Winter term 2017/2018

Mobile communications

Homework assignment 3: Fading, diversity, and combining

Due date: 2017-11-08

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Computer networks group

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1. **A look at various fading scenarios**

A mobile receiver moves at speed v and receives sinusoidal signals at frequency f_c. The signals arrive along two reflected paths which makes angles θ_1 and θ_2 with the direction of motion.

(a) Is the information sufficient to estimate the coherence time T_c and the coherence bandwidth W_c? If yes, express them in terms of the given parameters. If not, specify what is missing.

Solution:

What is the Coherence bandwidth?: the approximate maximum frequency interval over which two frequencies of a signal are likely to experience comparable or correlated amplitude fading.

What is the Coherence time?: Coherence time is the time duration over which the channel is considered to be not varying.

With the given information we can compute the Doppler spread

$$D_s = |f_1 - f_2| = \frac{fv}{c} |\cos\phi_1 - \cos\phi_2|$$

from which we can compute the coherence time $T_c \sim 1/D_s$.

Chapter 2: Parameters needed to compute Doppler Effect:

- Mobile terminal speed
- Transmission frequency
- Angle of arrival of wave with movement direction

There is not enough information to compute the coherence bandwidth $B_C \sim \frac{1}{50\sigma_t}$, σ_t = delay spread, as it depends on the delay spread which is not given. For this, we would need to know the difference in path length.

(b) Use T_c and W_c to characterize the following two environments and discuss the difference.

   i. Dense reflectors and scatterers in all directions from the receiver
   ii. Clustered reflectors and scatterers in a small angular range

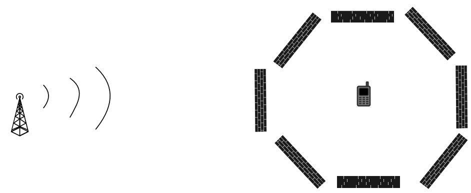

Abbildung 1: Dense scatterers

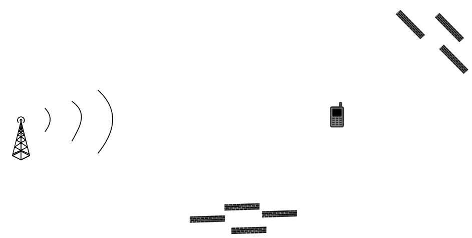

Abbildung 2: Clustered reflectors

Solution:

i. The signals are bouncing off the walls thus the interference situation between the walls is everywhere the same. Thus the coherence time is high. As due to similar scattering off multiple walls the difference in path lengths is low the delay spread is low, too. If the delay spread is low, the coherence bandwidth is high.
   This means the channel can be considered as not varying over time and equally stable over large chunks of bandwidth.

ii. As the mobile is moving the interference situation might change quickly. This leads to a short coherence time. Compared to the scenario before, the difference in path length is larger and so is the coherence bandwidth. So the channel can be considered as fast changing over time but due to the shorter coherence bandwidth a higher bandwidth used for transmission should suffice to get a good signal.

From the previous question we see that a larger angular range results in larger delay spread and smaller coherence time. Then, in the richly scattered environment the channel would show a smaller coherence time than in the environment where the reflectors are clustered in a small angular range.

2. **Compute coherence bandwidth**

   Let us look at a very simple, two path situation. Both paths from sender to receiver are assumed to have the same received signal strength; the only relevant effect is due to phase offset between these two paths, leading to constructive or destructive self-interference of the two copies of the same signal as it arrives over the two paths.

We are interested in how the chosen transmission frequency influences the received signal strength.

Assumptions and notation:

- Path lengths are d_A and d_B for two paths A and B. Let $d_B = d_A + \Delta d$.
- Speed of light $c = 300\,000\,000\,\mathrm{m/s}$.
- Let τ_A, τ_B denote propagation times along the respective paths.
- Let f, f_1, f_2 denote transmission frequencies of interest.
- Let $T = 1/f$ denote length of a period for a given frequency.
- Let n denote number of periods during a given time.
- Let $\phi(f)$ denote the *phase difference* at the receiver over these two paths A and B when using a communication frequency f.

Questions

(a) Derive an expression for the coherence bandwidth: the smallest bandwidth Δf such that changing transmission frequency from f to $f + \Delta f$ results in a significant change of received signal strength.

(b) Give examples!

(c) Relate your result to the discussion about root mean square delay (RMS) spread in class. Why is it plausible to conjecture that RMS generalizes your result? (No equations necessary.)

(d) Look at some concrete channel models (e.g., summarized here: `http://www.ericsson.com/res/thecompany/docs/journal_conference_papers/atsp/typical_urban_channel_model.pdf`). What do these numbers tell you?

Solution:

- Propagation time: $\tau_A = \frac{d_A}{c}$, $\tau_B = \frac{d_B}{c}$
- Difference in propagation time: $\Delta\tau = \tau_B - \tau_A = \frac{\Delta d}{c}$
- At transmission frequency f, how many periods happen during $\Delta\tau$?

$$n = \frac{\Delta\tau}{T} = \Delta\tau f$$

  Example: $\Delta\tau = 0.2\,\mathrm{s}$, $f = 3\,\mathrm{Hz}$. Then, $n = 0.6$ (a bit more than half a period).
- Normalize this to phase, measured in radian: $\phi(f) = 2\pi\Delta\tau f$. This is the offset of phase between the two paths, as seen when using frequency f.

- Main point: When switching from some transmission frequency f_1 to f_2, the phase offset will change from $\phi(f_1)$ to $\phi(f_2)$.

  What changes in the phase offset will lead to substantially different received power? I.e., what changes will turn a constructive interference into a destructive interference? Surely, changing the phase offset by π should do that, but already smaller changes. Hence, let's pick $\pi/2$ as the value when we regard the phase offset to have changed "substantially". (This is somewhat arbitrary, yes, but does not really matter for the order of magnitude of the result.)

  So: What is the smallest Δf such that $\phi(f_1) = \phi(f_1 + \Delta f_1) - \frac{\pi}{2}$?

  $$2\pi\Delta\tau f_1 = 2\pi\Delta\tau(f_1 + \Delta f) - \frac{\pi}{2}$$

  or, equivalently:

  $$\frac{\pi}{2} = 2\pi\Delta\tau\Delta f$$

  which yields

  $$\Delta f = \frac{c}{4\Delta d}$$

  Note: Δf does depend neither on d_A nor on f_1! Only the *difference* in path lengths is relevant.

3. **Channel Diversity and Outage**

   Consider a wireless receiver with M antennas that are arranged such that M independent Rayleigh fading channels are available at the receiver. We will use the random variable $\underline{\gamma}$ (random variables will be underlined in the following) to indicate the instantaneous SNR of a channel, or more specifically $\underline{\gamma}_i$ for the instantaneous SNR of channel i. Assume each channel i has an average SNR of Γ given as $\Gamma = \frac{E_b}{N_0}$.

   Since these are all Rayleigh-fading, we know that the probability density function $p_{\underline{\gamma}_i}$ is given by

   $$p_{\underline{\gamma}_i}(\gamma) = \frac{1}{\Gamma} \cdot e^{\frac{-\gamma}{\Gamma}}, \qquad \gamma \geq 0$$

   (a) Calculate the probability $\mathbb{P}\left[\underline{\gamma} \leq \gamma\right]$ that a channel has an instantaneous SNR smaller or equal to some constant γ.

   Solution: This is just the CDF of an exponentially distributed random variable:

   $$\mathbb{P}\left[\underline{\gamma} \leq \gamma\right] = \int_0^{\gamma} p(c)\underset{\cdot}{c} = \int_0^{\gamma} \frac{1}{\Gamma} e^{\frac{-c}{\Gamma}} \underset{\cdot}{c} = 1 - e^{-\frac{\gamma}{\Gamma}}$$

(b) Calculate the outage probability: the probability that all M channels *simultaneously* experience a SNR smaller or equal γ. (We use selection combining here.)

Solution:

$$\mathbb{P}\left[\text{outage}\right] = \mathbb{P}\left[\forall i = 1, \ldots, M : \underline{\gamma}_i \leq \gamma\right] = \left(1 - e^{-\frac{\gamma}{\Gamma}}\right)^M$$

(c) If γ_{th} ("th" for threshold) is defined to be the minimum SNR a wireless receiver can successfully work with, what does $\underline{\gamma}_i < \gamma \; \forall i \in \{1, ..., M\}$ mean?

Solution: That all of the M available channels experience too much noise or that the signal is too weak to successfully decode data from. Thus, the wireless receiver fails to receive any data.

(d) What is the probability p that *at least one* channel does not fail? (This corresponds to so-called selection combining.)

Solution:

$$p = Pr(\exists i : \gamma_i > \gamma_{\text{th}}) = 1 - Pr(\forall i : \gamma_i \leq \gamma_{\text{th}}) = 1 - \left(1 - e^{-\frac{\gamma}{\Gamma}}\right)^M$$

(e) Assume that (a) a wireless receiver can successfully decode a signal if the SNR is above $\gamma_{\text{th}} = 10$ dB and (b) the mean SNR of a channel is $\Gamma = 20$ dB.

Calculate the probability that a signal cannot be successfully decoded using either a single channel or four independent channels with selection combining. Compare these probabilities.

Solution:

$$\text{Note: } \frac{\gamma_{\text{th}}}{\Gamma} = 10\,dB - 20\,dB = -10\;dB = 0.1$$

$$\mathbb{P}\left[\gamma_1 \leq \gamma_{\text{th}}\right] = \left(1 - e^{-\frac{\gamma_{\text{th}}}{\Gamma}}\right) = \left(1 - e^{-0.1}\right) = 0.095$$

$$\mathbb{P}\left[\forall i : \gamma_i, ..., \gamma_M \leq \gamma_{\text{th}}\right] = \left(1 - e^{-\frac{\gamma_{\text{th}}}{\Gamma}}\right)^M = \left(1 - e^{-0.1}\right)^4 = 0.000082$$

(f) Using these results, under selection combining, the random variable describing the channel's behavior $\underline{\gamma}_{\text{SC}}$ is given by the maximum of the M individual channel's SNR; $\underline{\gamma}_{\text{SC}} = \max \underline{\gamma}_1, \ldots \underline{\gamma}_M$, with the selection combining channel's *effective* SNR CDF as just described.

What is the expected value Γ_{SC} of this effective SNR, corresponding to the expected value of an individual channel's SNR Γ?

Can the ratio between Γ_{SC} and Γ be interpreted as an SNR gain? How large is it?

Solution: We are looking for the expected value of $\underline{\gamma}_{\text{SC}}$ with CDF given by $\mathbb{P}\left[\underline{\gamma}_{\text{SC}} \leq \gamma\right] = \left(1 - e^{-\frac{\gamma}{\Gamma}}\right)^M$. We compute the expected value $\mathbb{E}\left[\underline{\gamma}_{\text{SC}}\right]$ by computing the density $p_{\underline{\gamma}_{\text{SC}}}(\cdot)$ from the CDF and then solving the usual integral for the expected value.

$$p_{\underline{\gamma}_{\text{SC}}}(\gamma) = \frac{\mathrm{d}}{\mathrm{d}\gamma} P_{\underline{\gamma}_{\text{SC}}}(\gamma) = \frac{M}{\Gamma}\left(1 - e^{-\frac{\gamma}{\Gamma}}\right)^{M-1} e^{-\frac{\gamma}{\Gamma}}$$

Summing up all possible values times the probability of that value gives us the expected value:

$$\Gamma_{\text{SC}} = \mathbb{E}\left[\underline{\gamma}_{\text{SC}}\right] = \int_0^\infty \gamma p_M(\gamma) d\gamma = \Gamma \int_0^\infty Mx\left(1 - e^{-x}\right)^{M-1} e^{-x} dx \quad \text{with } x = \frac{\gamma}{\Gamma}$$

As we are interested in the SNR **gain** archived by diversity, we have to calculate

$$\frac{\Gamma_{\text{SC}}}{\Gamma} = \sum_{k=1}^{M} \frac{1}{k}$$

4. **Combining schemes over multiple, independent fading channels**

   In class, we briefly discussed diversity and combining schemes. This programming assignment looks into the relative performance of these schemes when used over independent Rayleigh channels.

   Recall how combining over L independent diversity branches works:

   No combining Pick the signal of the first branch, ignoring all $L-1$ other branches. (Just for comparison.)

   Selection combining Pick the signal arriving over the branch with the strongest signal.

   Equal-gain combining Compute the combined signal as the average of all signals.

   Maximum-ratio combing Compute the combined signal as a power-weighted average of all signals.

   Your assignment is to write a simulation of these four schemes. Proceed as follows:

   - Use a modulation scheme like QPSK, represent the constellation points as complex numbers

- Vary the average SNR over a plausible range (you might have to experiment a bit here) and the number of diversity branches between 1 and (at least) 4.
- For each combination of SNR and number of branches, perform a number of repetitions (for plausible insights, you should choose modulation scheme and SNR ranges that give you error rates of down to 10^{-5} or lower – how many repetitions does that imply?)
- For each repetition, generate
  - a random constellation point (uniformly and independently chosen),
  - an exponentially distributed signal strength with expected value 1 (why?) for each diversity branch,
  - a AWGN noise value for each diversity branch with the average corresponding to the SNR level.

  Use each combining scheme to compute the combined signal and demodulate it (find the closest constellation point). Record whether you correctly or incorrectly decoded this symbol.
- Produce plots for resulting symbol error rate, shown over varying SNR, grouped by either number of diversity branches or by combining scheme. Use double-logarithmic plots for best visualisation.
- Interpret your results! Are they in line with your expectations?

Hints: With numpy and matplotlib, this is actually not much code. In particular, every combining scheme is doable in two lines of code (and probably even in a single one).