the energy per bit to the *average* noise power per Hertz, N_0. Discuss what the resulting E_b/N_0 ratio means for the quality of transmissions.

   Solution: Ratio of *mean* energy per bit to *mean* noise energy per bit (noise power per Hertz = energy!). Usually denoted in dB. Error probability decreases with increasing E_b/N_0.

   CAREFUL: The ratio talks about mean values. Errors can occur even at instantaneously high ratio (albeit unlikely, depends on coding); successful transmission can happen even at instantaneously low values (albeit very unlikely).

   Compare discussion about coding during class!

   (c) Explain why the bit error probability increases if higher-order modulations are used that carry many bits per symbol, e.g., Quadrature Phase Shift Keying (QPSK); all other parameters are assumed to be constant.

   (d) In which scenarios is it appealing to use higher-order modulations?

(e) Assume you are using a modulation with M bits per symbols and a symbol duratin T_S and a symbol bandwidth of δf. Relate the E_b/N_0 ratio to the Signal-to-Noise Ratio (SNR). How can you interpret the factor between these two ratios?

Solution: Notation: M is number of bits per symbol; T_S symbol time; Δf the used symbol bandwidth; E_S the energy per symbol.

Then recall: $E_b = E_S/M = (ST_S)/M$. $N_0 = N/\Delta f$. Hence:

$$\frac{E_b}{N_0} \cdot \frac{M}{T_S \Delta f} = \frac{S}{N}$$

E_b/N_0 refers to mean energy per bit per symbol time; SNR talks about mean power ratios and does not depend on symbol time or number of bits per symbol. The ratio $\frac{M}{T_S \Delta f}$ is the *spectral efficiency*, bits per second, per Hertz.

2. **Bit error rates of BPSK in fading channels; diversity benefits**

As we know from class, an AWGN channel (i.e., signal strength is fixed, noise is a Gaussian random variable) is charaterized by its *mean* SNR Γ. The bit error rate of BPSK over such a channel is approximated by the upper bound $P_{\text{BER}}^{\text{AWGN}} \leq \frac{1}{2}\exp(-\Gamma)$. We will work with this approximate error bound in the following.

Now let us assume that the channel is subject to Rayleigh-fading. Recall from class: the *instantaneous* SNR in a Rayleigh-fading channel is a random variable γ, which is exponentially distributed with mean $1/\Gamma$, i.e., $\gamma \sim \exp(1/\Gamma)$.

We want to investigate the resulting bit error rate over such a channel.

(a) Give an upper bound on the bit error rate of BPSK in presence of fading, $P_{\text{BER}}^{\text{Rayleigh}}$, as a function of Γ. Which impact does fading have on the bit error rate? Compare with the given result for the AWGN channel!

Hint: Understand that a particular realization of a fading channel behaves, for a short time (e.g., a bit time), like an AWGN channel with the randomly chosen signal strength. Then, use the law of total probability to compute the BER of the fading channel as an appropriately weighted sum of BERs in such AWGN channels. Make suitable independence assumptions!

Solution: We have: the distribution of the SNRs if the channel is a Rayleigh channel. As for the receiver the multipath effects are not distinguishable from white noise, the rayleigh channel seems like a AWGN channel to the receiver. So calculating the avg. BER is just summing up all BER values weighted with their probability.

$P_{\mathrm{BER}}^{\mathrm{Rayleigh}} = \int_0^\infty P_{\mathrm{BER}}^{\mathrm{AWGN}}(\gamma) f_\gamma(\gamma)\mathrm{d}\gamma \leq \frac{1}{2\Gamma}\int_0^\infty \exp\left(-\gamma(1+1/\Gamma\right)\mathrm{d}\gamma = 1/(2\Gamma+1) \sim 1/\Gamma$

AWGN: error rate decays exponentially in SNR, Rayleigh fading: only linear. Hence, increasing power in a Rayleigh channel has only marginal effect.

Note: we are implicitly assuming here that during a bit time, the channel does not change and for the next bit, a new, independent realization of the channel occurs. This is the so-called **block-fading model** – it is a stronger assumption than just looking at coherence time, but for a block duration on the order of the coherence time, the block-fading model often does give some good insights into order-of-magnitude behavior.

(b) We now employ L antennas at the receiver to benefit from receiver diversity (or any other form of L stochastically independent channels). We use maximum ratio combining, which allows us to model the *resulting* instantaneous SNR $\overline{\gamma}$ as a sum of random variables of each diversity branch, i.e., $\overline{\gamma} = \sum_{i=1}^{L} \gamma_i$.

What do you know about the CDF of the resulting random variable $\overline{\gamma}$? (This is easy to find in wikipedia or any stochastics textbook; look for the sum of iid exponential RVs.)

(For a bonus insight, do all the calculations below under the assumptions of selection combining.)

Solution: If all γ_i are independently and identically distributed (which means that the antennas are sufficiently spaced apart and path loss to the receive antenna is the same for any branch) then $\overline{\gamma}$ is Erlang-distributed, i.e., $\overline{\gamma} = \sum_{i=1}^{L} \gamma_i \sim \mathrm{Erl}(1/\Gamma, L)$.

Note: The probability density function of an Erlang-distributed r.v. X is $f_X(x) = \frac{(\lambda x)^{n-1}}{(n-1)!}\lambda \exp(-\lambda x)$ with parameters λ and n.

(c) Give an upper bound on the bit error rate of BPSK with L diversity branches in presence of fading, $P_{\mathrm{BER}}^{\mathrm{Rayleigh,L}}$, as a function of Γ. What is the effect of L?

Solution: We use the same approach as in the simple Rayleigh channel with $L = 1$ diversity branch.

Here, $\lambda = 1/\Gamma$ and $n = L$, thus $P_{\mathrm{BER}}^{\mathrm{Rayleigh,L}} = \int_0^\infty P_{\mathrm{BER}}^{\mathrm{AWGN}}(\overline{\gamma}) f_{\overline{\gamma}}(\overline{\gamma})\mathrm{d}\overline{\gamma} \leq \frac{1}{2\Gamma(L-1)!}\int_0^\infty (\overline{\gamma}/\Gamma)^{L-1} \exp\left(-\overline{\gamma}(1+1/\Gamma\right)\mathrm{d}\overline{\gamma} = ((L-1)!\Gamma)/(2(L-1)!\Gamma(\Gamma+1)^L) = 1/(2(\Gamma+1)^L) \sim 1/\Gamma^L$.

The error rate decreases as the Lth power of SNR; it is exponential *in* L, but not in SNR as it would be an AWGN channel.

(d) Is this just an effect of collecting more power via the multiple antennas? What would happen if the L channels to the L antennas were not stochastically independent, but rather perfectly identical?

What would happen when you use L antennas with stochastically independent **AWGN** channels (as opposed to fading channels)? Does it make sense to use multi-antenna systems in AWGN scenarios?

Solution: Power gain is just a constant; shifts curves left and right but does not changes the slope.

(e) For a mean SNR $0\,\mathrm{dB} \leq \Gamma \leq 30\,\mathrm{dB}$, use a log-log graph (i.e., logarithmic scale on both the SNR and the BER axis) to plot the bit error rate for the AWGN channel and the Rayleigh channel with $1 \leq L \leq 3$ diversity branches. Discuss your results.

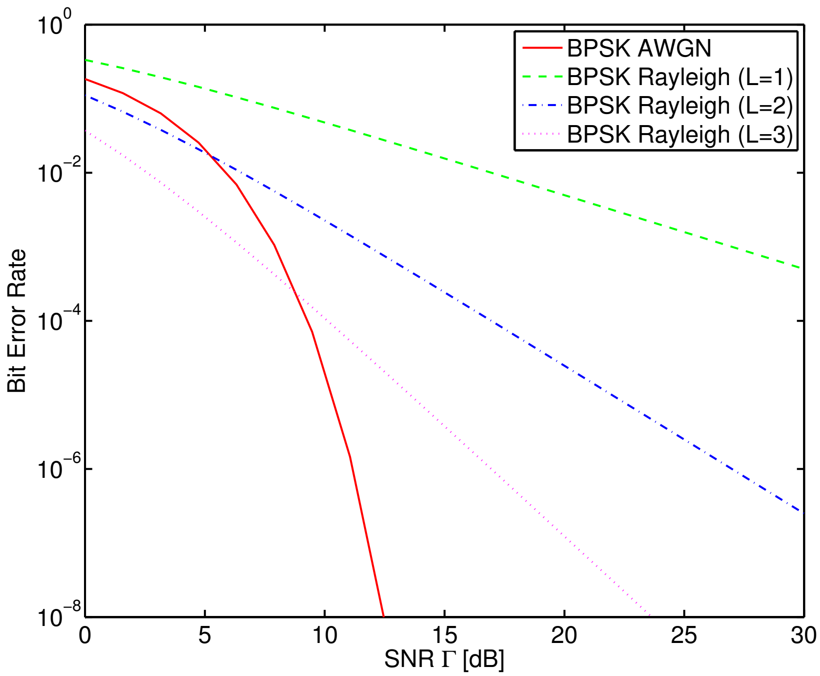
Solution: Example in MATLAB:

```
clear all
close all

SNRdB=linspace(0,30,20);
SNRlin=10.^(SNRdB./10);

BER_BPSK_AWGN=1./(2.*exp(SNRlin));
BER_BPSK_Rayleigh_L1=1./(2.*SNRlin+1);
BER_BPSK_Rayleigh_L2=1./(2.*SNRlin+1).^2;
BER_BPSK_Rayleigh_L3=1./(2.*SNRlin+1).^3;

figure;
semilogy(SNRdB,BER_BPSK_AWGN,'-r',SNRdB,BER_BPSK_Rayleigh_L1,'--g',..
        SNRdB,BER_BPSK_Rayleigh_L2,'-.b',SNRdB,BER_BPSK_Rayleigh_L3,'
xlabel('SNR \Gamma [dB]');
ylabel('Bit Error Rate');
legend('BPSK AWGN','BPSK Rayleigh (L=1)',...
    'BPSK Rayleigh (L=2)','BPSK Rayleigh (L=3)');
axis([min(SNRdB) max(SNRdB) 1e-8 1]);
```



3. **Simulate Direct Sequence Spread Spectrum**

   In this programming assignment, we develop a simple simulation of Direct-Sequence Spread Spectrum (DSSS) transmission. USe Python/numpy, Matlab, or some other tool of your choice.

   Proceed as follows:

   - Write a function that takes the following input:
     - An SNR level
     - A simple characterization of a multipath wireless channel as a sequence of attenuation values, one value for each delay of the channel by one chip length. (Recall the representation from slide 48 in Chapter 2a; extend that to multiple paths and different values of τ and h).
     - A chipping sequence
     - The number of chips per bit (or extract that from the chipping sequence)
     - The number of bits to simulate
   - Inside this function, do the following:
     - Generate random bits to transmit, according to the number of bits. Iid bits 0 or 1.

  - Spread this sequence of bits using the chipping sequence
  - Add up multiple copies of this chip sequence, one copy per delay of the channel, multiplied with the corresponding attenuation value. Obtain the chipping sequence after transmission through a multi-path channel.
  - To this sequence of chips, add additive white Gaussian noise according to the given SNR level.
  - Despread this noisy, multi-path-channel-transmitted chipping sequence. (Hint: scalar product!)
  - Compute which bits are erroneous after despreading, calculate bit error ratio.
- Call this function for varying values of SNR.
- Plot the resulting BER-over-SNR curves. Hint: double-logarithmic plot should give nice figures. Hint: You probably need to simulate quite a lot of bits to see anything meaningful for high SNR values.

Experiment with various parameters, e.g., the channel characteristics,

4. **Simulate Direct Sequence Spread Spectrum, part 2: Rake reveicer**

   In class, we briefly discussed the notion of a Rake receiver. Extend your function from the previous assignment to work as a Rake receiver. Use as additional parameters the desired number of rakes and the channel taps. Plot the same SNR curves as above, separately for varying number of rakes.

   What do you observe when you use more rakes than the channel has taps?

   If you are really courageous, try to estimate the channel taps from the received signal. This is not trivial, however!

   Solution: Compare for example dsss.py